CHARACTERISTIC STATISTICAL PROBLEMS

OF STOCHASTIC GEOMETRY

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Technical Report No. 130, Series 2
Department of Statistics
Princeton University
August 1977

This work was partially supported by the Office of Naval Research contract N00014-75-C-0453 awarded to the Department of Statistics, Princeton University, Princeton, New Jersey, and written while the author was a Guggenheim Fellow.

ABSTRACT

The Buffon needle problem and some variations are used to illustrate classical statistical methods of estimation and to lead into, and contrast with, the problems which arise when a sample of some random structure is the data. The flavor of these problems is conveyed largely by discussion of the simplest, and most described, case, that of point processes.

Note: *This technical report was originally delivered as a lecture at the Buffon Bicentenary Symposium on Geometrical Probability, Image Analysis, Mathematical Stereology and their relevance to the determination of Biological Structure, Paris, June 20-24, 1977.*

1. INTRODUCTION

The Buffon needle problem has its origins in gambling. But, unlike card and dice games, it requires some geometry and calculus. Thus it was a bold step in generalizing the idea of mathematical probability. The needle problem and minor variants have been studied by many authors up to the present day as a means of estimating π statistically. More sensibly, however, they provide a means of inferring the ratio of the length of the needle to the scale of the regular network onto which it is thrown. And of course this is the origin of modern methods for the sampling study of geometric bodies. Thus it seems most appropriate at this Symposium to begin my talk by going over this work which used only quite standard statistical ideas.

However when the needle is replaced by a more general probe or sampling window -- in other words, some set -- and the regular network is replaced by a random structure, we must face statistical problems of quite a different character. They are akin to those met in time series analysis which has a long history by statistical standards but in which there is still some confusion between the exploratory, modelling and confirmatory aspects and a lack of communication between probabilists and statisticians. The spatial problems are decidedly more difficult. There are not so many explicit probability models to get statistical experience with. One must consider the shape as well as the size of sampling windows. The data has a more awkward form. The second part of the lecture will therefore merely give some idea of these problems and their literature. Hopefully other speakers will address them in detail. But it is clear that statistical geometry or morphology is just beginning.

2. BUFFON PROBLEMS

The classical problem considers a parallel grid with spacing a and a needle of length ℓ where $\ell \leq a$. If the needle is tossed so that its position and orientation are random, the probability p that it cuts a grid line is given by

$$p = \frac{2}{\pi}\frac{\ell}{a} \quad . \tag{1}$$

Uspensky (1937), Kendall and Moran (1963), for example, give proofs of this and most of the results used below.

Define $\phi = \frac{1}{\pi}$, $r = \frac{\ell}{a}$ and suppose that n independent trials (tosses) yield C cuts. Then $\mathcal{L}(C)$ = Binomial (n,p) where $p = 2\phi r$. The statistical problems that arise are

(i) estimate ϕ, i.e., estimate π

(ii) estimate r, i.e., estimate a if ℓ is known

(iii) test that $r = r_0$ (known).

The most complete reference on (i) in this case and those given below is Perlman and Wichura (1975). Oddly no one seems to have considered (ii) and (iii). Trivial though they are, they are the prototypes of the real problems.

The likelihood of the data when c cuts are observed is $\binom{n}{c}p^c(1-p)^{n-c}$ and the number of cuts is a complete sufficient statistic for p. Thus we may assert: among all functions $f(c)$ such that $Ef(C) = p$, for all p in $(0,1)$, it is true that

$$\text{var } f(C) \geq \text{var } \frac{C}{n} = n\, p(1-p) \quad . \tag{2}$$

Thus the obvious estimator of p, $\hat{p} = C/n$ is, uniformly in p, the minimum variance unbiased estimator of p.

If one knows r and wishes to estimate π, the same is true of

$$\hat{\phi} = \frac{C}{2rn}, \quad \text{var}(\hat{\phi}) = \frac{\phi^2}{n}\left(\frac{1}{p} - 1\right). \tag{3}$$

However $\hat{\pi}_1 = 1/\hat{\phi}$ is biased. If n is large, we may argue that

$$\begin{aligned} E\hat{\pi}_1 &= E\frac{1}{\phi}\left[1 + \frac{\hat{\phi}-\phi}{\phi}\right]^{-1} \\ &\approx \frac{E}{\phi}\left\{1 - \frac{\hat{\phi}-\phi}{\phi} + \frac{(\hat{\phi}-\phi)^2}{\phi^2}\right\} \\ &= \pi\left\{1 - 0 + \frac{1}{n}\left(\frac{1}{p} - 1\right)\right\} \\ &\to \pi \quad \text{as} \quad n \to \infty . \end{aligned}$$

