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R. Brown and P.J. Higgins, ``Crossed complexes and non-abelian extensions", *Category theory proceedings, Gummersbach, 1981, Lecture Notes in Math.* 962 (ed. K.H. Kamps et al, Springer, Berlin, 1982), pp. 39-50.

The theory is developed in

R. Brown and T. Porter , ``On the Schreier theory of non-abelian extensions: generalisations and computations". *Proceedings Royal Irish Academy* 96A (1996) 213-227.

CROSSED COMPLEXES AND NON-ABELIAN EXTENSIONS

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Introduction

Crossed complexes may be thought of as chain complexes with operators from a group (or groupoid) but with non-abelian features in dimensions one and two. We start by surveying briefly their use.

The definition of crossed complex is motivated by the standard example, the *homotopy crossed complex* $\pi \underset{\sim}{X}$ of a filtered space $\underset{\sim}{X} : X_0 \subset X_1 \subset \ldots \subset X_n \subset X_{n+1} \subset \ldots \subset X$. Here $\pi_1 \underset{\sim}{X}$ is the fundamental groupoid $\pi_1(X_1, X_0)$ of homotopy classes rel $\dot{I}$ of maps $(I, \dot{I}) \to (X_1, X_0)$, with the usual groupoid structure induced by composition of paths. For $n \geq 2$, $\pi_n \underset{\sim}{X}$ is the family of relative homotopy groups $\pi_n(X_n, X_{n-1}, p)$ for all $p \in X_0$. For $n \geq 2$, there is an action of $\pi_1 \underset{\sim}{X}$ on $\pi_n \underset{\sim}{X}$, and there is a boundary map $\delta : \pi_n \underset{\sim}{X} \to \pi_{n-1} \underset{\sim}{X}$; there are also the initial and final maps $\delta^0, \delta^1 : \pi_1 \underset{\sim}{X} \to X_0$. The rules which are satisfied by all such $\pi \underset{\sim}{X}$ are taken as the defining rules for a crossed complex (§1). In particular, the rule $\delta\delta = 0$ shows the analogy with chain complexes. Of course the individual rules are commonly used in homotopy theory, without necessarily considering the total structure.

By a *reduced* crossed complex C we mean one in which C_0 is a point. These have been considered for some 35 years. They were called "group systems" by Blakers [2]. He writes that he follows a suggestion of Eilenberg in using these group systems to apply the homotopy addition lemma in his investigation of the relationship between the homology and homotopy groups of pairs. His proofs involve a functor from reduced crossed complexes to simplicial sets; the values of this functor have been shown recently by Ashley [1] to be *simplicial T-complexes*, and Ashley has proved the hard theorem that this functor gives an equivalence N between crossed complexes and simplicial T-complexes. This equivalence generalises the well known equivalence of chain complexes and simplicial abelian groups, due to Dold and Kan [27; Theorem 22.4], and the functor N generalises also the nerve of a groupoid, which we use in §3.

Reduced crossed complexes satisfying in each dimension a freeness condition were called "homotopy systems" by Whitehead [31, 32], and his main example was $\pi \underset{\sim}{K}$ where $\underset{\sim}{K}$ is the filtration of a CW-complex K by its skeletons. The paper [31] gives interesting relations between homotopy systems and chain complexes with operators: we shall generalise these results to crossed complexes in [10]. An overall consideration in the papers [30, 31, 32] is *realisability*. In §17 of [32] Whitehead sketches a proof of a theorem announced in §7 of [31], that if $\phi : C \to C'$ is a homotopy equivalence of finite dimensional homotopy systems, and C is realisable as $\pi \underset{\sim}{K}$ for some

CW-complex K, then C' is also realisable as $\pi \underset{\sim}{K}'$ and ϕ is realisable by a map $K \to K'$. The approach to simple homotopy theory in this section of [32] seems to have been ignored and indeed its predecessor [31] is not widely read.

Huebschmann, Holt and others (cf. [20, 17] and the historical note [26]) have shown how crossed complexes may be used to give an interpretation of all the cohomology groups $H^n(G; A)$ of a group G with coefficients in a G-module A. Lue has explained in [24] how related ideas had been developed earlier for varieties of algebras, rather than just for groups. However, the tie-up with classical cohomology was not made explicit (cf. p.172 of [24]).

