# The Third Man's Knotty Problems

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## Introduction

The involvement of man with knots certainly dates back tens of thousands of years. A house post, bound by rope, believed to be 30,000 years old, has been found in South America. More speculatively, it has been suggested that the Stone Age should be called the *Age of String*, and it is easy to conjecture how the functional and sometimes exquisite Stone Age tools could have been attached by string to form axes, arrowheads and knives. The solution of the problems involved in this tying would be an initial form of applied geometry, with a strong motivation to get it right.

Ancient fishermen solved several problems of a mathematical type when making nets. The knots used had to be (i) easily tied, (ii) non-slip, and (iii) form part of a repeating pattern. In the Museum of Antiquities in Helsinki, carefully preserved, is the oldest known net, the Antrea net, found in 1923 in a peat bog in Antrea, then part of Finland. It is thought that a boat capsized and lost all its gear. Constructed of willow bark fibre, the net is 30m by 1.5m with a 6cm mesh, and the connecting knot is still used today. The net is dated by carbon 14 and by pollen at 7,200 BC. The mathematization of just some of the understanding involved has taken nine millennia to emerge, and still there are some aspects almost untouched, such as the slippage of knots. Meanwhile, knots have been intimately linked with the development of civilisation, through housing, weapons, fishing, clothing, boating and a myriad of other ways. The metaphor of knots is found throughout literature. Knots are found in many forms of art but reach their peak in the famous illustrated manuscripts of the eighth century, such as the Book of Kells.

Knot theory is a form of geometry, without length or angle. For today's mathematicians it is a subject with fascinating examples and many attractive and unsuspected analogies and methods. For scientists, there are now applications to DNA replication, to statistical mechanics, to Feynman diagrams in quantum mechanics, to polymer chemistry, to the dynamics of weather systems and notions of chaos, and to many other areas of research. To teachers at all levels, knot theory can show some of the basic methods of mathematics, and how these are related to normal methods of discovery and analysis of the world around us.

The mathematical study of knots mushroomed in the 1920's with the discovery of invariants called the Alexander polynomials after their discoverer, the American topologist, J. W. Alexander. A second burst of activity in the 1980's was founded on the work of the New Zealander, Vaughan Jones, and led to a slew of polynomials found by groups of workers all over the world. One paper in 1985 announcing some new polynomials had five authors from three research groups, who had come to the same answer from different areas of mathematics, while a Polish team missed out through not being quick enough into print. Research teams in Tokyo and San Diego have now found connections between knots and energy, both elastic and electrostatic. The plot continues to grow more tangled!

An inspiration of much of this work comes from the pioneering work of Peter Guthrie Tait, Professor of Natural Philosophy in the University of Edinburgh over a hundred years ago.

## Who was Tait and how did he come to study the mathematics of knots?

During the second half of the nineteenth century Tait was the third man of natural philosophy. As a theoretical physicist he did not measure up to James Clerk Maxwell, the father of the electromagnetic theory of light and the kinetic theory of gases. Neither did he have that capacity for experimental physics possessed of the great thermodynamicist, William Thomson (Lord Kelvin), nor indeed the entrepreneurial drive necessary to convert the results of his endeavours into hard cash and royal honours. Nevertheless, in a long and distinguished career – forty-four years as professor – he worked closely with Maxwell and Kelvin, and tackled a wide range of physical and mathematical problems, most brilliantly those related to the flight of a golf ball and those in the topology of knots.

Tait's topology of knots, brought to the notice of the scientific community through papers of 1876, 1884 and 1885, was the second serious mathematical study of the subject. Seldom has an inspiring, important and significant work emanated from such a romantic and totally wrong speculation, namely the neglected, in fact justly neglected, vortex theory of the atom, propounded by Kelvin in a paper of 1867.

Recall that, for those in the middle of the nineteenth century who accepted the atomic theory, the problem was to explain the permanency of atoms, the vibrational properties, as shown by their spectral lines, and the variety of atoms, as shown by the periodic table of elements. (A finally convincing proof of the existence of atoms was perhaps not available till Einstein's 1905 theory of Brownian motion.) Tait acted as a catalyst to Kelvin's vortex theory in two ways, firstly by translating a paper of Helmholtz in which the apparent attraction and repulsion of vortices in certain fluids was noted, and then by spotting the same phenomenon in the aerodynamics of smoke rings. An inveterate pipe-smoker, he noticed that the smoke rings he blew came together, intermingled and deflected.

