SUBGROUPS OF FREE TOPOLOGICAL GROUPS AND FREE TOPOLOGICAL PRODUCTS OF TOPOLOGICAL GROUPS

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Introduction

Our objectives are topological versions of the Nielsen–Schreier Theorem on subgroups of free groups, and the Kurosh Theorem on subgroups of free products of groups.

It is known that subgroups of free topological groups need not be free topological [**2**, **6**, and **9**]. However we might expect a subgroup theorem when a *continuous* Schreier transversal exists, and we give such a result in the category of Hausdorff k_ω-groups (Theorem 8). In the same category, we give an open subgroup version of the Kurosh Theorem (Theorem 13).

The method of proof in both cases is a topological version of the groupoid method given by Higgins in [**8**]—that is, we use topological groupoids. The key steps are first of all to construct universal morphisms of topological groupoids, and secondly to prove that the pull-back by a covering morphism of a universal morphism is again universal. For the second step it is essential to know that the pullback of quotient maps of topological groupoids is again a quotient, and to obtain this we work in the category of Hausdorff k_ω-spaces. The chief technical work is then in constructing universal morphisms in this category—the results on k_ω-spaces needed are Propositions A1–A3 which are given in the Appendix.

This restriction to k_ω-spaces means that our theorems specialise only to countable versions of the abstract theorems. More general results can be given by using k-groupoids, but the proofs involve extra technicalities, and so we refer the interested reader to [**7**].

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2. *Universal k_ω-groupoids*

A *topological graph over* X consists of a space Γ of " arrows ", a space X of " objects " and continuous functions $\partial', \partial : \Gamma \to X$, $u : X \to \Gamma$ called the initial, final and unit functions respectively; these are to satisfy $\partial' u = \partial u = 1$. We usually confuse the graph with its space Γ of arrows, and also write $X = \mathrm{Ob}(\Gamma)$.

The graph Γ becomes a *topological category* (over X) if there is also given a continuous composition $\theta : (a, b) \mapsto ba$ with domain the set

$$\{(a, b) \in \Gamma \times \Gamma : \partial' b = \partial a\},$$

making Γ into a category in the usual sense. Finally such a topological category is a *topological groupoid* if it is abstractly a groupoid and the inverse map $a \mapsto a^{-1}$ is continuous. Morphisms of topological graphs, categories or groupoids are defined in the obvious way.

Any space X defines a topological groupoid with arrows X, objects X, and $\partial' = \partial = u = 1_X$; this topological groupoid is also written X.

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A morphism $\theta : G \to H$ of topological groupoids is called *universal* if the following diagram is a pushout of topological groupoids

$$\begin{array}{ccc} \mathrm{Ob}(G) & \xrightarrow{\mathrm{Ob}(\theta)} & \mathrm{Ob}(H) \\ \downarrow & & \downarrow \\ G & \xrightarrow[\theta]{} & H \end{array}$$

in which the vertical morphisms are the obvious inclusions. Clearly such a topological morphism, if it exists, is uniquely determined up to isomorphism by G and $\mathrm{Ob}(\theta)$: $\mathrm{Ob}(G) \to \mathrm{Ob}(H)$.

We now prove the existence of universal topological morphisms under certain restricted circumstances.

Let G be a topological groupoid, and $\sigma : \mathrm{Ob}(G) \to X$ a continuous function into a topological space X. For each $n > 0$, let $G^n = G \times \ldots \times G$, and let $G^0 = X$. Then $W_\sigma(G) \subseteq \bigsqcup_{n \geqslant 0} G^n$ is defined to be the set of all (*non-reduced*) " *words* " $(a_n, \ldots, a_1)$ of length $n \geqslant 0$ such that a_{i+1}, a_i are " σ-composable ", i.e. $\sigma\partial(a_i) = \sigma\partial'(a_{i+1})$, $1 \leqslant i \leqslant n$. We give $W_\sigma(G)$ its topology as a subspace of $\bigsqcup_{n \geqslant 0} G^n$, and with this topology it is easy to see that $W_\sigma(G)$ is a topological category over X—the initial and final maps are $\partial'(a_n, \ldots, a_1) = \sigma\partial'(a_1)$, $\partial(a_n, \ldots, a_1) = \sigma\partial(a_n)$ and $\partial'(x) = \partial(x) = x$; the unit map is inclusion of X; and composition is simply juxtaposition, i.e.