We have given in [6, 7] a colimit theorem for the homotopy crossed complex of a union of filtered spaces. This theorem includes the usual Seifert-van Kampen theorem on the fundamental groupoid of a union of spaces; it also includes the Brouwer degree theorem ($\pi_n S^n = \mathbb{Z}$), the relative Hurewicz theorem, and a subtle theorem of J.H.C. Whitehead on free crossed modules [31; §16]. The proof of the colimit theorem in [7] involves in an essential way two other categories equivalent to crossed complexes, namely ω-groupoids and cubical T-complexes [6, 8]. With simplicial T-complexes [1], ∞-groupoids [9] and poly-T-complexes [22], there are now five categories known to be equivalent to crossed complexes, the proofs in each case being highly non-trivial.

The papers [16, 18] give other work on crossed complexes.

One of our aims here is to show how the *homotopy* addition lemma (which plays a key rôle in the work of Blakers [2] and of the authors [6, 7]) is also important in the *cohomology* of a group G. We do this by showing that the *standard crossed resolution* CG, which is constructed algebraically in [20] and applied further in [21], in fact arises as πBG, the homotopy crossed complex of the classifying space of G. The boundary maps in CG are determined by the homotopy addition lemma.

Our further aim is an exposition of the Schreier theory of non-abelian extensions. Much has been written on non-abelian extensions and cohomology, (cf. [5, 12, 13 23] and the further references there), but it is notable that, while there are accounts in several books on group theory, texts on homological algebra remain largely silent on the subject, presumably because there is no known exposition using chain complexes, on which expositions of the abelian case are rightly based. Here we show that the non-abelian features of crossed complexes allow an exposition closer to the abelian case, involving morphisms and homotopies. We strengthen the theory, by presenting an equivalence of groupoids which on components induces the usual one-one correspondence of sets. We also generalise the theory, to extensions of groupoids rather than just groups, and to "free" equivalences of extensions.

1. Crossed Complexes

We recall from [6] the definition of the category (here denoted XC) of crossed complexes.

A crossed complex C (over a groupoid) is a sequence

$$\cdots \longrightarrow C_n \xrightarrow{\delta} C_{n-1} \longrightarrow \cdots \longrightarrow C_2 \xrightarrow{\delta} C_1 \overset{\delta^0}{\underset{\delta^1}{\rightrightarrows}} C_0$$

satisfying the following axioms:

(1.1) C_1 is a groupoid with C_0 as its set of vertices and δ^0, δ^1 as its initial and final maps.

We write $C_1(p,q)$ for the set of arrows from p to q $(p,q \in C_0)$ and $C_1(p)$ for the group $C_1(p,p)$.

(1.2) For $n \geq 2$, C_n is a family of groups $\{C_n(p)\}_{p \in C_0}$ (equivalently, C_n is a totally disconnected groupoid over C_0) and for $n \geq 3$ the groups $C_n(p)$ are abelian.

(1.3) The groupoid C_1 operates on the right of each $C_n (n \geq 2)$ by an action denoted $(x,a) \mapsto x^a$. Here if $x \in C_n(p)$ and $a \in C_1(p,q)$, then $x^a \in C_n(q)$. (Thus $C_n(p) \cong C_n(q)$ if p and q lie in the same component of the groupoid C_1.)

We use additive notation for all groups $C_n(p) (n \geq 2)$ and for the groupoid C_1, and we use the same symbol 0 for all their identity elements.

(1.4) For $n \geq 2$, $\delta : C_n \to C_{n-1}$ is a morphism of groupoids over C_0 and preserves the action of C_1, where C_1 acts on the groups $C_1(p)$ by conjugation: $x^a = -a + x + a$.

(1.5) $\delta\delta = 0 : C_n \to C_{n-2}$ for $n \geq 3$ (and $\delta^0\delta = \delta^1\delta : C_2 \to C_0$, as follows from (1.4)).

(1.6) If $c \in C_2$, then δc operates trivially on C_n for $n \geq 3$ and operates on C_2 as conjugation by c, that is

$$x^{\delta c} = -c + x + c \quad (x, c \in C_2(p)) .$$

In the case when C_0 is a single point, we call C a *reduced* crossed complex.

We observe that the above laws make each $C_2(p)$ a crossed module over $C_1(p)$; we take the laws up to dimension two as defining C_2 as a *crossed module over the groupoid* (C_1, C_0), or, simply, as a *crossed* C_1*-module*. Let $n \geq 3$. Then $C_n(p)$ is a module over $C_1(p)$, and we take the laws (1.1) - (1.3) as defining C_n as *module over the groupoid* (C_1, C_0), or, simply, as a C_1*-module*.

A *morphism* $f : C \to D$ of crossed complexes is a family of morphisms of groupoids $f_n : C_n \to D_n$, compatible with the boundary maps $C_n \to C_{n-1}$, $D_n \to D_{n-1}$ and the actions of C_1, D_1 on C_n, D_n. We denote by XC the resulting category of crossed complexes.

