ON THE SECOND RELATIVE HOMOTOPY GROUP OF AN ADJUNCTION SPACE:

AN EXPOSITION OF A THEOREM OF J. H. C. WHITEHEAD

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Dedicated to the memory of a friend and colleague, Peter Stefan

1. *Introduction*

Let $X = X_0 \cup \{e_\lambda^2\}_{\lambda \in \Lambda}$ be a space obtained by attaching 2-cells to X_0, and let $x_0 \in X_0$. In his 1941 paper [4], J. H. C. Whitehead attempted an algebraic description of the second homotopy group $\pi_2(X, x_0)$. In his 1946 paper [5] his results were reformulated (with some corrections of definitions) in terms of a precise algebraic description of the second relative homotopy group

$$A = \pi_2(X, X_0, x_0).$$

In his 1949 paper [6], a different exposition of part of the proof was given, and also the result was codified finally in saying that *the group A is the free crossed* $\pi_1(X_0, x_0)$*-module on the* 2*-cells.*

One difficulty in obtaining this theorem by standard methods of algebraic topology is that it is a non-abelian result. A proof has recently been given by Ratcliffe, using a homological characterisation of free crossed modules [3]. The theorem also is a special case of the generalisation to dimension 2 of the Seifert–van Kampen Theorem [1], where free crossed modules arise as very special cases of pushouts of crossed modules. Indeed one aim of the program completed in [1] was to formulate a generalisation of Whitehead's theorem, and to prove it by verification of a universal property. A list of papers which apply the theorem is also given in [1].

In spite of these other proofs and generalisations, Whitehead's proof still has interest. It has a curious structure, a modern-looking use of transversality, and a clever interplay of the relations for a knot group and the rules of a crossed module. Also the ideas have some relevance to difficult problems in combinatorial group theory involving identities between relations. (I hope to give elsewhere an exposition of some ideas of Peter Stefan in this area.†) But the proof is difficult to read, for reasons which include its originality of conception, and the change in notation and formulation over the years 1941–49. I hope therefore that it will prove useful to present a straightforward account of Whitehead's proof; my own contribution is simply that of presentation in a modern, and uniform notation and terminology.

Thanks are due to the late Peter Stefan for discussions on some of this material, and to Johannes Huebschmann for helpful comments on an earlier draft.

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† This work is planned to appear in [8]. There are related ideas in [7].

2. *Results*

We assume X_0 to be path-connected. Let $u_\lambda : S^1 \to X_0$ be the attaching map for the cell e_λ^2, $\lambda \in \Lambda$. Let $A = \pi_2(X, X_0, x_0)$ as above, and let $B = \pi_1(X_0, x_0)$. Let $\alpha_\lambda \in B$ be represented by $-t_\lambda + u_\lambda + t_\lambda$ where t_λ is a path in X_0 joining $u_\lambda(1)$ to x_0. Let $a_\lambda \in A$ be determined by the characteristic map for e_λ^2 joined to x_0 by the path t_λ chosen already. Then the boundary map $\partial : A \to B$ satisfies $\partial(a_\lambda) = \alpha_\lambda$.

The group B acts on A so that A with ∂ becomes a crossed B-module. We assume the notation and terminology for crossed modules from [1]; in particular we assume the notion of free crossed module.

Whitehead's result is:

THEOREM. *The crossed module* (A, B, ∂) *given above is free on the elements* a_λ *with* $\partial a_\lambda = \alpha_\lambda$, $\lambda \in \Lambda$.

The proof is in several steps.

3.

Let (C, B, d) be the free crossed B-module on generators c_λ with $dc_\lambda = \alpha_\lambda$, $\lambda \in \Lambda$. Then there is a unique morphism $\theta : C \to A$ of crossed B-modules such that $\theta(c_\lambda) = a_\lambda$, $\lambda \in \Lambda$. Whitehead proves that θ is an isomorphism.