$$\theta((a_n, \ldots, a_1), (b_m, \ldots, b_1)) = (b_m, \ldots, b_1, a_n, \ldots, a_1),$$

$$\theta((a_n, \ldots, a_1), x) = (a_n, \ldots, a_1),$$

$$\theta(y, (a_n, \ldots, a_1)) = (a_n, \ldots, a_1)$$

and

$$\theta(x, x) = x.$$

If G, X are Hausdorff k_ω-spaces (see Appendix), then using Proposition A.3 it is easy to prove that $W_\sigma(G)$ is also a Hausdorff k_ω-space.

A (*reduced*) *word* of length $n \geqslant 0$ in G is a word $(a_n, \ldots, a_1)$ such that no a_i is an identity and a_{i+1}, a_i are σ-composable but not composable in G, i.e.

$$\sigma\partial'(a_{i+1}) = \sigma\partial(a_i)$$

but

$$\partial'(a_{i+1}) \neq \partial(a_i),$$

$1 \leqslant i \leqslant n$. There is a well-known process [**8**; p. 73] of assigning to each (non-reduced) word a unique reduced word called its *reduced form*. Let $U_\sigma(G)$ denote the set of all reduced words of length $n \geqslant 0$ in G, with its topology as a quotient space of $W_\sigma(G)$; i.e. the canonical map $p : W_\sigma(G) \to U_\sigma(G)$, which sends each word to its reduced form, is a quotient map.

THEOREM 1. *If G, X are Hausdorff k_ω-spaces then $U_\sigma(G)$ with its quotient topology is a Hausdorff k_ω-space. Further $U_\sigma(G)$ is a topological groupoid over X, and $\sigma^* : G \to U_\sigma(G)$, $a \mapsto p(a)$, is the universal topological morphism induced by σ.*

Proof. By [**8**; p. 73], $U_\sigma(G)$ is (abstractly) the universal groupoid generated by σ. The continuity of ∂', ∂, u on $U_\sigma(G)$ follows from the fact that p is a quotient map.

Similarly, the continuity of the inverse on $U_\sigma(G)$ follows from the continuity of the anti-automorphism on $W_\sigma(G)$, $(a_n, \ldots, a_1) \mapsto (a_1^{-1}, \ldots, a_n^{-1})$. So the only problem is continuity of composition in $U_\sigma(G)$, and it is at this stage that we need to restrict the spaces involved.

Essentially we require the restriction of $p \times p$ mapping the domain of composition in $W_\sigma(G)$ to the domain of composition in $U_\sigma(G)$ to be a quotient map. Now the domain of composition in $W_\sigma(G)$ is a closed $(p \times p)$-saturated subset of $W_\sigma(G) \times W_\sigma(G)$; so it is sufficient that $p \times p$ be a quotient map. This is implied by Proposition A.2 once we have proved $U_\sigma(G)$ a Hausdorff k_ω-space, and this follows from Proposition A.1 and the following

LEMMA 2. *The relation* $R = \{(w, w') : w, w' \in W_\sigma(G), p(w) = p(w')\}$ *is closed in* $W_\sigma(G) \times W_\sigma(G)$.

The proof is given in §4.

Thus $U_\sigma(G)$ is a topological groupoid. It follows easily that $\sigma^* : G \to U_\sigma(G)$, $a \mapsto p(a)$, is a morphism of topological groupoids, and is universal.

Note. It is proved in [3] that if G is any topological groupoid and $\sigma : \mathrm{Ob}(G) \to X$ is any continuous function, then the universal topological groupoid $U_\sigma(G)$ exists. In fact, if R is the relation given by Lemma 2, then in the terminology of [3], $U_\sigma(G)$ is the topological category $W_\sigma(G)$ with relations R. However, this extra generality would not help us towards our main objectives, since we have been unable to prove Theorem 4 below for arbitrary spaces.