By restriction of structure, we have categories of modules, and of crossed modules (over groupoids). Let $f : C \to D$ be a morphism of crossed modules. If f_0 is the identity (as happens throughout §5) we write f as a pair (f_1, f_2). If f_0 and f_1 are the identity, we call f a *morphism of crossed* C_1*-modules*.

Suppose given a crossed module C, a set X and function λ from X to the union of the $C_1(p)$, $p \in C_0$. Then we say C is the *free crossed* C_1*-module on generators* $[x] \in C_2$ with $\delta[x] = \lambda x$ for all $x \in X$, if such elements $[x]$ are given, and for any other crossed C_1-module C' and elements $[x]' \in C_2'$ with $\delta[x]' = \lambda x$ for all $x \in X$, there is a unique morphism $f : C \to C'$ of crossed C_1-modules such that $f[x] = [x]'$ for all $x \in X$. This definition becomes the usual one in the reduced case [31]. Analogously to the group case, (an exposition of which is given in [11]), C_2 is constructed, given C_1, X and λ, as the groupoid with generators $x^a \in C_2(q)$ for all $x \in X$, $a \in C_1(\delta^0\lambda x, q)$, and $q \in C_0$, with the usual relations $-x^a + y^b + x^a = y^{b-a+\lambda x+a}$ where these make sense, and with $[x] = x^0$, $\delta(x^a) = -a + \lambda x + a$.

A module over C_1 can be regarded as a crossed C_1-module C with $\delta : C_2 \to C_1$ trivial. Such a C_1-module is *free on generators* $[x] \in C_2(p_x)$, $x \in X$, if it is a free crossed C_1-module on these generators with $\delta[x]$ equal to the zero at p_x, $x \in X$.

Let C be a crossed complex. Its *fundamental groupoid* $\pi_1 C$ is the quotient [15] of the groupoid C_1 by the normal, totally disconnected subgroupoid δC_2. The rules for a crossed complex give C_n, for $n \geq 3$, the induced structure of $\pi_1 C$-module.

A crossed complex C is *free* if C_1 is a free groupoid (on some graph X_1), C_2 is a free crossed C_1-module (for some $\lambda : X_2 \to C_1$), and for $n \geq 3$, C_n is a free $\pi_1 C$-module (on some X_n).

A crossed complex C is *exact* if for $n \geq 2$

$$\operatorname{Ker}(\delta : C_n \longrightarrow C_{n-1}) = \operatorname{Im}(\delta : C_{n+1} \longrightarrow C_n) .$$

If C is exact and G is a groupoid, then C together with an isomorphism $\pi_1 C \to G$ (or, equivalently, C with a quotient morphism $C_1 \to G$ whose kernel is δC_2) is called a *crossed resolution* of G. It is a *free* crossed resolution if also C is free.

Let G be a groupoid. A free crossed resolution of G may be constructed as follows. Let X be any subgraph of G generating G and let (C_1, C_0) be the free groupoid on X, with quotient morphism $\phi : C_1 \to G$. Let R be any set and let $w : R \to C_1$ be a function to the union of the $C_1(p)$, $p \in C_0$, such that the normal closure of the image of w is $\operatorname{Ker} \phi$. (The triple $(X; R, w)$ is a *presentation* of G.) Let C_2 be the free crossed C_1-module determined by w. Then $\kappa = \operatorname{Ker}(\delta : C_2 \to C_1)$ is the G*-module of identities* for the presentation (cf. [11]). Choose any free G-resolution $\to C_n \to \cdots \to C_3 \to \kappa$ of κ by G-modules; this may be spliced into $\delta : C_2 \to C_1$ to give a free crossed resolution of G. (Such a construction for groups is used in [20, 21].)

As explained in the introduction, a key example of a crossed complex is the homotopy crossed complex $\pi \underline{X}$ of a filtered space $\underline{X}$. Let $\underline{X}$ be the filtered space defined by the skeletons of a CW-complex X; then $\pi \underline{X}$ is a free crossed complex. (This is due to Whitehead [31; §16] in the reduced case, from which the more general case follows. A simple proof of freeness in dimension two is given in [11].) Further $\pi_1 \underline{X}$ is $\pi_1(X, X_0)$, and the homology of $\pi \underline{X}$ (i.e. $\operatorname{Ker} \delta / \operatorname{Im} \delta$) is for $n \geq 2$ isomorphic to the family of groups $H_n(\tilde{X}_p)$, $p \in X_0$, where $\tilde{X}_p$ is the universal cover of X based at p (cf. [32; Footnote 41]). In particular, if X is aspherical (i.e. $\pi_n X = 0$ for $n \geq 2$), then $\pi \underline{X}$ is exact, and so it is a free crossed resolution of $\pi_1(X, X_0)$.