By choosing a small 2-disc σ_λ^2 in e_λ^2 and adding the complement of $\mathring{\sigma}_\lambda^2$ in e_λ^2 to X_0 (this does not change the homotopy type of the triple (X, X_0, x_0)) we may assume that each e_λ^2 is the interior of a 2-disc $\sigma_\lambda^2 \subset X$ whose boundary $\dot{\sigma}_\lambda^2$ is contained in X_0. We also assume that the path t_λ used above joins a base point x_λ of $\dot{\sigma}_\lambda^2$ to x_0.

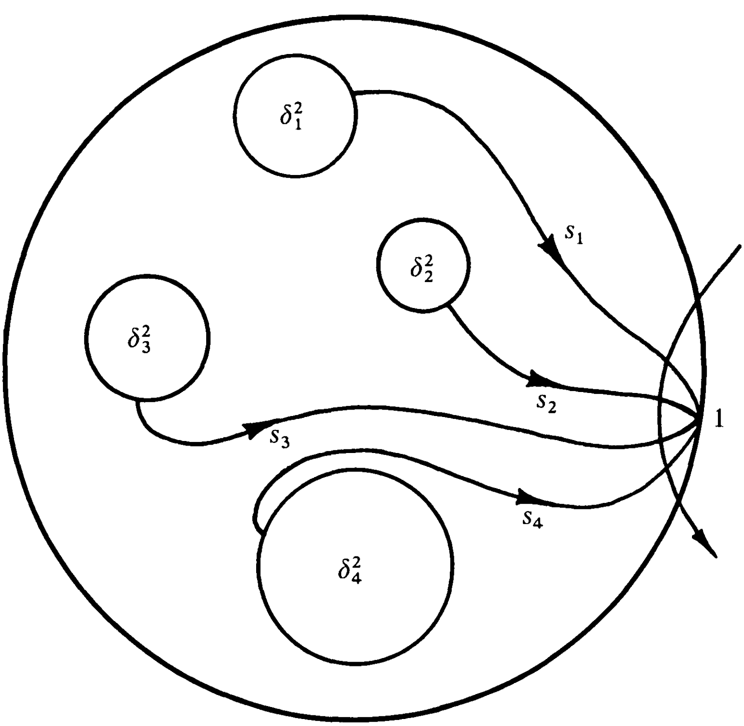

Fig. 1

A *transverse map* $f:(E^2, S^1, 1) \to (X, X_0, x_0)$ is a map of triples such that each $f^{-1}(\sigma_\lambda^2)$ is a finite set $\delta_{1\lambda}^2, \ldots, \delta_{r_\lambda\lambda}^2$ of disjoint 2-discs in $E^2 \setminus S^1$ such that each $\delta_{i\lambda}^2$ is mapped diffeomorphically by f to σ_λ^2. (Whitehead uses the term *normal map*.) It is a standard kind of deduction from transversality theory that every element of $A = \pi_2(X, X_0, x_0)$ may be represented by a transverse map (for more details on transversality, see [**2**; Chapter VII]).

Let f be transverse as above. Let $x_{i\lambda}$ be the unique point of the disc $\delta_{i\lambda}^2$ such that $f(x_{i\lambda}) = x_\lambda$. Let $\{s_{i\lambda}\}$ be a set of paths in E^2 such that $s_{i\lambda}$ joins $x_{i\lambda}$ to 1 and such that the $s_{i\lambda}$ meet each other only at the end point 1, and meet one of the discs only in the other end point $x_{i\lambda}$. We now order the $s_{i\lambda}$ round 1 and relabel them as $s_1, \ldots, s_q$; we label the corresponding $\delta_{i\lambda}^2$ as $\delta_1^2, \ldots, \delta_q^2$ and write the λ such that $f(\delta_i^2) = \sigma_\lambda^2$ as λi.