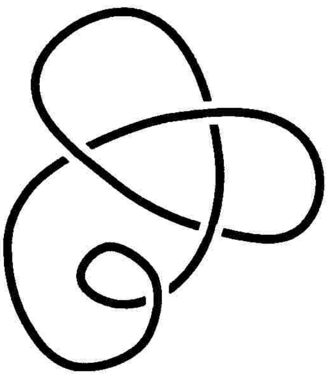
The one problem remaining was the variety of atoms. It was Kelvin who suggested that the variety could come from knotted or linked vortices. He explained in his paper how *knotted and linked models of vortex atoms were shown to the Society, the infinite variety of which is more than sufficient to explain the affinities and allotropies of all known matter.* So it fell to Tait to find a classification. *May you ..... prosper and disentangle your formulae in proportion as you entangle your worbles*, enthused Maxwell.

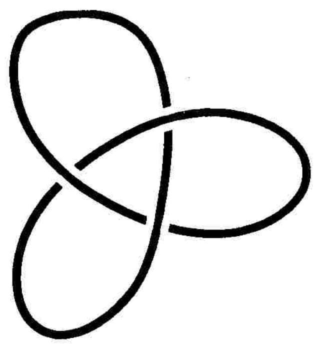
The relation with chemistry quickly disappeared from sight, but Tait's fascination with knots only grew. At the outset, Tait believed that no work had been done previously in this area, and it was not until he had made significant progress that Maxwell sent him a copy of the first of two works by Johann Benedict Listing, *Vorstudien zur Topologie* of 1847. Topology, as a mathematical term, was born in the title of this book! (Interestingly, in 1834 Listing was a student of Gauss at Göttingen and just the previous year, to be specific in a note dated 22 January 1833, the master had written that *an important task arising tangentially from Geometria Situs* [now Combinatorial Topology]....... *will be that of counting the linkings of two closed or continuous lines.*) Listing had given examples of many of the forms that Tait had drawn but he had made no effort to trawl for all possible forms. Nevertheless, to some extent, Tait's investigations had been anticipated and he recorded the German's priority in a tribute published in Nature in 1883 as well as in an address given to the Edinburgh Mathematical Society later the same year.

## Highlights from Tait's papers, with classroom applications

### (1) Knottiness

Two knots are deemed to be the same as each other if one can be moved into the other by twisting, manipulating and moving it around, but without cutting. Consequently, establishing ways to distinguish between knots by finding invariants is a central theme of topology. In his first paper Tait's starting point was his concept of *knottiness*, the minimum number of intersections a knot possesses. Additional twists which can be simply removed he called *nugatory crossings*, nugatory meaning inoperative or redundant. The *trefoil* below, for example, is shown on the left as a knot with four intersections one of which is a nugatory crossing and on the right with the nugatory crossing removed to establish that it has knottiness three at most.

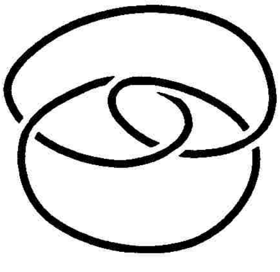




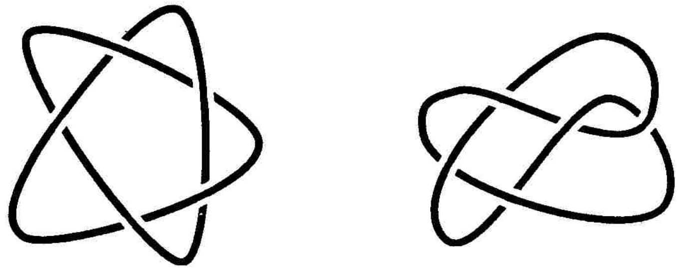
All knots can be given in a symbolic form dubbed a *scheme*. One method of moving from knot to scheme is to traverse the knot calling the first intersection met A, the second B (unless it is A again) and so on. Alternatively, call the first over-crossing A, the second over-crossing B and so on. Whichever method is used, a number of procedures can be developed for the simplification of a knot's projection through the manipulation of the algebraic scheme. Adopting the first method for example, if the same letter occurs twice in succession then there is a nugatory crossing. In the figure above left the scheme is ABBCD, with the nugatory crossing being at B. To these algebraic methods Tait added one from *rubber-*

*sheet geometry* as topology is often called - the *flype*. Flyping is achieved by imagining the knot stretched around the surface of a sphere so that it can be viewed from inside or outside.

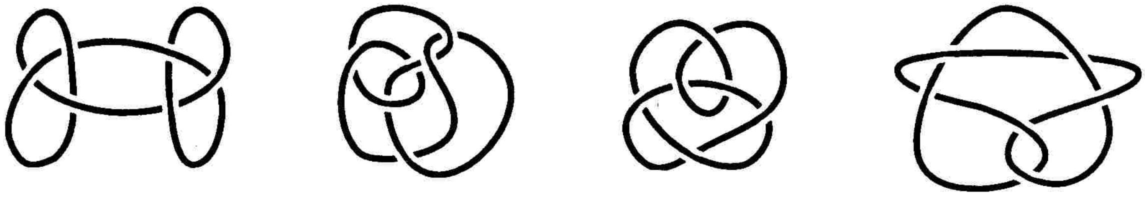
Now armed with a variety of approaches to his investigation Tait went about finding all the different forms of knots of given knottiness. His results are as follows: There are no knots with knottiness one or two, the trefoil is the sole knot with knottiness three, while for knottiness four, five and six the number of distinct forms is 1, 2 and 4 respectively.