2. The homotopy addition lemma

This is a basic, but not so easy to prove, lemma in homotopy theory. Intuitively, it expresses the idea that "the boundary of a simplex is the composite of its faces". Its formulation involves all the structural elements of the homotopy crossed complex, and so for completeness we state it here.

Let Δ^n be the standard n-simplex with ordered set of vertices $\{v_0, v_1, \ldots, v_n\}$, and let Δ^n have its filtration by skeletons Δ^n_r. Then $\pi_n(\Delta^n, \Delta^n_{n-1}, v_n)$ is for $n > 1$ an infinite cyclic group with generator σ, say. The unique arrow of $\pi_1(\Delta^1, \Delta^1_0)$ from v_0 to v_1 is also written σ. The face maps $\partial_i : \Delta^{n-1} \to \Delta^n$ then determine elements $\partial_i\sigma \in \pi_{n-1}\underline{\Delta}^n$, and the map $u : \Delta^1 \to \Delta^n$, which sends v_0, v_1 to v_{n-1}, v_n respectively, determines $u\sigma \in \pi_1\underline{\Delta}^n$.

Proposition 1. *(The homotopy addition lemma) The elements* σ *may be chosen so that the boundary*

$$\delta : \pi_n(\Delta^n, \Delta^n_{n-1}, v_n) \longrightarrow \pi_{n-1}(\Delta^n_{n-1}, \Delta^n_{n-2}, v_n)$$

is given by

$$\delta\sigma = \begin{cases} -\partial_1\sigma + \partial_0\sigma + \partial_2\sigma & \text{if } n = 2, \\ \partial_0\sigma - (\partial_3\sigma)^{u\sigma} - \partial_1\sigma + \partial_2\sigma & \text{if } n = 3, \\ \sum_{i=0}^{n-1} (-1)^i \partial_i\sigma + (-1)^n(\partial_n\sigma)^{u\sigma} & \text{if } n \geq 4. \end{cases} \quad \square$$

For a proof of this result, see for example [29]. A corresponding cubical form of the homotopy addition lemma is given for ω-groupoids as Lemma 7.1 of [6].

3. The standard crossed resolution

Let G be a groupoid. There is a well-known simplicial set NG [28], the *nerve* of G, in which N_nG is the set of composable elements $(u_1, \ldots, u_n)$ of G^n, i.e. of n-tuples of elements u_i of G such that $u_i + u_{i+1}$ is defined for $1 \leq i < n$. The geometric realisation $X = |NG|$ is known as the classifying space BG of G. The simplicial structure on NG induces a structure of CW-complex on X, and so the homotopy crossed complex $\pi\underline{X}$ (for the skeletal filtration on X) is defined. We write CG for this crossed complex and call it the *standard crossed resolution* of G.

Proposition 2. *Let* G *be a groupoid. Then* CG *is a free crossed resolution of* G *and has the following structure.*

(i) $C_0G = G_0$; C_1G *is the free groupoid on the sub-graph* G^* *consisting of all the vertices and all the non-identity arrows of* G.
The basis element of C_1G *corresponding to* $u \in G^*$ *is written* $[u]$, *and this notation is extended to* G *by setting* $[0_p] = 0_p$.

(ii) C_2G *is the free crossed* C_1G*-module on generators* $[u,v] \in C_2G(\delta^1 v)$ *with*

$$\delta[u,v] = -[u+v] + [u] + [v]$$

for all $(u,v) \in N_2G^*$ *(the composable pairs of* G^**)*.

(iii) *For* $n \geq 3$, C_nG *is the free* G*-module on generators* $[u_1, \ldots, u_n] \in C_nG(\delta^1 u_n)$ *for all* $(u_1, \ldots, u_n) \in N_nG^*$.
We also let $[u_1, \ldots, u_n] \in C_nG$ *be the identity at* $\delta^1 u_n$ *if* $(u_1, \ldots, u_n) \in N_nG$ *and some* $u_i = 0$.

(iv) $\delta : C_3G \to C_2G$ *is given by*

$$\delta[u,v,w] = [v,w] - [u,v]^{[w]} - [u+v,w] + [u,v+w],$$

for all $(u,v,w) \in N_3G$.

