ASYMPTOTIC INDEPENDENCE OF CERTAIN STATISTICS CONNECTED WITH THE EXTREME ORDER STATISTICS IN A BIVARIATE DISTRIBUTION

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SUMMARY. The exact distribution of extremes in a sample and its limiting forms are well known in the univariate case. The limiting form for the largest observation in a sample was derived by Fisher and Tippet (1928) as early as 1927 by a functional equation, and that for the smallest was studied by Smirnov (1952). Though the joint distribution of two extremes has not been fully studied yet Sibuya (1960) gave a necessary and sufficient condition for the asymptotic independence of two largest extremes in a bivariate distribution. In this paper a necessary and sufficient condition for the asymptotic independence of two smallest observations in a bivariate sample has been derived, and the result has been used to find the condition for the asymptotic independence of any pair of extreme order statistics, one in each component of the bivariate sample. This result is further extended to find the condition for asymptotic independence of the pair of distances between two order statistics, arising from each component.

1. Introduction and notations

Let $F(x, y)$ be a continuous distribution function of (X, Y) with marginal distribution functions $F_1(x)$ and $F_2(y)$. Let the observations of a random sample (X_i, Y_i) $i = 1, 2, \ldots, n$ be ordered component-wise so that

$$Z_1^{(n)} \leqslant Z_2^{(n)} \leqslant \ldots Z_n^{(n)}$$

and

$$W_1^{(n)} \leqslant W_2^{(n)} \leqslant \ldots W_n^{(n)}$$

are the ordered values of X_i and Y_i, $i = 1, 2, \ldots, n$ respectively. Further, let

$$G_{k,l}^{(n)}(x, y) = P[Z_k^{(n)} \leqslant x,\ W_l^{(n)} \leqslant y]$$
$$G_k^{(n)}(x) = P[Z_k^{(n)} \leqslant x]$$
$$G_l'^{(n)}(y) = P[W_l^{(n)} < y]$$

where k and l are positive integers such that

$$\lim_{n\to\infty} k/n = \lim_{n\to\infty} l/n = 0.$$

Finkelshtein (1953) in the particular case ($k = 1, l = 1$) proved that for suitable choice of constants a_n, b_n, c_n and $d_n(a_n > 0,\ c_n > 0)$, the covergence of the sequence of distribution functions $G_{11}^{(n)}(a_nx+b_n,\ c_ny+d_n)$ to a distribution function $G_{11}(x, y)$ is a necessary and sufficient condition for the simultaneous convergence of each of the functions:

$$\begin{aligned} U_n(x) &= nF_1(a_nx+b_n) \\ V_n(y) &= nF_2(c_ny+d_n) \qquad \ldots \quad (1) \\ W_n(x, y) &= nF(a_nx+b_n,\ c_ny+d_n) \end{aligned}$$

to $U(x)$, $V(y)$ and $W(x, y)$ respectively where the functions $U(x)$, $V(y)$ and $W(x, y)$ are nondecreasing, non-negative functions of x and y such that $U(-\infty) = V(-\infty) = 0$, $U(+\infty) = V(+\infty) = \infty$.

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Throughout the paper $G_{k,l}(x, y)$, $G_k(x)$ and $G'_l(y)$ are used for the asymptotic distribution functions of $(Z_k^{(n)}, W_l^{(n)})$, $Z_k^{(n)}$ and $W_l^{(n)}$ respectively. More explicitly

$$G_{k,l}^{(n)}(x, y) = \lim_{n \to \infty} P[Z_k^{(n)} < a_n x + b_n,\ W_l^{(n)} < c_n y + d_n]$$

$$G_k(x) = \lim_{n \to \infty} P[Z_k^{(n)} < a_n x + b_n]$$

and
$$G'_l(y) = \lim_{n \to \infty} P[W_l^{(n)} < c_n y + d_n]$$

where a_n, b_n, c_n and d_n are constants as mentioned above. Further, for the sake of brevity we define

$$f_n(n_1, n_2, n_3, n_4) = \begin{cases} \dfrac{n!}{\prod\limits_{i=1}^{4} n_i!} [F(x, y)]^{n_1} [F_1(x) - F(x, y)]^{n_2} [F_2(y) - F(x, y)]^{n_3} \\ \qquad [1 - F_1(x) - F_2(y) + F(x, y)]^{n_4} \quad \text{for } 0 \leqslant n_i \leqslant n \\ 0 \qquad \text{otherwise.} \end{cases} \qquad \ldots (2)$$