Further it follows from (3) that

$$\text{var}\ \hat{\pi}_1 \approx \frac{1}{n\phi^2}\left(\frac{\pi a}{2\ell} - 1\right). \tag{4}$$

Thus if one is set on estimating π this way and can "design the experiment" one should take $\ell = a$ since this choice minimizes (4). In this case $\text{var}\ \hat{\pi}_1 \approx 5.63/n$. Lazzerini (1901) conducted such an experiment with $n = 3408$ and found $\hat{\pi}_1 - \pi = 3 \times 10^{-7}$. As Kendall and Moran (ibid.) suggest, he must have stopped when he noticed the remarkable and fortuitous accuracy!

In fact we know ϕ and are more likely to want to know a. By the same arguments used above, $\hat{a} = \frac{2\ell\phi n}{c}$, and we will have $E\hat{a} \to a$ as $n \to \infty$ and

$$\text{var}(\hat{a}) \approx \frac{a^3}{n}\left(\frac{\pi}{2\ell} - \frac{1}{a}\right). \tag{5}$$

Again if we can choose the needle size we should try to make it near to but less than a. Thus the effort to optimize may lead to bias since (1) is false when $\ell > a$. Trivially if n is large, $\hat{a}$ is Gaussian so tests are easy to make.

Instead of a parallel grid, Laplace considered a rectangular grid, the A lines being a apart, the B lines being b apart. He showed that the probability that the needle cuts at least one line is

$$\frac{2\ell(a+b) - \ell^2}{\pi a b} , \tag{6}$$

a fascinating formula whose direct derivation is tricky so that it seems easier to get it from a more general Crofton argument. To illustrate my points here set $a = b$ and

$$\frac{\ell}{a} = \frac{\ell}{b} = r , \quad \phi = \frac{1}{\pi} .$$

Introducing the notation

$$P_{\overline{A}B} = \text{Prob(needle cuts a B line but not an A line)},$$

$$P_{\overline{AB}} = \text{Prob(needle cuts neither an A nor a B line)},$$

etc., formula (6) is clearly $1 - P_{\overline{AB}}$. Further if

$$P_A = \text{Prob(needle cuts an A line)}$$
$$P_B = \text{Prob(needle cuts a B line)}$$

then

$$P_A = P_{AB} + P_{A\overline{B}} = 2r\phi ,$$

$$P_B = P_{AB} + P_{\overline{A}B} = 2r\phi ,$$

and

$$P_{AB} + P_{A\overline{B}} + P_{\overline{A}B} + P_{\overline{AB}} = 1.$$

These three equations plus (6) for $1 - P_{\overline{A}\overline{B}}$ yield

$$P_{AB} = r^2\phi \ , \ P_{\overline{A}B} = P_{A\overline{B}} = r(2-r)\phi \tag{7}$$

and for brevity we follow Perlman and Wichura in writing

$$P_{\overline{A}\overline{B}} = 1 - (4r-r^2)\phi = 1 - m\phi \ . \tag{8}$$

Let n trials yield results (in an obvious notation) $\underline{N} = \{n_{AB}, n_{A\overline{B}}, n_{\overline{A}B}, n_{\overline{A}\overline{B}}\}$. Then

$$\mathcal{L}(N) = \text{4-nomial } (n; P_{AB}, P_{A\overline{B}}, P_{\overline{A}B}, P_{\overline{A}\overline{B}}) \ .$$

Thus the likelihood of the data is proportional to

$$L = P_{AB}^{\ n_{AB}} \ P_{A\overline{B}}^{\ n_{A\overline{B}}} \ P_{\overline{A}B}^{\ n_{\overline{A}B}} \ P_{\overline{A}\overline{B}}^{\ n_{\overline{A}\overline{B}}} \ .$$

Defining

$$\begin{aligned} N_0 &= n_{\overline{A}\overline{B}} = \#\text{ no cuts} \\ N_1 &= n_{A\overline{B}} + n_{\overline{A}B} = \#\text{ 1 cuts} \\ N_2 &= n_{AB} = \#\text{ 2 cuts} \\ n &= N_0 + N_1 + N_2 \end{aligned}$$

and using (7) and (8), L may be written as

$$L = (1-m\phi)^{N_0} \ \phi^{N_1+N_2} \ r^{2N_2+N_1} \ (2-r)^{N_1}. \tag{9}$$

Thus if r is known, N_0 or $N_1 + N_2$ is a complete sufficient statistic for ϕ and $L(N_1+N_2) = \text{Binomial}(n, m\phi)$ so the story of the estimator of π follows the previous pattern. The resulting estimator $\hat{\pi}_2$ has, for n large, a variance equal to $0.47/n$ so that the extra work in using a square grid yields an estimator which is 12 times as efficient as that for the parallel grid.