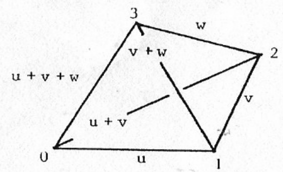
(v) *For* $n \geq 4$, $\delta : C_nG \to C_{n-1}G$ *is given by*

$$\delta[u_1, \ldots, u_n] = [u_2, \ldots, u_n] + \sum_{i=1}^{n-1} (-1)^i [u_1, \ldots, u_i + u_{i+1}, \ldots, u_n] + (-1)^n [u_1, \ldots, u_{n-1}]^{[u_n]}. \quad \square$$

This proposition follows from the homotopy addition lemma, the standard description of the face operators in NG, and the fact that BG is aspherical [28]. Note that if G is a group, then Proposition 2 shows CG to be the same as the standard (inhomogeneous) crossed resolution of G as defined in §9 of [20]. We have now shown how CG arises geometrically.

The curious formula for $\delta : C_3G \to C_2G$ should be noted; the values of this δ are in a family of (generally) non-abelian groups. There is a functor assigning to a crossed complex C a chain complex ΔC with operators from π_1C [10]; for this functor $\Delta(CG)$ is the bar resolution of G (cf. [25]), for the group case). However Δ abelianises C_2 and so loses information.

The 3-simplices of NG may be pictured as



(cf. p.12 of [25]). Now NG is a T-complex in which every n-simplex is thin for $n > 1$. Every T-complex has a groupoid structure in dimension 1, and the above picture illustrates the 3-simplex used to prove associativity [1] of this groupoid. This suggests the link between $\delta : C_3G \to C_2G$ and associativity in extension theory.

4. Homotopies

The notion of homotopy has a similar importance for crossed complexes to that for chain complexes. However, because of the more complicated structure of crossed complexes, there are several possible conventions for the definition of homotopy, and there are also two levels of generality (corresponding to free and based homotopy in the topological case). Our definition follows from the cubical homotopy addition lemma in the algebra of ω-groupoids [6], applied to a natural notion of homotopy for ω-groupoids, a topic which we hope to develop elsewhere.

Let $f, g : C \to D$ be morphisms of crossed complexes. A *homotopy* $\theta : f \simeq g$ is a family of functions $\theta_n : C_n \to D_{n+1} (n \geq 0)$ with the following properties.

(4.1) If $p \in C_0$, then $\theta_0 p \in D_1(fp, gp)$. If $x \in C_1(p,q)$, then $\theta_1 x \in D_2(gq)$. If $n \geq 2$ and $x \in C_n(q)$, then $\theta_n x \in D_{n+1}(gq)$.

(4.2) $\theta_1 : C_1 \to D_2$ is a derivation over g_1, that is if $x + y$ is defined in C_1 then

$$\theta_1(x + y) = (\theta_1 x)^{gy} + \theta_1 y \quad , \quad \text{where } gy = g_1 y .$$

(4.3) For $n \geq 2$, $\theta_n : C_n \to D_{n+1}$ is an operator morphism over g_1, that is, if $a \in C_1(p,q)$, $x \in C_n(p)$, $y \in C_n(q)$, then

$$\theta_n(x^a + y) = (\theta_n x)^{ga} + \theta_n y \quad \text{where } ga = g_1 a .$$

(4.4) If $x \in C_1(p,q)$ then

$$gx = -\theta_0 p + fx + \theta_0 q - (\delta\theta_1 x) .$$

(4.5) If $n \geq 2$, and $x \in C_n(q)$ then

$$gx = (fx)^{\theta q} - \theta_{n-1}\delta x - \delta\theta_n x \; , \quad \text{where } \theta q = \theta_0 q .$$

(A similar definition, but with different conventions, is given in the reduced case by Whitehead [23]. For further comments, see Remark 4 at the end of the paper.)

A homotopy $\theta : f \simeq g$ is said to be *rel* C_0 if $\theta_0 p$ is an identity for all $p \in C_0$ (so that in consequence $f_0 = g_0$). (It is these homotopies, with different conventions, which are used by Huebschmann [20].) For emphasis, the more general kinds of homotopy are sometimes called *free* homotopies.

If $\theta : f \simeq g$, $\theta' : g \simeq h$ are (free) homotopies, their composite $\phi = \theta + \theta'$ is defined by $\phi_0 p = \theta_0 p + \theta'_0 p$, $p \in C_0$, and if $n \geq 1$ and $x \in C_1(p,q)$ or $x \in C_n(q)$, then $\phi_n x = \theta'_n x + (\theta_n x)^{\theta' q}$. It is easily checked that ϕ is a homotopy $f \simeq h$.