2. Lemmas and theorems

Lemma 1 : *If $G_{11}^{(n)}(a_n x + b_n,\ c_n y + d_n)$ converges then*

$$G_1(x) = 1 - e^{-U(x)} \qquad \ldots (3)$$

and

$$G'_1(y) = 1 - e^{-V(y)} \qquad \ldots (4)$$

Proof : $G_1^{(n)}(a_n x + b_n) = P[Z_1^{(n)} \leqslant a_n x + b_n] = 1 - \left[1 - \dfrac{U_n(x)}{n}\right]^n$.

Since convergence of $G_{11}^{(n)}(a_n\ x + b_n, c_n\ y + d_n)$ assures the convergence of $U_n(x)$, to $U(x)$ the result (3) follows.

Proof for (4) is similar.

Theorem 1 : *If the sequence $G_{11}^{(n)}(a_n x + b_n,\ c_n y + d_n)$ converges, then*

$$\lim_{n \to \infty} [1 - F(a_n x + b_n,\ c_n y + d_n)]^n = \frac{[1 - G_1(x)][1 - G'_1(y)]}{1 - G_1(x) - G'_1(y) + G_{11}(x, y)}. \qquad \ldots (5)$$

Proof :

$$\begin{aligned} G_{1,1}(x, y) &= \lim_{n \to \infty} P[Z_1^{(n)} < a_n x + b_n,\ W_1^{(n)} < c_n y + d_n] \\ &= 1 - \lim_{n \to \infty} [\{1 - F_1(a_n x + b_n)\}^n + \{1 - F_2(c_n y + d_n)\}^n - \{1 - F_1(a_n x + b_n) \\ &\qquad - F_2(c_n y + d_n) + F(a_n x + b_n,\ c_n y + d_n\}^n] \\ &= 1 - \lim_{n \to \infty} \left\{1 - \frac{U_n(x)}{n}\right\}^n - \lim_{n \to \infty} \left\{1 - \frac{V_n(y)}{n}\right\}^n \\ &\qquad + \lim_{n \to \infty} \left\{1 - \frac{U_n(x) + V_n(y) - W_n(x, y)}{n}\right\}^n \\ &= 1 - \exp(-U(x)) - \exp(-V(y)) + \exp(-U(x) - V(y) + W(x, y)) \\ &= G_1(x) + G'_1(y) - 1 + [1 - G_1(x)][1 - G'_1(y)] \exp(W(x, y)). \end{aligned} \qquad \ldots (6)$$

Also,

$$\lim_{n\to\infty}[1-F(a_nx+b_n,\ c_ny+d_n)]^n = \lim_{n\to\infty}\left[1-\frac{W_n(x,y)}{n}\right]^n = \exp(-W(x,y)). \quad \dots (7)$$

From (6) and (7), (5) follows immediately.

Theorem 2 : *A necessary and sufficient condition for asymptotic independence of $Z_1^{(n)}$ and $W_1^{(n)}$ is that*

$$F(a_nx+b_n,\ c_ny+d_n) = o(1/n). \quad \dots (8)$$

Proof : Since the left hand expression of (5) is equal to 1 if and only if $F(a_nx+b_n,\ c_ny+d_n) = o(1/n)$, the result follows.

At the end of the paper it has been shown that for a bivariate normal distribution condition (8) is satisfied and so the smallest observations in the two components of a sample from a bivariate normal population are asymptotically independent.