However to estimate r knowing π is quite different. The practical method is to choose r to maximize the likelihood (9). Setting $\partial \log L/\partial r$ equal to zero leads to the equation

$$\frac{N_0}{1 - \frac{4r-r^2}{\pi}} \; \frac{2r-4}{\pi} + \frac{2N_2+N_1}{r} - \frac{N_1}{2-r} = 0 \qquad (10)$$

which must be solved iteratively to yield $\hat{r}$. The standard theory of maximum likelihood estimation gives us an asymptotic formula for $\hat{r}$,

$$\text{var } \hat{r} \sim 1 \Big/ - \left(\frac{\partial^2 \log L}{\partial r^2}\right)_{\hat{r}} \qquad (11)$$

and asymptotic normality of $\hat{r}$ so that tests can be made.

The rectangular grid follows the same pattern with $r_1 = \frac{\ell}{a}$, $r_2 = \frac{\ell}{b}$, L is a function of r_1 and r_2 which are estimated by solving $\partial \log L/\partial r_1 = 0$, $\partial \log L/\partial r_2 = 0$. Other regular networks do not introduce the need for further techniques.

Above we considered only the case of $\ell \le a = b$. The case when ℓ is <u>much</u> greater than a is simple and instructive to consider. Let then this long needle intersect the B lines at an angle θ. Then θ is uniformly distributed on $(0,\pi/2)$. If we define

$$\left.\begin{aligned} N_A &= \#\ A \text{ lines cut} \simeq r\cos\theta\,, \\ N_B &= \#\ B \text{ lines cut} \simeq r\sin\theta\,, \\ N &= \#\text{ lines cut} \simeq r(\cos\theta + \sin\theta)\,, \end{aligned}\right\} \qquad (12)$$

we have

$$EN_A = \frac{2}{\pi}\int_0^{\pi/2} r \cos\theta \, d\theta = \frac{2r}{\pi} = E(N_B) \ ,$$

$$EN = \frac{4r}{\pi} \ ,$$

$$EN^2 = E(r^2 + 2r^2 \cos\theta \sin\theta) \ ,$$

$$= r^2\left(1 + \frac{2}{\pi}\right) \ ,$$

so

$$\operatorname{var} N = r^2\left(1 + \frac{2}{\pi} - \left(\frac{4}{\pi}\right)^2\right) \ .$$

Hence if we make n throws and find $\bar{c}$ as the average number of cuts, it will be an unbiased estimator of $E(N)$. Hence the estimator of π that is suggested, following our ealier work, is

$$\hat{\pi}_3 = \frac{4r}{\bar{c}}$$

and an easy calculation shows that

$$\operatorname{var} \hat{\pi}_3 \approx \frac{\pi^4}{16r^2} \ \frac{\operatorname{var} N}{n}$$

$$\approx \frac{0.0095}{n} \ .$$

While this seems a great improvement, we will show below that one can do better still with this experiment.

To estimate r from $\bar{c}$, the "natural" method is to set $r = \frac{\pi}{4}\bar{c}$ with $\operatorname{var} \hat{r} \approx \frac{r^2}{n} \frac{\pi^2}{4^2}\left(1 + \frac{2}{\pi} - \frac{16}{\pi^2}\right)$. This estimate too can be improved because neither is the maximum likelihood estimator, as was true of our first three examples.