In the next section we will be considering only crossed complex morphisms $C \to D$ which are the identity on $C_0 = D_0$. Therefore we write

$$(C, D)_f \quad \text{and} \quad (C, D)$$

for the groupoids which have such morphisms as objects, and whose arrows are respectively the free, and the rel C_0, homotopies. The sets of components of these groupoids are thus the respective sets of homotopy classes of morphisms over $C_0 = D_0$, and they are written respectively

$$[C, D]_f \quad \text{and} \quad [C, D] .$$

5. Non-abelian extensions

Throughout this section, G and A will be groupoids such that $G_0 = A_0$ and A is totally disconnected (i.e. A is a family $A(p)$, $p \in A_0$, of groups). An *extension* of A by G is a pair

$$A \xrightarrow{i} E \xrightarrow{p} G$$

of morphisms of groupoids satisfying the following properties.

(5.1) $E_0 = G_0$ and i and p are the identity on objects.

(5.2) p is a quotient morphism of groupoids.

(5.3) i maps A isomorphically onto $\operatorname{Ker} p$.

(That p is a quotient morphism means that p induces an isomorphism $E/\operatorname{Ker} p \to G$; for more details see [15].) For such an extension, conjugation in E induces an action of E on A making A a crossed E-module. This can be extended trivially to a crossed complex $\underline{E} : \ldots \to 0 \to 0 \to A \to E$ which is (with the quotient morphism p) a crossed resolution of G.

A *free equivalence* of such extensions of A by G is a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{i} & E & \xrightarrow{p} & G \\ \zeta\downarrow & & \downarrow\eta & & \downarrow = \\ A & \xrightarrow{i'} & E' & \xrightarrow{p'} & G \end{array} \qquad (*)$$

such that ζ is an isomorphism; this implies that η also is an isomorphism. Such a free equivalence is an *equivalence* if ζ is the identity. We can thus form two (large) groupoids

$$\underline{\mathrm{Ext}}_f(G, A) \quad \text{and} \quad \underline{\mathrm{Ext}}(G, A) ,$$

both having objects the extensions of A by G, but having arrows respectively the free equivalences and the equivalences of extensions.

For any groupoid A there is a groupoid $\operatorname{Act} A$ of actions on the vertex groups of A. Here $\operatorname{Act} A$ has the same objects as A, and an arrow in $\operatorname{Act} A$ from p to q is an isomorphism $A(p) \to A(q)$ of groups. There is a conjugation map $\partial : A \to \operatorname{Act} A$. Under our assumption that A is totally disconnected, this map and the action of $\operatorname{Act} A$ on A determine a crossed complex

$$\longrightarrow 0 \longrightarrow \ldots \longrightarrow 0 \longrightarrow A \xrightarrow{\partial} \operatorname{Act} A$$

which we write χA. If $A \xrightarrow{i} E \xrightarrow{p} G$ is an extension with associated crossed resolution $\underline{E}$ of G, then the action of E on A by conjugation induces a morphism $(\sigma, 1) : \underline{E} \to \chi A$ (where $\sigma : E \to \operatorname{Act} A$). A free equivalence as in (*) induces an isomorphism $(\xi, \zeta) : \chi A \to \chi A$ where $\xi : \operatorname{Act} A \to \operatorname{Act} A$ is given by $a^{\xi\beta} = \zeta((\zeta^{-1}a)^{\beta})$; further $\sigma'\eta = \xi\sigma : E \to \operatorname{Act} A$.

<u>Theorem 3.</u> *There are canonical equivalences of groupoids*

$$e_f : (CG, \chi A)_f \longrightarrow \underline{\mathrm{Ext}}_f(G, A) ,$$

$$e : (CG, \chi A) \longrightarrow \underline{\mathrm{Ext}}(G, A) .$$

Proof. The morphism e is the restriction of e_f. We give the proof only for e_f. Also, since this result is a reformulation of standard theory, we do not give full details. [Some of the calculations are given in [14], §15.1 for the group case, and for equivalence rather than free equivalence, but with differences in notation as follows : for Hall's H, N, G read our G, A, E; his factor set $(u,v) \in N$ becomes our morphism $k : C_2G \to A$; his automorphism $a \mapsto a^u$ of N for $u \in G$ becomes our morphism $h : C_1G \to \text{Act } A$; his choice $u \mapsto \bar{u}$ of coset representatives becomes our morphism $\ell : C_1G \to E$; his function $\alpha : H \to N$ becomes our derivation $\alpha : C_1G \to A$.]