Lemma 3 :

$$G^{(n)}_{k+1,l}(x,y) = G^{(n)}_{k,l}(x,y)-\binom{n}{k}[F_1(x)]^k[1-F_1(x)]^{n-k}$$

$$+\sum_{m=0}^{l-1}\sum_{\mu=0}^{n} f_n(\mu, k-\mu, m-\mu, n-m-k+\mu). \quad \dots (9)$$

Proof : Since $G^{(n)}_{k,l}(x,y) = \sum_{r=k}^{n}\sum_{m=l}^{n}\sum_{\mu=0}^{n} f_n(\mu, r-\mu, m-\mu, n-r-m+\mu)$

$$G^{(n)}_{k,l}(x,y)-G^{(n)}_{k+1,l}(x,y)$$

$$=\sum_{m=l}^{n}\sum_{\mu=0}^{n} f_n(\mu, k-\mu, m-\mu, n-k-m+\mu)$$

$$=\sum_{m=0}^{n}\sum_{\mu=0}^{n} f_n(\mu, k-\mu, m-\mu, n-k-m+\mu)-\sum_{s=0}^{l-1}\sum_{\mu=0}^{n} f_n(\mu, k-\mu, m-\mu, n-k-m+\mu)$$

$$=\binom{n}{k}\{F_1(x)\}^k\{1-F_1(x)\}^{n-k}-\sum_{m=0}^{l-1}\sum_{\mu=0}^{n} f_n(\mu, k-\mu, m-\mu, n-k-m+\mu).$$

Lemma 4 :

$$G'_l(y) = 1-\sum_{m=0}^{l-1}\frac{V(y)^m\exp(-V(y))}{m!}. \quad \dots (10)$$

Proof :

$$G'_l(y) = P[W_l^{(n)} \leqslant y] = 1-P[W_l^{(n)} > y]$$

$$= 1-\sum_{m=0}^{l-1}\binom{n}{m}[F_2(y)]^m[1-F_2(y)]^{n-m}.$$

Thus $$G_l'^{(n)}(c_ny+d_n) = 1-\sum_{m=0}^{l-1}\binom{n}{m}\left[\frac{V_n(y)}{n}\right]^m\left[1-\frac{V_n(y)}{n}\right]^{n-m}$$

so that taking the limit,

$$G_l'(y) = 1 = \sum_{m=0}^{l-1} \frac{V(y)^m e^{-V(y)}}{m!}.$$

Theorem 3 : *If $Z_1^{(n)}$ and $W_1^{(n)}$ are asymptotically independent, then for any $k \geqslant 1$, $l \geqslant 1$, $Z_k^{(n)}$ and $W_l^{(n)}$ are asymptotically independent.*

Proof : From Theorem 2, if $Z_1^{(n)}$ and $W_1^{(n)}$ are asymptotically independent, then

$$F(a_nx+b_n,\ c_ny+d_n) = o(n^{-1}).$$

The theorem will be proved by induction.

Let $Z_k^{(n)}$ and $W_l^{(n)}$ be asymptotically independent, i.e.

$$\begin{aligned} G_{k,l} &= \lim_{n\to\infty} G_{k,l}^{(n)}(a_nx+b_n,\ c_ny+d_n) \\ &= \lim_{n\to\infty} G_k^{(n)}(a_nx+b_n) \lim_{n\to\infty} G_l'^{(n)}(c_ny+d_n) \\ &= G_k(x)\, G_l'(y). \end{aligned}$$

From Lemma 2 we have

$$\begin{aligned} G_{k+1,l}^{(n)}(a_nx+b_n,\ c_ny+d_n) &= G_{k,l}^{(n)}(a_nx+b_n,\ c_ny+d_n) \\ &\quad -\binom{n}{k} F_1(a_nx+b_n)^k\,[1-F_1(a_nx+b_n)]^{n-k} \\ &\quad +\sum_{m=0}^{l-1}\sum_{\mu=0}^{m} f_n'(\mu,\ k-\mu,\ m-\mu,\ n-m-k+\mu) \end{aligned}$$

where a prime on f denotes that x and y are replaced by a_nx+b_n and c_ny+d_n respectively. It can be easily proved that

$$\lim_{n\to\infty} \binom{n}{k} F_1(a_nx+b_n)^k\,[1-F_1(a_nx+b_n)]^{n-k} = \frac{U(x)^k e^{-U(x)}}{k!}. \qquad \dots \quad (11)$$