From (12), $N = \sqrt{2}\, r \cos\left(\theta - \frac{\pi}{4}\right)$ so that $\text{Prob}(N \le k) = \text{Prob}(\cos\psi \le k/r\sqrt{2})$ where $\mathcal{L}(\psi)$ is uniform on $(-\pi/4, \pi/4)$. Let $\psi_0 = \cos^{-1}\frac{k}{r\sqrt{2}}$. Then

$$\begin{aligned} \text{Prob}(N \le k) &= 2\,\text{Prob}(\psi > \psi_0)\,, \\ &= \frac{4}{\pi}\left(\frac{\pi}{4} - \psi_0\right)\,, \end{aligned} \tag{13}$$

so that the probability density of N at k is the partial derivative of (13) with respect to k, namely

$$\frac{4}{\pi}\,\frac{1}{\sqrt{2r^2 - k^2}}\,.$$

Thus given counts $k_1,\ldots,k_n$ in n trials, their likelihood is

$$\prod_1^n \frac{4}{\pi}\,\frac{1}{\sqrt{2r^2 - k_i^2}}$$

provided $r \le$ all the k_i's $\le \sqrt{2}\, r$, and zero otherwise. Thus the maximum likelihood estimate of r is

$$r^* = \frac{1}{\sqrt{2}}\max(k_1,\ldots,k_n)\,, \tag{14}$$

not $\hat{r} = \pi\bar{c}/4$. It may be shown that the variance of r^* is of order n^{-2}, not order n^{-1} like that for $\hat{r}$. Thus for large n, r^* is a very much better estimator than $\hat{r}$. This shows dramatically that the usual practice in geometrical statistics of obtaining estimators by equating theoretical and observed means may be <u>very</u> inefficient. So much for the "long needle." Other details may be found in Diaconis (1976) and in a forthcoming monograph by H. Solomon.

Buffon's needle may be used to obtain a connection with a quite different aspect of geometrical statistics. Let us analyze the tossing of the needle. Suppose now that an origin is marked on one of the lines of the parallel grid and that the needle is thrown so that its center rests on the plane a distance X from the marked line. Let $\mathcal{L}(X)$ = Gaussian $(0,\sigma^2)$ so it has probability density

$$f(x) = (\sigma\sqrt{2\pi})^{-1}\exp(-x^2/2\sigma^2) .$$

It is then clear that if Y is the distance from the center of the needle to the nearest line below it,

$$\begin{aligned} \text{Prob}(x < Y \leq x + dx) &= \sum_{\nu=-\infty}^{\infty} f(x - \nu a)dx \\ &= g(x)dx \text{ , say.} \end{aligned} \tag{15}$$

The density $g(x)$ is concentrated on $(0,a)$ and $g(x)$ is periodic, period a. Thus we may write

$$g(x) = \sum_{j=-\infty}^{\infty} g_j \exp(-2\pi i j x/a) . \tag{16}$$

It is shown that (see Hartman and Watson (1974)) this density can be very well approximated by

$$\left(2\pi I_0(\kappa)\right)^{-1}\exp \kappa \cos (2\pi x/a) \tag{17}$$

where κ is a suitably chosen function of σ and that by certain randomizing an exact result may be obtained. One of the commonest distributions for describing non-uniformly distributed angles (which we would need if we wished to give the needle a preferential orientation) is the von Mises distribution

$$\left(2\pi I_0(\kappa)\right)^{-1} \exp \kappa \cos \theta \; . \tag{18}$$

Our final two examples lead into random structures. Suppose that the spacings of the parallel grid are identically and independently distributed (I.I.D.) with some density function $h(a)$ which is zero when $a < a_0$. Consider a needle of length $\ell \leq a_0$ tossed at random. Then

$$\begin{aligned} &\text{Prob(center of the needle falls in a space, } a < \text{space} < a + da) = \frac{a h(a) da}{\int_{a_0}^{\infty} a h(a) da} \\ &\qquad = \frac{a h(a) da}{E(a)} \; . \end{aligned}$$

Since

$$\text{Prob(cut}|\text{space } a) = \frac{2}{\pi}\frac{\ell}{a} \, ,$$

$$\text{Prob(cut)} = \int_{a_0}^{\infty} \frac{2}{\pi}\frac{\ell}{a} \cdot \frac{a h(a) da}{E(a)}$$

$$= \frac{2}{\pi}\frac{\ell}{E(a)} \; .$$

Thus by repeating this experiment the <u>only</u> thing we can learn about the spacings is $E(a)$. There is, for example, no way one can check whether they are I.I.D. This would require a long needle.