4 intersections



5 intersections



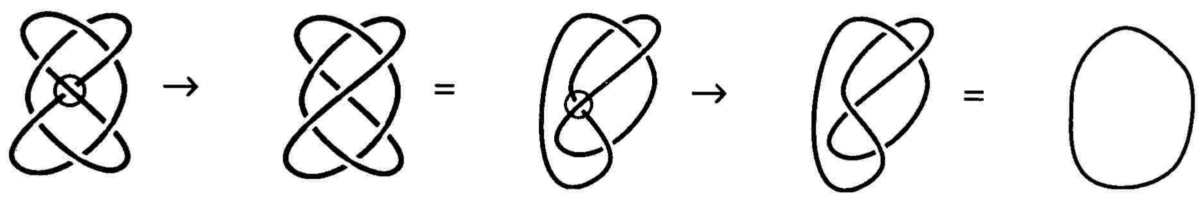
6 intersections

Tait also managed to find the 8 forms of knot with knottiness 7, but was dissuaded from further extension by Cayley and Muir whose analysis of a permutation problem had shown that if he were to successfully arrive at all the distinct forms he would need to consider 4738 configurations of knots with eightfold knottiness and no less than 43387 configurations with ninefold knottiness. Understandably Tait felt that he could not justify the time to undertake this Herculean task but thought that a relatively simple machine could be built to search for solutions.

After completing the first paper Tait became aware of T. P. Kirkman's investigations on the properties of polyhedra, research which seemed to parallel Tait's work. Indeed, Kirkman had been able to confirm the number of distinct forms with up to sevenfold knottiness and to take a census of the knots with knottiness eight and nine. Now armed with two methods for discovering new designs, Tait was able to make one or two corrections to Kirkman's results and illustrate all the forms with knottiness eight or nine in his second paper, and with knottiness ten in his final paper of 1885.

**(2) Beknottedness**

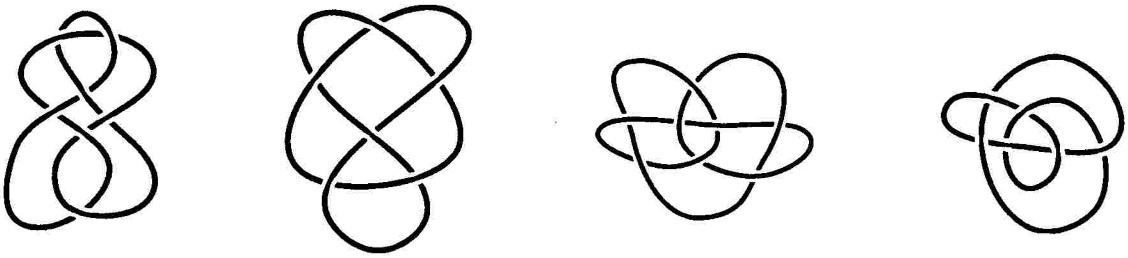
In his second paper Tait concentrated on a different concept, one which he had done little more than allude to earlier, that of *beknottedness*. The notion of beknottedness has a subtlety. First, for any diagram of a knot, its beknottedness is the minimum number of cuts and reconnections (or, equivalently, the changes from over-crossing to under-crossing or vice-versa) which are required to reduce the diagram to that of a simple loop. Secondly, the beknottedness of a knot is the minimum of the beknottedness of all possible diagrams of a knot. Notice this favourite mathematical trick of taking the minimum! The difficulty of calculating this invariant arises from the complicated nature of its definition, involving apparently all possible diagrams of the knot. Here is a suitably convoluted example. Note that we ring the crossing to be altered, and use an equals sign between two forms to indicate that one may be deformed into the other without cutting.



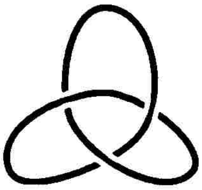

beknottedness = 2

Here are some ideas for use in class!

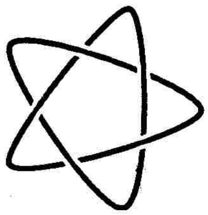
*(1) Establish the beknottedness of each of these knots.*



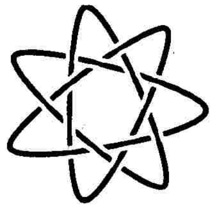
*(2) For the designs below find the connection between the order of rotational symmetry and beknottedness.*



trefoil



cinquefoil



septfoil

.......and so on.