Let $\xi_i \in B$ be represented by $-t_{\lambda i}+f(s_i)$, and let ε_i be ± 1 according as f maps δ_i^2 to $\sigma_{\lambda i}^2$ in an orientation preserving or reversing manner. We define

$$\psi(f) = (c_{\lambda 1}^{\varepsilon_1})^{\xi_1} \ldots (c_{\lambda q}^{\varepsilon_q})^{\xi_q} \in C\,. \tag{1}$$

The homotopy addition lemma in dimension 2 can be formulated as the statement: f is a representative of $\theta\psi(f)$. *This proves that θ is surjective.*

Note also that if $\xi \in B$ is represented by the restriction of f to $(S^1, 1) \to (X_0, x_0)$, then

$$\xi = \xi_1^{-1}\alpha_{\lambda 1}^{\varepsilon_1}\xi_1 \ldots \xi_q^{-1}\alpha_{\lambda q}^{\varepsilon_q}\xi_q$$

and it follows that

$$d\psi(f) = \xi\,.$$

4.

Group presentations of free crossed modules are given in [**6**; p. 255] and [**1**; p. 207]. It follows from these that any element of C can be written in the form of the right hand side of (1) for some ε_i, ξ_i, λi. Hence for any $c \in C$ there is a transverse map f_c such that $\psi(f_c) = c$.

We now start on the proof that θ is injective. Let $c \in C$ be such that $\theta(c) = 0$. Let f_c be a transverse map such that $\psi(f_c) = c$. Since f_c represents $\theta(c)$, which is 0, there is a null-homotopy of f_c and so f_c extends to a map

$$H:(CE^2, CS^1, C1) \to (X, X_0, x_0)$$

of the cone on the triple $(E^2, S^1, 1)$.

By transversality theory, H may be deformed rel E^2 into a map such that for each $\lambda \in \Lambda$, $H^{-1}(\sigma_\lambda^2)$ is a disjoint union of solid tubes $\delta^2 \times I$ and solid tori $\delta^2 \times S^1$, the union of all these for all $\lambda \in \Lambda$ forming a linkage L in CE^2. We may also assume that the projection from p, the vertex of the cone, to the base, makes this linkage regular in the sense of knot theory, so that there are only over—and under—passes as in Figure 2.

Let $G = \pi_1(CE^2 \setminus L, p)$ be the group of the linkage L. Let s be the path in CE^2 joining 1 to p along $C1$, and let $g \in G$ be represented by $-s+S^1+s$.

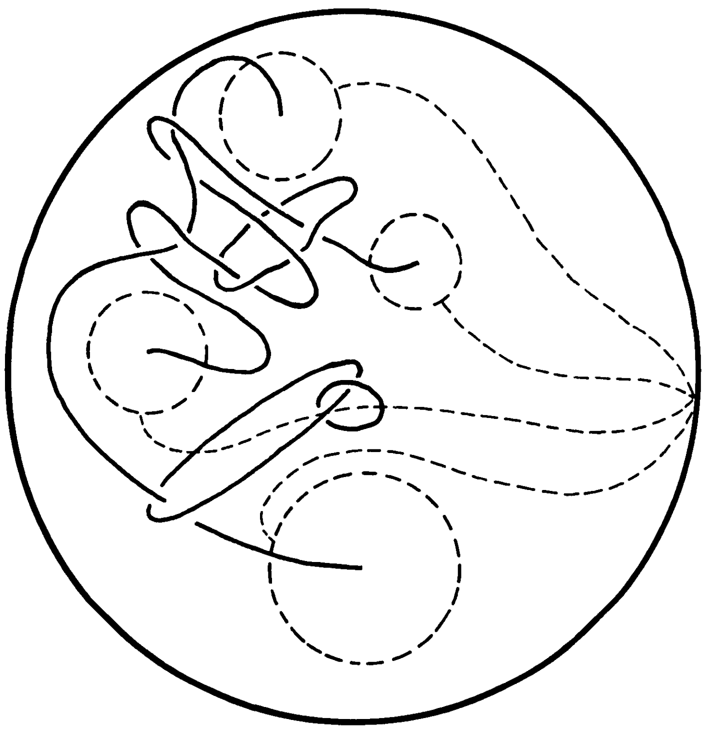

Fig. 2

CLAIM. *There is a homomorphism of groups* $\phi : G \to C$ *such that* $\phi(g) = c$.