We now consider free topological groupoids. Let Γ be a topological graph. The *(Graev) free topological groupoid on* Γ is a topological groupoid $F(\Gamma)$ together with a topological graph morphism $i : \Gamma \to F(\Gamma)$ such that if $f : \Gamma \to H$ is any topological graph morphism into a topological groupoid H then there is a unique topological groupoid morphism $f^* : F(\Gamma) \to H$ such that $f^* i = f$. Clearly if $F(\Gamma)$ exists then it is uniquely determined by Γ. Also given a non-based topological graph Γ' (so that only ∂', ∂ are part of the structure) we can form a new graph Γ with $\mathrm{Ob}(\Gamma) = \mathrm{Ob}(\Gamma')$, $\Gamma = \Gamma' \sqcup \mathrm{Ob}(\Gamma')$ and $u : \mathrm{Ob}(\Gamma) \to \Gamma$ the inclusion. The (Graev) free topological groupoid on Γ is then called the *Markov free topological groupoid on* Γ' (compare a similar distinction for free topological groups [6 and 13]).

If X is a topological space with base point e, then X defines a topological graph with arrows X, objects $\{e\}$ and $u : \{e\} \to X$ inclusion, and $F(X)$ is the (Graev) free topological group on X.

PROPOSITION 3. *If* Γ *is a topological graph which is a Hausdorff* k_ω*-space, then the (Graev) free topological groupoid* $F(\Gamma)$ *exists and is also a Hausdorff* k_ω*-space.*

Proof. Define a topological groupoid $\Gamma^{(2)}$ as follows: $\mathrm{Ob}(\Gamma^{(2)})$ is the space Γ of arrows of Γ, and the arrows of $\Gamma^{(2)}$ are pairs (a, b) of arrows of Γ with the same initial point. The initial and final maps of $\Gamma^{(2)}$ are the projections $(a, b) \mapsto a$, $(a, b) \mapsto b$, and composition is given by $(b, c).(a, b) = (a, c)$.

Let $\sigma : \mathrm{Ob}(\Gamma^{(2)}) \to \mathrm{Ob}(\Gamma)$ be the final map ∂. Let $\sigma^* : \Gamma^{(2)} \to U_\sigma(\Gamma^{(2)})$ be the universal morphism, and let $i : \Gamma \to U_\sigma(\Gamma^{(2)})$ be the identity on objects and be given on arrows by $a \mapsto \sigma^*(u\partial'(a), a)$. Then it is easy to check that i satisfies the required universal property, and so $F(\Gamma) = U_\sigma(\Gamma^{(2)})$.

Notice that our definitions and constructions do not coincide with those in the abstract case given in [**8**]—the construction there is not so convenient in the topological case.

The definition and construction of the free topological product of a countable family of Hausdorff k_ω-groupoids are the exact analogues of those for the abstract case, which are given in [**8**; p. 79], and so we do not repeat them here. (Note that a countable disjoint union of k_ω-groupoids is k_ω.)

A morphism $q : \tilde{G} \to G$ of topological groupoids is a *(topological) covering morphism* [**4**] if $(q, \partial') : \tilde{G} \to G \mathbin{\tilde{\times}} \mathrm{Ob}(\tilde{G})$ is a homeomorphism, where $G \mathbin{\tilde{\times}} \mathrm{Ob}(\tilde{G})$ is the pull-back of $\partial' : G \to \mathrm{Ob}(G)$ and $\mathrm{Ob}(q) : \mathrm{Ob}(\tilde{G}) \to \mathrm{Ob}(G)$. Our key result for the subgroup theorems is the following which is a topological version of Theorem 8 of [**8**].

THEOREM 4. *Suppose given a pull-back diagram of topological groupoids*

$$\begin{array}{ccc} \tilde{B} & \xrightarrow{\tilde{\theta}} & \tilde{G} \\ \downarrow & & \downarrow q \\ B & \xrightarrow{\theta} & G \end{array}$$

in which q is a covering morphism, B, G, $\tilde{G}$ are Hausdorff k_ω-spaces and $\theta : B \to G$ is universal. Then $\tilde{\theta} : \tilde{B} \to \tilde{G}$ is also universal.