A morphism $CG \to \chi A$ over $G_0 = A_0$ is determined by a pair of morphisms $h : C_1G \to \text{Act } A$, $k : C_2G \to A$ such that k is an operator morphism over h , and such that the equations

$$h\delta = \partial k \ , \ k\delta = 0$$

hold. (These equations are equivalent to the first two equations in Theorem 15.1.1 of [14], and indeed $k\delta = 0$ is, by Proposition 2, equivalent to the "factor set" condition

$$k[u + v,w] + k[u,v]^{h[w]} = k[u,v + w] + k[v,w] \ ,$$

for all $(u,v,w) \in N_3G^*$.) Given such a morphism $CG \to \chi A$, an extension E of A by G is defined by setting $E_0 = G_0$ and for $p, q \in G_0$, letting $E(p,q)$ be the set of pairs (u,a) such that $u \in G(p,q)$, $a \in A(q)$, with addition

$$(u,a) + (v,b) = (u + v, \ k[u,v] + a^{h[v]} + b) \ ,$$

for $v \in G(q,r)$, $b \in A(r)$. The verification that E is a groupoid is left to the reader (cf. p.220 of [14]). We write $E = e(h,k)$.

Suppose now given two morphisms $CG \to \chi A$ over G_0 , which we write as pairs (h,k) , (h',k') as above. Let $\theta : (h,k) \simeq (h',k')$ be a (free) homotopy, and write $\beta = \theta_0$, $\alpha = \theta_1$. Then α is a derivation over h' and if $u \in G(p,q)$, $v \in G(q,r)$, we have

$$h'[v] = -\beta q + h[v] + \beta r - \partial\alpha[v] \ ,$$
$$k'[u,v] = k[u,v]^{\beta r} - \alpha\delta[u,v] \ .$$

A straightforward calculation shows that

$$k'[u,v] + \alpha\delta[u,v] = -\alpha[u + v] + k'[u,v] + (\alpha[u])^{h'[v]} + \alpha[v]$$

(and this verifies that our definition of equivalence agrees with that on p.221 of [14]). Define

$$e_f(\theta) : e(h,k) \longrightarrow e(h',k')$$
$$(u,a) \longmapsto (u, \ \alpha[u] + a^{\beta q}) \ , \ u \in G(p,q) \ , \ a \in A(q) \ .$$

Then $e_f(\theta)$ is an isomorphism of groupoids which, with the automorphism $A \to A$ given by $a \mapsto a^{\beta q}$, $a \in A(q)$, defines a free equivalence of extensions. Conversely, any free equivalence

$$\begin{array}{ccccc} A & \longrightarrow & e(h,k) & \longrightarrow & G \\ \downarrow{\scriptstyle\zeta} & & \downarrow{\scriptstyle\eta} & & \downarrow{=} \\ A & \longrightarrow & e(h',k') & \longrightarrow & G \end{array}$$

arises in the above way if $\beta : G_0 \to \text{Act } A$ is defined by $\beta(q) = \zeta|A(q)$, and $\alpha : C_1G \to A$ is defined by extending to a derivation over h' the function $\alpha' : G \to A$ defined by $\eta(u,0) = (u,\alpha'u)$.

Finally, we show that any extension $A \xrightarrow{i} E \xrightarrow{p} G$ of A by G is equivalent to some $e(h,k)$. Let $\phi : C_1G \to G$ be the quotient morphism and consider the crossed complex $\underline{E}$ obtained by trivial extension of the crossed E-module A . Consider the diagram

$$\begin{array}{ccccccc} C_3G & \xrightarrow{\delta} & C_2G & \xrightarrow{\delta} & C_1G & \xrightarrow{\phi} & G \\ \downarrow & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle \ell} & & \downarrow{=} \\ 0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & G \\ & & \downarrow{=} & & \downarrow{\scriptstyle\sigma} & & \\ & & A & \xrightarrow{\partial} & \text{Act } A & & \end{array}$$

The crossed complex CG is free, while $\underline{E}$ is exact, and both have G as fundamental groupoid. So the identity on G has a lift $(\ell,k) : CG \to \underline{E}$. Let $h = \sigma\ell$. Then (h,k) is a morphism $CG \to \chi A$ over G_0 and an equivalence of extensions $e(h,k) \to E$ is defined by $(u,a) \mapsto \ell[u] + ia$. □

Thus the crossed complex approach is successful because some of the difficulties in non-abelian extension theory are so-to-speak compressed into the standard crossed resolution (a kind of universal example) and in particular into the formula for $\delta : C_3G \to C_2G$.