COROLLARY. $c = 0$.

Proof. Since the linkage L does not meet the boundary of CE^2 except in the interior of the base, the loop $-s+S^1+s$ may be deformed up the cone to p. So $g = 0$. Hence $c = 0$.

This shows that to prove the theorem it is sufficient to prove the above claim. This proof occupies the next two sections.

5. *Construction of* $\phi : G \to C$

Let $\{L_j\}$ be the (finite) set of overpasses of L. The Wirtinger presentation for G has one generator g_j for each overpass L_j and a relation $g_k = g_j^{-1} g_l g_j$ at each crossing. A precise description of g_j is as follows.

By taking the tubes of the linkage L slightly smaller, we have that $H(L_j)$ is contained in the interior of a disc $\sigma^2_{\mu j}$, say, of X. Let v_j be a loop going once round L_j and mapped by H homeomorphically and preserving orientation to $\dot{\sigma}^2_{\mu j}$. Let p_j be the unique point on v_j such that $H(p_j) = x_{\mu j}$ (the base point on $\dot{\sigma}^2_{\mu j}$). Let w_j be a path from p_j to p which goes round v_j as necessary and then up a cone line to p.

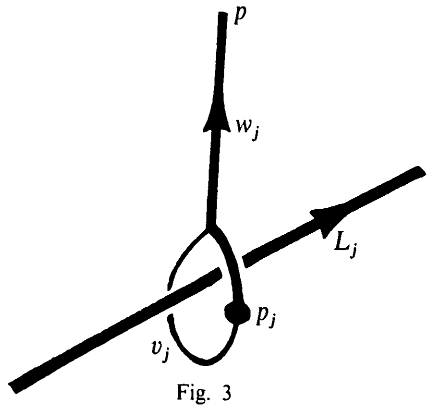

Fig. 3

Let $g_j = [-w_j+v_j+w_j] \in G$, $\eta_j = [-t_{\mu j}+H(w_j)] \in B$. We define ϕ on the generators of G by

$$\phi(g_j) = (c_{\mu j})^{\eta_j}$$

and verify that ϕ annihilates the relations.

Consider the relation $g_k = g_j^{-1} g_l g_j$ defined by a crossing with overpass L_j. Let z_{lk} be a path joining p_l to p_k and mapped by H into $\dot{\sigma}^2_{\mu l}(=\dot{\sigma}^2_{\mu k})$. Then for some $\varepsilon = \pm 1$, the paths $-w_j+\varepsilon v_j+w_j$ and $-w_l+z_{lk}+w_k$ both go close to and once round L_j in the same direction and so are equivalent. Also $-t_{\mu l}+H(z_{lk})+t_{\mu k}$ represents an element of B of the form $\alpha^n_{\mu l}$. Hence the path $H(-w_l+z_{lk}+w_k)$, which is equivalent to

$$(H(-w_l)+t_{\mu l})-t_{\mu l}+H(z_{lk})+t_{\mu k}+(-t_{\mu k}+H(w_k)),$$

represents $\eta_l^{-1}\alpha^n_{\mu l}\eta_k$.

Similarly, $H(-w_j+\varepsilon v_j+w_j)$ represents $\eta_j^{-1}\alpha^\varepsilon_{\mu j}\eta_j$. Hence

$$\eta_j^{-1}\alpha^\varepsilon_{\mu j}\eta_j = \eta_l^{-1}\alpha_{\mu l}\eta_k . \tag{2}$$

We wish to verify in C that

$$(c_{\mu k})^{\eta_k} = (c_{\mu j}^{-1})^{\eta_j}(c_{\mu l})^{\eta_l}(c_{\mu j})^{\eta_j} . \tag{3}$$