Proof. Let $\sigma = \mathrm{Ob}(\theta) : \mathrm{Ob}(B) \to \mathrm{Ob}(G)$; then we may assume $G = U_\sigma(B)$. Let $\tilde{\sigma} = \mathrm{Ob}(\tilde{\theta}) : \mathrm{Ob}(\tilde{B}) \to \mathrm{Ob}(\tilde{G})$. The proof of Theorem 8 of [**8**] constructs in essence an algebraic isomorphism $\Theta : U_{\tilde{\sigma}}(\tilde{B}) \to \tilde{G}$ such that $\Theta\tilde{\sigma}^* = \tilde{\theta}$. We have to prove Θ a topological isomorphism.

To this end, let $p : W_\sigma(B) \to G$, $\tilde{p} : W_{\tilde{\sigma}}(\tilde{B}) \to U_{\tilde{\sigma}}(\tilde{B})$ be the canonical quotient maps, and consider the pullback diagram

$$\begin{array}{ccc} P & \xrightarrow{\bar{p}} & \tilde{G} \\ \downarrow & & \downarrow q \\ W_\sigma(B) & \xrightarrow[p]{} & G \end{array}$$

We construct below a topological isomorphism Φ in a commutative diagram

$$\begin{array}{ccc} W_{\tilde{\sigma}}(\tilde{B}) & \xrightarrow{\Phi} & P \\ \tilde{p}\downarrow & & \downarrow \bar{p} \\ U_{\tilde{\sigma}}(\tilde{B}) & \xrightarrow[\Theta]{} & \tilde{G} \end{array}$$

Now $\tilde{p}$ is a quotient map. Also by Proposition A.2, $p \times 1 : W_\sigma(B) \times \tilde{G} \to G \times \tilde{G}$ is a quotient map. Since P is closed and $(p \times 1)$-saturated, it follows easily that $\bar{p}$ is a quotient map. Hence Θ is a topological isomorphism.

It remains to construct Φ. Let s be the inverse of $(q, \partial') : \tilde{G} \to G \mathbin{\tilde{\times}} \mathrm{Ob}(\tilde{G})$. Then, regarding pullbacks as subspaces of products, we set

$$\Phi\big((b_n, \tilde{a}_n), \ldots, (b_1, \tilde{a}_1)\big) = \big((b_n, \ldots, b_1), \tilde{a}_n \ldots \tilde{a}_1\big)$$

and

$$\Phi^{-1}((b_n, \ldots, b_1), \tilde{a}) = ((b_n, \tilde{a}_n), \ldots, (b_1, \tilde{a}_1))$$

where

$$\tilde{a}_1 = s(\theta b_1, \partial' \tilde{a}), \text{ and } \tilde{a}_{i+1} = s(\theta b_{i+1}, \partial \tilde{a}_i), \qquad 1 \leqslant i < n.$$

Of course the same formulae on reduced words define Θ and Θ^{-1}—the further work on p.113–4 of [**8**] is to use the solution of the word problem to show these maps are well-defined.

COROLLARY 5. *Let $q : \tilde{G} \to G$ be a covering morphism of Hausdorff k_ω-groupoids.* (i) *If G is the free topological groupoid on a Hausdorff k_ω-graph Γ, then $\tilde{G}$ is the free topological groupoid on the Hausdorff k_ω-graph $\tilde{\Gamma}$, where $\tilde{\Gamma}$ is the pullback of the inclusion $i : \Gamma \to G$ and $q : \tilde{G} \to G$.* (ii) *If G has a free decomposition $G = \underset{\lambda}{*} G_\lambda$ as a free topological product of a countable family of Hausdorff k_ω-groupoids $\{G_\lambda\}$, then $\tilde{G}$ has a free decomposition $\tilde{G} = \underset{\lambda}{*} \tilde{G}_\lambda$ as a free topological product of Hausdorff k_ω-groupoids $\{\tilde{G}_\lambda\}$, where $\tilde{G}_\lambda$ is the pullback of the inclusion $i_\lambda : G_\lambda \to G$ by $q : \tilde{G} \to G$.*