Thus let us consider an infinitely long needle and suppose that we could know, after it is tossed at random onto an arbitrary parallel grid, the sequence of spaces on the needle between line crossings, $\{s_i\}$. If the needle makes an angle θ (which we do not know) and the grid spacings are a_i then $a_i = s_i \sin \theta$, for all positive and negative integers i. From only the sequence $\{s_i\}$ we can check all properties of the $\{a_i\}$ sequence that do not depend upon scale, e.g., that it is I.I.D., stationary, etc.

The cut points on the needle form a Point Process in one dimension. The discussion of Point processes in space is the largest aspect of geometric probability and statistics in the modern sense, the topic to which we now turn.

3. WHAT IS STATISTICAL GEOMETRY?

Everyone is fairly clear what is meant by geometry but statistics is less well defined. It has a number of facets - (i) exploring data for regularities, i.e., patterns; (ii) estimating "population" characteristics from a sample; (iii) testing hypotheses; (iv) designing sampling plans to be effective and efficient (usually by including a random element).

Probability models enter (i) to (iv) in several ways:

a) by a scientific mechanism or model,

b) by assumption,

c) via a random sampling plan,

and to different degrees. In (i) they may not enter explicitly at all.

In statistical geometry our data will be a sample from some geometrical "population."

Such definitions do not convey much, so we now give some examples of problems and the groups that pursue them. (A) Grenander's books on "Pattern Synthesis" (1976) cover a vast area in a novel way not represented at all at this Symposium, and I think they are of basic importance. He has developed an abstract way of generating and distorting patterns and then restoring them. The latter is of course statistical. He gives a wealth of diverse examples; one of the simplest is discussed below. (B) Classical problems such as may be found in the Kendall and Moran (1963) book. (C) The publications of the Fontainebleau School of Mathematical Morphology represent a different line again. Their major achievement seems to me to be the wedding of the image analyzer and mathematical description of the objects scanned. While much practical work is done, the publications deal more with the mathematical theory than with the statistical aspects. (D) The Stochastic Geometry pursued in Cambridge by D. G. Kendall (see, e.g., Harding and Kendall, 1974) and others overlaps theoretically with the French School but has, it seems, purely mathematical motivations. Like so much of this literature, it has not been reduced to a level of mathematical simplicity for practical statistical use. (E) The Point Process literature, stemming from Bartlett (see, e.g., 1963, 1964, 1976) originated in practical statistical problems and mainly in one dimension. It is now pursued at a highly mathematical level by Krickeberg (1977) and other Europeans in many dimensions. Earlier practical work in Forestry, especially Matern's (1960) has led to many papers -- see, e.g., the issues of Biometrika. Ripley's recent papers (1977a, 1977b) have a combination of theory and practice and extensive bibliographies.

While there are many mechanisms for generating point processes in time, the few that do so in space are summarized by Ripley (1977a), for example. The main emphasis, in line with second order stationary processes, is the definition and estimation of functions that control the enhancement or inhibition of neighboring points. Here, as in time series analysis, there is a large exploratory element. Even if there were parametric models, it would rarely be possible to write down the likelihood of the data so that the time-honored statistical methods illustrated earlier cannot be used. Computers are essential for almost all calculations, unlike the Buffon problems, e.g., variances must usually be found by simulating.

In practice we will often want random sets, rather than the Poisson fields of points, lines, flats, etc. that are most often discussed. In his 1967 book Matheron made one of the early models that can be dealt with easily -- the Boolean scheme. Here I.I.D. copies of a random set K_i are attached to a Poisson field of points $\{x_i | x_i \varepsilon X\}$ in a vector space to obtain the random set $A = \bigcup_{x_i \varepsilon X} (K_i + x_i)$. Such a set is intuitively stationary, i.e., spatially homogeneous though not necessarily isotropic. If we know, for any fixed set B,

$$q(B) = \mathrm{Prob}(K \cap B = \emptyset)$$

then

$$\mathrm{Prob}(A \cap B = \emptyset) = \exp\{-\lambda \int_{R^P} (1-q(B+\xi))\, d\xi\} \tag{19}$$

where λ is the intensity of the Poisson process. In this Symposium Coleman went further in this construction than (19) which is the zero term of a Poisson distribution.

Time series analysis is about 100 years old. It began as a practical endeavor, became very mathematical and it is only recently that practical books and programs have been readily available. The time lag for this subject could be greatly shortened if theoreticians would make the effort to write for practical users and not only for other mathematicians.