Now the right-hand side of (3) is

$$(c_{\mu l})^{\eta_l \eta_j^{-1}\alpha_{\mu j}\eta_j} = (c_{\mu l})^{\alpha^{\varepsilon n}_{\mu l}\eta_k} \quad \text{by (2)},$$

$$= (c_{\mu l})^{\eta_k} \quad \text{since } c^{\partial c} = c ,$$

$$= (c_{\mu k})^{\eta_k} \quad \text{since } \mu k = \mu l .$$

This proves (3) and thus that $\phi : G \to C$ is a well-defined morphism. Note further that H determines $H_* : G \to \pi_1(X_0) = B$, and that $H_*(g_j) = \eta_j^{-1}\alpha_{\mu j}\eta_j$, $j \in J$. It follows that

$$d\phi = H_* . \tag{4}$$

6. *Proof that* $\phi(g) = c$

We now generalise our representation of elements of G corresponding to the overpasses L_j.

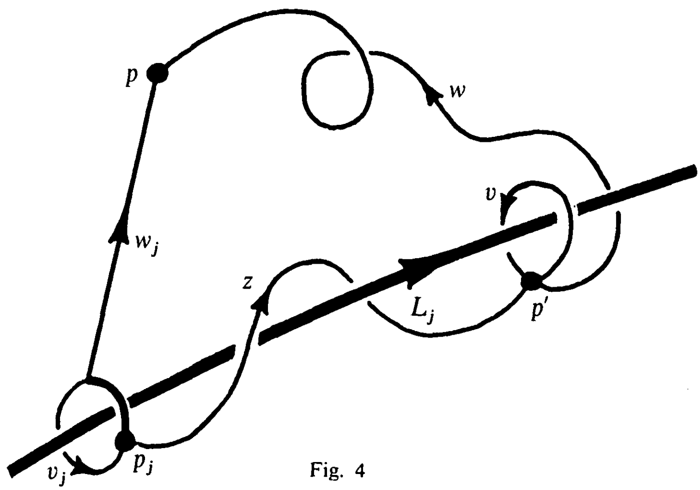

Fig. 4

Let v be any small loop round L_j which is mapped by H homeomorphically and preserving orientation to $\dot\sigma^2_{\mu j}$; let p' be the unique point of v with $H(p') = x_{\mu j}$. Let w be any path in $CE^2 \setminus L$ joining p' to p.

LEMMA. *If* $-w+v+w$, $-t_{\mu j}+H(w)$ *represent* $h \in G$, $\zeta \in B$ *respectively, then* $\phi(h) = (c_{\mu j})^{\zeta}$.

Proof. Let z be a path joining p_j to p' and mapped by H into $\dot\sigma^2_{\mu j}$. Then h is represented also by

$$-(z+w)+v_j+(z+w).$$

So

$$h = k^{-1}g_jk \quad \text{where} \quad k = [-w_j+z+w],$$

and

$$\phi(h) = e^{-1}\phi(g_j)e \quad \text{where} \quad e = \phi(k)$$

$$= (c_{\mu j})^{\eta_j d(e)}.$$

But

$$d(e) = d\phi k = H_*(k) = [-H(w_j)+H(z)+H(w)], \text{ using (4).}$$

Hence

$$\eta_j d(e) = [-t_{\mu j}+H(w_j)-H(w_j)+H(z)+H(w)]$$

$$= [-t_{\mu j}+H(z)-t_{\mu j}]+[-t_{\mu j}+H(w)]$$

which is of the form $\alpha^m_{\mu j}\zeta$, for some m. The lemma follows.

Recall now that g is represented by $-s+S^1+s$. Then $g = g_1 \dots g_q$ where g_i is represented by

$$-s-s_i+\delta_i^2+s_i+s\,.$$

By the lemma, $\phi(g_i) = (c_{\lambda i}^{\varepsilon_i})^{\zeta_i}$ where

$$\zeta_i = [-t_{\lambda i}+H(s_i+s)]$$

$$= [-t_{\lambda i}+f_c(s_i)]$$

which is the element ξ_i defined just before (1). It follows from (1) that $\phi(g) = c$, and Whitehead's proof of his theorem is complete.

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