Proof. (i) We have to prove that the following diagram is a pullback,

$$\begin{array}{ccc} \tilde{\Gamma}^{(2)} & \xrightarrow{\ \tilde{\theta}\ } & \tilde{G} \\ {\scriptstyle q^{(2)}}\downarrow & & \downarrow{\scriptstyle q} \\ \Gamma^{(2)} & \xrightarrow[\ \theta\ ]{} & G \end{array}$$

in which $\theta(a, b) = i(b).i(a)^{-1}$, $\tilde{\theta}((a, \tilde{a}), (b, \tilde{b})) = \tilde{b}.\tilde{a}^{-1}$ and $q^{(2)}((a, \tilde{a}), (b, \tilde{b})) = (a, b)$. If $\Gamma^{(2)} \ltimes \tilde{G}$ is the pullback of θ and q, then the commutativity of the above diagram gives a topological morphism $\Psi : \tilde{\Gamma}^{(2)} \to \Gamma^{(2)} \ltimes \tilde{G}$, $((a, \tilde{a}), (b, \tilde{b})) \mapsto ((a, b), \tilde{b}.\tilde{a}^{-1})$, and it is easily checked that Ψ^{-1} is given by

$$((a, b), \tilde{c}) \mapsto \Big(\big(a, s(i(a)^{-1}, \partial' \tilde{c})^{-1}\big), \big(b, s(i(b)^{-1}, \partial \tilde{c})^{-1}\big)\Big).$$

The argument for (ii) is similar.

3. *The subgroup theorems*

Let X be a topological space. The *tree groupoid* $X \times X$ has object space X and arrows pairs (x, y) of elements of X with composition $(y, z).(x, y) = (x, z)$. A topological groupoid T is called a tree groupoid on X if it is isomorphic over X to $X \times X$.

Let G be a topological groupoid over X. Then G is *globally trivial* if G contains a wide, tree topological subgroupoid. In such a case we can define, for any $x_0 \in X$, a morphism $r : G \to G\{x_0\}$ of topological groupoids retracting G onto $G\{x_0\}$.

For example, let G be a topological group, and H a subgroup. Then G operates on the space G/H of left cosets of G, and we can form the translation groupoid, $\tilde{G} = \mathrm{Tr}(G, H)$ as in [**8**]; thus $\mathrm{Ob}(\tilde{G}) = G/H$ and the space of arrows is $G \times G/H$, with ∂', ∂ the projection to G/H and the operation respectively. The composition in $\tilde{G}$ is $(h', g'H)\,(h, gH) = (h'h, gH)$. Clearly the product topology on $\tilde{G}$ makes $\tilde{G}$ a topological groupoid and the projection $q : \tilde{G} \to G$ makes $\tilde{G}$ a topological covering groupoid of G.

PROPOSITION 6. *Under the above assumption,* Tr(G, H) *is globally trivial if and only if the projection* $G \to G/H$ *has a continuous section.*

The proof is trivial.

Of course to obtain a subgroup theorem for free topological groups a section of $G \to G/H$ has to be chosen carefully.

Let Γ be a subgraph of G, and T a wide tree subgroupoid of G. We say Γ *conjoins* T if T is abstractly freely generated by the subgraph Γ_T of T comprising all the non-identity arrows of T which are also in Γ.

PROPOSITION 7. *Let G be the free topological groupoid on a subgraph Γ, and suppose G contains a wide tree subgroupoid T such that Γ conjoins T. Let $x_0 \in \mathrm{Ob}(G)$, and let $r : G \to G\{x_0\}$ be the retraction defined by T. Then $G\{x_0\}$ is the free topological group on $r(\Gamma)$.*

The proof follows easily from the abstract case [**1**; p. 277] and the fact that r is a retraction.

Now we have our main result on free topological groups, namely an analogue of the usual Nielsen-Schreier theorem.

THEOREM 8. *Let G be the Graev free topological group on a Hausdorff k_ω-space X, and let H be a closed subgroup of G admitting a continuous Schreier transversal with respect to X. Then H is a Graev free topological subgroup of G.*

Here a continuous Schreier transversal with respect to X for H is a continuous section $s : G/H \to G$ of the projection $p : G \to G/H$ such that set $\{s(\bar{a}) : \bar{a} \in G/H\}$ is a Schreier transversal with respect to X in the usual sense.

Proof. We form the covering groupoid $\mathrm{Tr}(G, H) = \tilde{G}$ with covering morphism $q : \tilde{G} \to G$. By Corollary 5, $\tilde{G}$ is the Graev free topological groupoid on $\Gamma = q^{-1}(X)$. The assumptions on s imply that Γ conjoins the wide tree subgroupoid T generated by $\{(s(\bar{a}), H) : \bar{a} \in G/H\}$. The theorem then follows from Proposition 7.