Also,

$$\begin{aligned} \sum_{m=0}^{l-1}\sum_{\mu=1}^{m} f_n'(\mu, k-\mu, m-\mu, n-m-k+\mu) &= \sum_{m=0}^{l-1}\sum_{\mu=1}^{m} f_n'(\mu, k-\mu, m-\mu, n-m-k+\mu) \\ &\quad + \sum_{m=0}^{l-1} f_n'(o, k, m, n-m-k). \qquad \dots \quad (12) \end{aligned}$$

The first expression on the right can be expressed as

$$\sum_{m=0}^{l-1}\sum_{\mu=1}^{m} \frac{n!}{\mu!(k-\mu)!\,(m-\mu)!\,(n-m-k+\mu)!} \left[\frac{W_n(x,y)}{n}\right]^{\mu} \left[\frac{U_n(x)-W_n(x,y)}{n}\right]^{k-\mu}$$

$$\left[\frac{V_n(y)-W_n(x,y)}{n}\right]^{m-\mu} \left[1-\frac{U_n(x)+V_n(y)-W_n(x,y)}{n}\right]^{n-m-k+\mu}$$

which under the condition $F(a_n x+b_n,\ c_n y+d_n) = \frac{1}{n}\ W_n(x, y) = o(n^{-1})$, approaches zero as $n \to \infty$.

The second expression on the right of (12) can be expressed as

$$\sum_{m=0}^{t-1} \frac{1}{k!\ m!\ (n-m-k)!} \left[\frac{U_n(x)-W_n(x, y)}{n}\right]^k \left[\frac{V_n(y)-W_n(x, y)}{n}\right]^m$$

$$\left[1-\frac{U_n(x)+V_n(y)-W_n(x, y)}{n}\right]^{n-m-k}$$

which after simplification and taking the limit as n approaches infinity reduces to

$$\frac{U(x)^k\ e^{-U(x)}}{k!} \sum_{m=0}^{l-1} \frac{V(y)^m\ e^{-V(y)}}{m!}.$$

Hence from (9) and (10) it can be easily shown that

$$\lim_{n\to\infty} G^{(n)}_{k+1,\, l}\,(a_n x+b_n,\ c_n y+d_n) = \left[G_k(x)-\frac{U(x)^k\ e^{-U(x)}}{k!}\right] G'_l\,(y),$$

which proves that if $Z_k^{(n)}$ and $W_l^{(n)}$ are asymptotically independent, so are $Z_{k+1}^{(n)}$ and $W_l^{(n)}$.

Similarly it can be shown that $Z_k^{(n)}$ and $W_{l+1}^{(n)}$ are also asymptotically independent. Thus, by induction the proof of the theorem is complete.

An immediate consequence of the above theorem is that if $F(a_n x+b_n,\ c_n y+d_n) = o(n^{-1})$ then $Z_k^{(n)}$ and $W_l^{(n)}$ are asymptotically independent.

In the following the asymptotic independence of distances $d_{1,k}^{(n)} = Z_{k+1}^{(n)}-Z_1^{(n)}$ and $d_{1,l}'^{(n)} = W_{l+1}^{(n)}-W_1^{(n)}$, under the same condition will be proved by the following two lemmas.

Lemma 5 :

$$\lim_{n\to\infty} P[Z_1^{(n)} > a_n x+b_n,\ Z_k^{(n)} > a_n x'+b_n] = \sum_{i=0}^{k-1} e^{-U(x')} \frac{[U(x')-U(x)]^i}{i\ !} \qquad \ldots \quad (13)$$

Proof : Since $Z_1^{(n)} > a_n x+b_n$, it follows that all the X_i's are greater than $a_n x+b_n$. Letting A_i be the event such that i be the number of X_i's which are less than $a_n x'+b_n$:

$$P(A_i) = \binom{n}{i}[F_1(a_n x'+b_n)-F_1(a_n x+b_n)]^i\ [1-F_1(a_n x'+b_n)]^{n-i}.$$