Before we leave beknottedness, it is worth noting that in 4-space over-crossings can be changed into under-crossings without the need for a cut and hence knots do not exist in 4-space – a fact discovered by Felix Klein.

**(3) Möbius Bands**

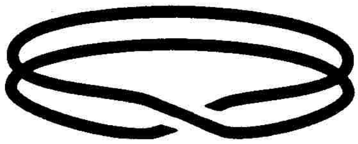
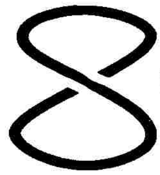
The name of Möbius is most familiar today in connection with the one-sided surface he discovered when analysing the geometry of polyhedra in pursuit of a prize offered by the Paris Academy of Sciences. He made his discovery of the band around September 1858 and published it in 1865. Listing found the same surface in July 1858 and published it in 1861. A classic case of simultaneous independent discovery.

Tait treated the bands in two ways, both of which may originate with him. Certainly, at no point in his papers does he credit either Möbius or Listing with priority. The second is essentially that which is so familiar to devotees of Martin Gardner, of cutting up the middle throughout their length bands of increasing numbers of half twists, to discover what forms are produced. His conclusions were:

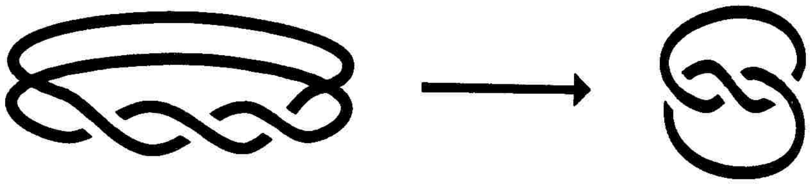
> *That which has half a twist, having originally only one edge, and that edge not being cut through in the process of splitting, remains a closed curve... a clear coil of two turns.*
>
> *That with one whole twist splits... into two interlinking single coils each having one whole twist.*
>
> *That with three half twists gives, when split, the trefoil knot, and when flattened out it has three whole twists.*
>
> *From two whole twists we get two single coils twice linked, each with two whole twists.*

Tait also noted that beginning with the standard band of one half twist *one continued cut, which takes off a strip constantly equal to one quarter of the original breadth of the slip, gives a half twist ring of half breadth, intersecting once a double twist ring of quarter breadth. A second cut splits the wider ring into one smaller than the narrow one, but there is now double linking.*

Tait's first treatment is perhaps less well known but sits comfortably with his ideas on knottiness and beknottedness. It would constitute an ideal starting point for investigation by pupils today. Concentrating now on the edges of the band, as if they are made of slender wire, he described what amounts to

flattening the shape  into  by for example holding the

upper and lower bands at the back and pulling them apart. The result is a curve with zero knottiness and zero beknottedness – a simple loop. Similarly, the whole twist band is flattened into two linked curves with knottiness 2 and unit beknottedness, and the three half twist band into a single curve with knottiness 3 and unit beknottedness and so on.



## Postscript

Over the years, a team at the University College of Wales, Bangor, (R. Brown, N.D. Gilbert, T. Porter), has been developing a variety of popular material on knots. Their exhibition **Mathematics and Knots** took four years to develop, and it featured prominently in the Pop Maths Roadshow which toured Britain, Scotland included, in 1989 and 1990. It is still available for hire, or in some cases loan. A brochure[1] of the exhibition with full text is also available and an A3 card version is for sale.

A relatively recent addition to the team, Heather McLeay, has been developing knot material for classroom use as part of a Welsh Office funded project. Designed as sources of investigative work for GCSE they would be suitable for use with Standard Grade classes. An article by Heather McLeay is in the January 1991 edition of *Mathematics in School* and she has recently published *The Mathematics of Knots* (Tarquin). Knot material has also been used at various times by each of the team in events such as Royal Institution Masterclasses, and workshops and lectures for teachers, children and the general public.

At such an event, one boy would not believe that knots had anything to do with mathematics, until some string tied on a copper knot according to a formula was shown to slip off the knot. "Where did you get that formula?" he demanded. Others are delighted to find that mathematics is more that pluses and minuses. Knots continue to fascinate and to reveal new aspects of mathematics. The manipulation of knot diagrams – real Tait territory – is even becoming to be regarded as a king of 'higher dimensional algebra'. What could be more wonderful, or awful, than that?

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1 Beware of differences in terminology. The Bangor group uses the more modern terms *crossing number* for Tait's *knottiness* and *unknotting number* for Tait's *beknottedness*.