We conclude with the simplest instance of problems in Grenander's book (ibid.). It illustrates (i) how a finite window is different from a finite sample of I.I.D. observations, (ii) the use of Fourier analysis. It is the restoration of a linear lattice whose points have been independently displaced. The complete set of points is

$$X_\nu = a + \nu\xi + n_\nu \qquad (\nu = \ldots, -1, 0, 1, \ldots)$$

where a is a phase, ξ = the lattice spacing (unknown) and n_ν is the noise. The window is the interval (0,L). When the noise is small with respect to L, almost all the points that should be in (0,L) will be there and no two points will have their true order inverted. Then we have an ordinary regression problem in estimating a and ξ. When the noise is not small, successive X points may not have successive indices and the "wrong" points may be in the window -- this illustrates point (i). (This model is essentially the same as that set up by D. G. Kendall (1974) to detect a unit of measurement in an archeological site.)

Here one automatically thinks of Fourier analysis. To save time, set $a = 0$ and define,

$$\phi(\omega) = \frac{1}{L} \sum_{X_\nu \varepsilon (0,L)} \exp i\, \omega X_\nu, \tag{20}$$

$$m(\omega) = E\phi(\omega) \;,$$

$$= \frac{1}{L} \int_0^L \exp(i\omega x) p(x) dx,$$

where

$$p(x) = \sum_{\nu} f(x - \nu\xi),$$

$$f(x) = \text{density of the noise } n_{\nu}$$

and

$$p(x) = \sum_{k} p_k \exp(-2\pi ik/\xi).$$

Then

$$m(\omega) = \sum_{k} p_k \frac{\exp\left[\{\omega - \frac{2\pi k}{\xi}\}iL\right] - 1}{iL\{\omega - \frac{2\pi k}{\xi}\}} .$$

As $L \to \infty$,

$$m(\omega) \to 0, \ \omega \neq \text{multiple of } 2\pi k/\xi$$

$$\to p_k, \ \omega = \text{multiple of } 2\pi k/\xi .$$

Thus we would hope to see a pattern of peaks near the points $2\pi k/\xi$ from which we would first try to see <u>if</u> there <u>is</u> a pattern, and if so to estimate ξ.

$$\text{var } \phi(\omega) \sim \frac{1}{L}\frac{1}{\xi}\left[1 - |f^*(\omega)|^2\right]$$

where f^* is the Fourier transform of f so that

$$p_k = \frac{1}{\xi} f^*\left(\frac{2\pi k}{\xi}\right) .$$

Thus $\text{var}(\phi(\omega))$ may also help us learn about the noise since this is governed by f.

The same computation with a Poisson process leads to

$E(\phi) = \lambda \dfrac{\exp(iL\omega) - 1}{iL\omega}$, $\text{var } \phi = \dfrac{\lambda}{L}$, a very different picture also seen with all renewal processes. Thus ϕ does not differentiate between stationary point processes but one hopes that the variance might. If we define

$$N(x) = \#\text{ points in } (0,x) ,$$

we have

$$\phi(\omega) = \frac{1}{L}\int_0^L \exp(i\omega x)\,dN(x) ,$$

$$|\phi(\omega)|^2 = \frac{1}{L^2}\int_0^L\!\!\int \exp\left(i\omega(x-y)\right)dN(x)\,dN(y) .$$

Assume with Bartlett (1976) that

$$\begin{aligned} E\left(dN(x)\right)^2 &= \lambda dx \\ EdN(x)dN(y) &= \{\lambda^2 + W(x-y)\}dxdy . \end{aligned} \tag{21}$$

If $W(\cdot)$ in (21) is identically zero, the points are Poisson. Let $W(v) = W(-v)$. It is clear that if $W(v)$ is positive, a point at y means that there is, relative to the Poisson process, more chance of having a point at $y + v$, i.e., enhancement. Negative values mean inhibition. Now

$$E|\phi|^2 \sim \frac{\lambda}{L} + \lambda^2 + \frac{1}{L}\int_{-\infty}^{\infty} W(v)\exp(i\omega v)\,dv \tag{22}$$

which verifies our notion that knowledge of $|\phi|^2$ should yield information about the function $W(v)$.

In this use of Fourier analysis one should note that the F.F.T. cannot be used; it is hard to adjust for bias and finite L and hard to find the variance of $|\phi|^2$, even in R'. The vagueness in these last two paragraphs is to some extent unavoidable. When dealing with unknown functions, one simply <u>has</u> to use judgment, try various tricks with the computer -- there cannot be any simple and apparently clear cut methods of the t-test type.

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