Thus it is obvious that

$$P[Z_1^{(n)} > a_n x+b_n\ ;\ Z_k^{(n)} > a_n x'+b_n] = \sum_{i=0}^{k-1} \binom{n}{i} \left[\frac{U_n(x')-U_n(x)}{n}\right]^i \left[1-\frac{U_n(x')}{n}\right]^{n-i}$$

and taking the limit as $n\to\infty$,

$$\lim_{n\to\infty} P[Z_1^{(n)} > a_n x+b_n;\ Z_k^{(n)} > a_n x'+b_n] = \sum_{i=0}^{k-1} e^{-U(x')} \frac{[U(x')-U(x)]^i}{i!}.$$

Lemma 6 : *If*

$$F(a_n x+b_n, c_n y+dn) = o(n^{-1}),$$

then

$$\lim_{n\to\infty} P_n[Z_1^{(n)} > x,\ Z_k^{(n)} > x';\ W_1^{(n)} > y,\ W_l^{(n)} > y']$$

$$= \lim_{n\to\infty} P_n[Z_1^{(n)} > x,\ Z_k^{(n)} > x'] \lim_{n\to\infty} P_n[W_1^{(n)} > y,\ W_l^{(n)} > y']$$

where subscript n on P means that x, x', y *and* y' *have been replaced by* $a_n x+b_n$, $a_n x'+b_n$, $c_n y+d_n$, $c_n y'+d_n$ *respectively.*

Proof : Let $A_{\mu\nu\sigma}$ be the set defined as follows :

$$A_{\mu\nu\sigma} = \{\omega : \mu \text{ of } \omega_i\text{'s} \in R_1,\ \nu \text{ of } \omega_i\text{'s} \in R_2,\ \sigma \text{ of } \omega_i\text{'s} \in R_3 \text{ and } n-\mu-\nu-\sigma \text{ of } \omega\text{'s} \in R_4\}$$

where

$$R_1 = [X > x',\ y < Y \leqslant y']$$

$$R_2 = [x < X \leqslant x',\ y < Y \leqslant y']$$

$$R_3 = [x < X \leqslant x',\ Y > y']$$

$$R_4 = [X > x',\ Y > y'],$$

so that

$$P(A_{\mu\nu\sigma}) = \frac{n!}{\mu!\,\nu!\,\sigma!(n-\mu-\nu-\sigma)!} P[R_1]^{\mu}\, P[R_2]^{\nu}\, P[R_3]^{\sigma}\, P[R_4]^{n-\mu-\nu-\sigma} \qquad \ldots \quad (14)$$

Thus

$$P_n^{(n)}[Z_1^{(n)} > x,\ Z_k^{(n)} > x';\ W_1^{(n)} > y,\ W_l^{(n)} > y'] = \sum_{\nu=0}^{k-1} \sum_{\mu=0}^{l-1} \sum_{\sigma=0}^{\min(k,l)-1} P_n(A_{\mu-\sigma,\,\nu-\sigma,\,\sigma}).$$

Expressing $P_n(R_1)$, $P_n(R_2)$, $P_n(R_3)$ and $P_n(R_4)$ in terms of $U_n(x)$, $V_n(y)$ and $W_n(x, y)$ and taking the limit, it follows immediately that

$$\lim_{n\to\infty} P_n[Z_1^{(n)} > x,\ Z_k^{(n)} > x';\ W_1^{(n)} > y,\ W^{(n)} > y] = \sum_{\nu=0}^{k-1} \sum_{\mu=0}^{l-1} \sum_{\sigma=0}^{\min(k,l)-1} \frac{1}{\sigma!(\mu-\sigma)!(\nu-\sigma)!}$$

$$\times \lim_{n\to\infty} \{[V(y')-V(y)-W(x', y')+W(x', y)]^{\mu-\sigma} [W(x', y')-W(x', y)-W(x, y')+W(x, y)]^{\sigma}$$

$$\times [U(x')-U(x)-W(x', y')+W(x, y)]^{\nu-\sigma} \exp(-U(x)-V(y)+W(x', y')\}.$$