COROLLARY 9. *If G is the Graev free topological group on a Hausdorff k_ω-space X, and H is an open subgroup of G, then H is a Graev free topological subgroup of G.*

In this case H is also closed, and G/H is discrete. Hence the construction of the section s required by Theorem 8 can be carried out as in the abstract case.

Notice that a closed subgroup of finite index is open, so that Corollary 9 applies to this case. Also if X is a countable discrete space then G is just the (abstract) free group on the set X with the discrete topology, and any subgroup H is open. So we have the usual Nielsen-Schreier result for subgroups of the free group on a countable set. As mentioned in the introduction, a more general version of Theorem 8 is given in [**7**] using the notion of k-groupoid—this version specialises to the usual theorem for subgroups of the free group on an arbitrary set.

In view of [**6**; p. 328], Corollary 8 implies:

COROLLARY 10. *An open subgroup of a Markov free topological group on a Hausdorff k_ω-space is a Markov free topological group if and only if it is disconnected.*

The following corollaries are proved in [**11**], [**9**], respectively, by different methods.

COROLLARY 11. *Let G be a Hausdorff k_ω-space and let H be the kernel of the canonical morphism $F(G) \to G$. Then H is a Graev free topological group.*

In this case the section $s : G \to F(G)$ simply sends an element of G to its corresponding generator.

COROLLARY 12. *Let X be a closed subset of the real line R such that $0 \in X$, and the non-zero elements of X are positive. Then the commutator subgroup C of $F(X)$ is a Graev free topological group.*

Proof. As base point of X we take 0. Let $AF(X)$ be the Graev free abelian topological group on X, so that we have an exact sequence

$$1 \to C \to F(X) \to AF(X) \to 1.$$

Since X is a Hausdorff k_ω-space, $F(X)$ is a quotient of the topological monoid of words in elements of X or $(-X)$; let $\bar{X} = X \cup (-X) \subseteq R$. Then we have quotient mappings, where $W = \bigsqcup_{n \geqslant 0} (\bar{X})^n$,

$$q_1 : W \to F(X),$$
$$q_2 : W \to AF(X).$$

Let $s' : W \to W$ be given by $(\varepsilon_1 x_1, \ldots, \varepsilon_n x_n) \mapsto (\tau_1 y_1, \ldots, \tau_n y_n)$, where $(\tau_1 y_1, \ldots, \tau_n y_n)$ is the rearrangement of $(\varepsilon_1 x_1, \ldots, \varepsilon_n x_n)$ to be in ascending order, $\tau_1 y_1 \leqslant \ldots \leqslant \tau_n y_n$. Then s' is continuous, and defines a section $s : AF(X) \to F(X)$ of p satisfying the conditions of Theorem 8. Hence C is a free topological group.

In contrast to Corollary 12, F. Clarke has proved that the commutator subgroup C of the Graev free topological group on S^n is not the free topological group on any countable CW-complex for $n \geqslant 2$. The method is to show that the classifying space B_C has a cohomology ring with non-trivial cup-products. However it is proved in **[7]** (using results of Milnor **[15]**) that the classifying space of the free topological group on a countable CW-complex has the homotopy type of a suspension, and so has trivial cup products.

We now give our version of the Kurosh theorem.

THEOREM 13. *Let $G = \ast_{\lambda \in \Lambda} G_\lambda$ be the free topological product of a countable family of topological groups which are Hausdorff k_ω-spaces. Then any open subgroup H of G is a free topological product*

$$H = (\ast_{\lambda, \mu} H_{\lambda\mu}) \ast F$$

where

(i) *each $H_{\lambda\mu} (\lambda \in \Lambda, \mu \in M_\lambda)$ is of the form $H \cap x_{\lambda\mu} G_\lambda x_{\lambda\mu}^{-1}$ where as μ varies in M_λ, $x_{\lambda\mu}$ runs through a (suitably chosen) set of double coset representatives of HxG_λ.*

(ii) *F is the Graev free topological group on a countable discrete space.*

Proof. We assume that the reader is familiar with Higgins' groupoid proof of the Kurosh theorem [**8**; p. 118].