Since $W(x, y)$, $W(x', y)$, $W(x, y')$ and $W(x', y')$ are $o(1/n)$ by the condition of lemma, the expression within the summation sign will be zero unless $\sigma = 0$. Therefore

$$\lim_{n\to\infty} P_n[Z_1^{(n)} > x,\ Z_k^{(n)} > x';\ W_1^{(n)} > y,\ W^{(n)} > y']$$

$$= \sum_{\nu=0}^{k-1} \sum_{\mu=\sigma}^{l-1} \frac{1}{\mu!\,\nu!} \exp(-U(x')-V(y'))(U(x')-U(x))^{\nu}(V(y')-V(y))^{\mu}$$

Comparing the right hand expression with Lemma 5, Lemma 6 follows immediately.

In view of the above lemma the random vector $\boldsymbol{Z}^{(n)} = \begin{pmatrix} Z_1^{(n)} \\ Z_k^{(n)} \end{pmatrix}$ is asymptotically independent of $\boldsymbol{W}^{(n)} = \begin{pmatrix} W_1^{(n)} \\ W_l^{(n)} \end{pmatrix}$.

Theorem 4 : *If* $F(a_n x+b_n,\ c_n+d_n)=o(1/n)$ *then the distance* $d_{1,k-1}^{(n)}= Z_k^{(n)}-Z_1^{(n)}$ *is asymptotically independent of* $d_{1,l-1}^{(n)} = W_l^{(n)}-W_1^{(n)}$.

Proof : Under the condition of the theorem since the random vectors $\boldsymbol{Z}^{(n)}$ and $\boldsymbol{W}^{(n)}$ are asymptotically independent, the Borel functions $Z_k^{(n)}-Z_1^{(n)}$ and $W_l^{(n)}-W_1^{(n)}$ are also asymptotically independent.

3. Application

In this section we apply the result of Theorem 2 to a bivariate normal distribution. Without *loss* of generality we can take the variance-covariance matrix of the form

$$A = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \qquad 0 < \rho < 1,$$

so that
$$f(x, y) = \frac{1}{2\pi(1-\rho^2)^{\frac{1}{2}}} \exp\left[-\frac{1}{2(1-\rho^2)}(x^2-2\rho xy+y^2)\right]$$

and
$$F(x, y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(u, v)\, dv\, du.$$

For marginal distribution $F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-\frac{1}{2}u^2)du$ it is known (Cramér, 1946, p. 374) that the distribution of $nF(x)$ converges to a proper distribution. This result after algebraic manipulation reduces to the fact that if

$$a_n = (2 \log n)^{\frac{1}{2}} = c_n$$

and
$$b_n = -2 \log n+\tfrac{1}{2}(\log \log n+\log 4\pi) = d_n,$$

$F(a_n x+b_n)$ converges to a proper distribution function. Using the result (Cramér, 1946, p. 290) we can write

$$F(a_n x+b_n,\ c_n y+d_n) = = \sum_{\nu=0}^{\infty} \frac{f^{(\nu)}(x')f^{(\nu)}(y')}{\nu\,!}\,\rho^{\nu}$$

where
$$x' = a_n x+b_n, \quad y' = c_n y+d_n \qquad \ldots \quad (15)$$

and where $f^{\nu}\ (x)$ is ν-th derivative of $f(x)$.

It is immediate that $\lim_{n \to \infty} n\, \phi_n(x, y) = n \exp[-\frac{1}{2}(2 \log n)(x^2+y^2)+\rho] = 0$, hence in order to prove that $F(a_n x+b_n,\ c_n y+d_n) = o(1/n)$ it is sufficient to prove that infinite series (15) and

$$\phi_n(x, y) = \exp[-\tfrac{1}{2}(2 \log n)(x^2+y^2)] \sum_{\nu=0}^{\infty} \frac{\rho^\nu}{\nu!} \quad \ldots \quad (16)$$

are of the same order when n approaches ∞.

Noting that $f^{(r)}(x) = -xf^{(r-1)}(x)-(r-1)f^{(r-2)}(x)$ the term by term ratio of the two series approaches 1 as n approaches ∞, and thus the result is proved. Thus two extreme order statistics one in each component of a bivariate normal distribution are asymptotically independent.

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