Let $\tilde{G} = \mathrm{Tr}(G, H)$ be the translation groupoid, and $q : \tilde{G} \to G$ the topological covering morphism described in §3. Then, by Corollary 5, $\tilde{G}$ is the free topological product $\ast_\lambda \tilde{G}_\lambda$ where $\tilde{G}_\lambda = q^{-1}(G_\lambda)$.

LEMMA 14. *For each* $\lambda \in \Lambda$, *the components* $\tilde{G}_{\lambda\mu}(\mu \in M_\lambda)$ *of* $\tilde{G}_\lambda$ *are globally trivial subgroupoids of* $\tilde{G}_\lambda$, *and* $\tilde{G}_\lambda = \underset{\mu \in M_\lambda}{*} \tilde{G}_{\lambda\mu}$.

This follows trivially from the abstract case and the fact that $\mathrm{Ob}(\tilde{G}_\lambda)$ is a discrete space.

An easy consequence of Lemma 14 is that for each $\lambda \in \Lambda$, $\mu \in M$ we can choose a vertex group $K_{\lambda\mu}$ and a wide tree topological subgroupoid $S_{\lambda\mu}$ of $\tilde{G}_{\lambda\mu}$ such that $\tilde{G}_{\lambda\mu} = K_{\lambda\mu} * S_{\lambda\mu}$. Thus $\tilde{G} = K * S$, where $K = \underset{\lambda,\mu}{*} K_{\lambda\mu}$ is totally intransitive (i.e. the only non-empty " hom sets " of K are vertex groups), and $S = \underset{\lambda,\mu}{*} S_{\lambda\mu}$ is a free topological groupoid on Γ (say). Clearly S is a wide transitive subgroupoid of G; and in fact we have

LEMMA 15. *S contains a wide tree topological subgroupoid T which conjoins* Γ.

Proof. Since $\mathrm{Ob}(\tilde{G})$ is a discrete space, we can choose T exactly as in the abstract case [**8**; p. 119].

If $r : \tilde{G} \to H$ is the retraction defined by T, then Higgins [**8**; p. 119] proves that H is (abstractly) the free product $(\underset{\lambda,\mu}{*} H_{\lambda\mu}) * F$, where $H_{\lambda\mu} = r(K_{\lambda\mu})$ and $F = r(S)$ is a vertex group of S. The topological result now follows from Proposition 7 and the fact that r is a continuous retraction.

Finally, we prove that S is a countable discrete space. It is clear that each $S_{\lambda\mu}$ is countable and discrete, but since $\mathrm{Ob}(\tilde{G}_\lambda)$ is a subspace of $\mathrm{Ob}(\tilde{G}) = G/H$, it is countable, so $\tilde{G}_\lambda$ has at most a countable number of components (i.e. M_λ is countable) and $S = \underset{\lambda,\mu}{*} S_{\lambda\mu}$ is a countable discrete space. Hence F is countable and discrete.

It is also notable that in [5] Gildenhuys and Ribes prove in the category of pro-C-groups a version of the Kurosh theorem which implies an open subgroup version of the Nielsen-Schreier Theorem (in the same category). It would be interesting to have a version of Theorem 13 which implied Theorem 8.

4. *Proof of Lemma* 2

For any topological spaces Y, Z we will denote a function f with domain D_f a subset of Y, and values in Z, by $f: Y \mapsto Z$; we will say f is continuous if $f | D_f$ is continuous in the usual sense.

Let W_n be the subset of $W_\sigma(G)$ consisting of all the (non-reduced) words of length $\leqslant n$ in G. Then $W_\sigma(G)$ has the weak topology with respect to $\{W_n\}_{n \geqslant 0}$, so that $W_\sigma(G) \times W_\sigma(G)$ has the weak topology with respect to $\{W_n \times W_n\}_{n \geqslant 0}$, and a necessary and sufficient condition for R to be closed is that $R_n = R \cap (W_n \times W_n)$ is closed in $W_n \times W_n$ for each $n \geqslant 0$. We prove this by induction.

Let Δ_0 be the diagonal in $X \times X$; and for each $n > 0$, let Δ_n be the diagonal in $W_n \times W_n$. Then, since $W_\sigma(G)$ is Hausdorff, each Δ_n, $n \geqslant 0$, is closed.

LEMMA 16. R_1 *is closed.*

Proof. R_1 is a subset of $(G \times G) \sqcup (G \times X) \sqcup (X \times G) \sqcup (X \times X)$. Now

$$R_1 \cap (G \times G) = \Delta_G \cup (u \times u)(\sigma \times \sigma)^{-1}(\Delta_0),$$

where Δ_G is the diagonal in $G \times G$, and is closed since G is Hausdorff, while u is a homeomorphism onto a closed subset of G(since $\partial' u = 1$ and G is Hausdorff), and so is proper, whence $u \times u$ is a closed map.

Next $R_1 \cap (X \times X) = \Delta_0$ which is closed.

Finally $R_1 \cap (G \times X) = (u \times 1)(\sigma \times 1)^{-1}(\Delta_0)$, which again is closed since u is proper, and similarly $R_1 \cap (X \times G) = (1 \times u)(1 \times \sigma)^{-1}(\Delta_0)$ is closed.

Now for the inductive step, suppose R_{n-1} is closed in $W_{n-1} \times W_{n-1}$; then we prove that R_n is closed in $W_n \times W_n$. To this end, for each $1 \leqslant i < n$, let $\theta_i : W_n \mapsto W_{n-1}$, $(a_n, \ldots, a_1) \mapsto (a_n, \ldots, a_{i+1} . a_1, \ldots, a_1)$, be given by composition of the ith and $(i+1)$th co-ordinates whenever possible; and for each $1 \leqslant j \leqslant n$, let $\pi_j : W_n \mapsto W_{n-1}$, $(a_n, \ldots, a_1) \mapsto (a_n, \ldots, \tilde{a}_j, \ldots, a_1)$, be given by omission of the jth co-ordinate whenever it is an identity. Let $\theta_0 = \pi_0 : W_n \mapsto W_{n-1}$ be given by the identity on W_{n-1}. Then it is easy to see that each θ_i, π_j is continuous and that G Hausdorff implies that each θ_i, π_j has closed domain.

Our proof is now completed by

LEMMA 17. *$R_n = \Delta_n \cup \bigcup_{\phi} \phi^{-1}(R_{n-1})$, where ϕ runs through all $\lambda \times \mu$, with λ, μ either a θ_i, $0 \leqslant i < n$ or a π_j, $0 \leqslant j \leqslant n$.*

Proof. This follows from the fact that the non-diagonal elements of R_n are obtained from elements of R_{n-1} by the inverse of the reduction process [**8**; p. 73].

5. *Appendix*

The purpose of this section is to give the basic results we have used on k_ω-spaces.

A topological space X is a *k_ω-space* if it has the weak topology with respect to some countable increasing family $X_0 \subseteq X_1 \subseteq \ldots \subseteq X_n \subseteq \ldots$ of compact subspaces whose union is X; we then call $\{X_n\}_{n \geqslant 0}$ a *k_ω-decomposition* of X. We are more concerned with Hausdorff k_ω-spaces.

PROPOSITION A.1. *Let X be a Hausdorff k_ω-space and $p : X \to Y$ a quotient map. Then the following are equivalent:*

(i) *The graph of the equivalence relation R associated with p is closed.*

(ii) *Y is a Hausdorff k_ω-space.*

The proof is given in [**12**; Proposition 4.25].

PROPOSITION A.2. *If $p : X \to Y$, $p' : X' \to Y'$ are quotient maps of Hausdorff k_ω-spaces X, X', Y, Y' then $p \times p' : X \times X' \to Y \times Y'$ is also a quotient map of Hausdorff k_ω-spaces.*

This follows from the fact that (a) any Hausdorff k_ω-space is a k-space, (b) that the weak product of quotient maps of k-spaces is a quotient [**12**], and (c) the following proposition:

PROPOSITION A.3. *If $\{X_n\}_{n \geqslant 0}$, $\{Y_n\}_{n \geqslant 0}$ are k_ω-decompositions of Hausdorff k_ω-spaces X, Y respectively, then $\{X_n \times Y_n\}_{n \geqslant 0}$ is a k_ω-decomposition of the Hausdorff space $X \times Y$. That is, $X \times Y$ is a Hausdorff k_ω-space.*

The proof, which is essentially that of [**14**; Lemma 2.1], is given in [**12**; page 16].

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