By standard homotopy arguments, we obtain from Theorem 3;

<u>Corollary 4.</u> *Let* C *be any free crossed resolution of the groupoid* G . *Then there are equivalences of groupoids*

$$e'_f : (C, \chi A)_f \longrightarrow \underline{\text{Ext}}_f(G, A),$$
$$e' : (C, \chi A) \longrightarrow \underline{\text{Ext}}(G, A) \ . \quad \square$$

<u>Corollary 5.</u> *Let* $N \xrightarrow{i} F \xrightarrow{p} G$ *be an extension of groupoids such that* F *is free. Let* $\underline{P}$ *denote the crossed resolution of* G *obtained by trivial extension of the crossed* F*-module* N . *Then there are equivalences of groupoids*

$$e''_f : (\underline{P}, \chi A)_f \longrightarrow \underline{\text{Ext}}_f(G, A) \ ,$$
$$e'' : (\underline{P}, \chi A) \longrightarrow \underline{\text{Ext}}(G, A) \ .$$

Proof. Let C be a free crossed resolution of G such that $C_1 = F$ and $N = \delta(C_2)$. Then the projection $C \to \underline{P}$ induces isomorphisms

$$(\underline{P}, \chi A)_f \to (C, \chi A)_f \ , \ (\underline{P}, \chi A) \to (C, \chi A) \ . \quad \square$$

An interesting special case of Corollary 5 is when A is centreless, i.e. when $\partial : A \to \text{Act } A$ is injective. Then a morphism $\underline{P} \to \chi A$ is determined by a morphism $h : F \to \text{Act } A$ of groupoids such that $h(r)$ is a conjugation of A for each r in a set normally generating N .

The above methods also enable one to give a crossed complex version of a generalisation of Dedecker's work on non-abelian cohomology and extensions [12]. Let G , A be as above and suppose given a crossed Π-module A (where Π is a groupoid with $\Pi_0 = G_0$) . A Π-*extension* of A by G is an extension $A \xrightarrow{i} E \xrightarrow{p} G$ as above together with a morphism of crossed modules

$$\begin{array}{ccc} A & \xrightarrow{i} & E \\ \| & & \downarrow \sigma \\ A & \xrightarrow{\partial} & \Pi \end{array}.$$

In fact if, by extending trivially, we regard these crossed modules as crossed complexes $\underline{E}$ and $\chi_\Pi A$ respectively, then the above diagram is a morphism $(\sigma, 1) : \underline{E} \to \chi_\Pi A$ of crossed complexes.

Define a *conjugation* $\chi_\Pi A \to \chi_\Pi A$ to be an isomorphism (ξ, ζ) for which there is a function β from Π_0 to the union of the $\Pi(q)$, $q \in \Pi_0$, such that $\xi(x) = -\beta p + x + \beta q$, $x \in \Pi(p,q)$ and $\zeta(a) = a^{\beta q}$, $a \in A(q)$. Define a *free equivalence* of Π-extensions to be an isomorphism $(\eta, \zeta) : \underline{E} \to \underline{E}'$ over the identity on G, and a conjugation $(\xi, \zeta) : \chi_\Pi A \to \chi_\Pi A$ such that $\sigma'\eta = \xi\sigma$. Such free equivalences form under composition a groupoid $\underline{\text{Ext}}^\Pi_f(G, A)$. The *(strict) equivalences* are those in which (ξ, ζ) is the identity.

We have the following generalisation of Theorem 3.

Theorem 6. *There are equivalences of groupoids*

$$e_f : (CG, \chi_\Pi A)_f \longrightarrow \underline{\text{Ext}}^\Pi_f(G, A),$$

$$e : (CG, \chi_\Pi A) \longrightarrow \underline{\text{Ext}}^\Pi(G, A). \quad \square$$

The proof is similar to that of Theorem 3. (Dedecker's result is the bijection induced by e on components when G, A, Π are groups.)

Remark 1. Huebschmann has proved related results for the group case. On p.309 of [20] he shows (for groups) that given a morphism $\underline{P} \to \chi_\Pi A$ (where $\underline{P}$ is as in Corollary 5), one can define an extension $A \to E \to G$ by taking E to be the coequaliser of two maps $A \rightrightarrows F \ltimes A$ (the semi direct-product). In a letter to Dedecker [19] he relates such morphisms $\underline{P} \to \chi_\Pi A$ to Dedecker's 2-cocycles.

Remark 2. A theory of extensions and cohomology of T-algebras is given in [23], using internal category objects for T-algebras. This generalises Dedecker's theory, and also includes the above equivalence e of groupoids. The results do not include extensions of groupoids, nor free equivalences, and crossed complexes are not used.

Remark 3. If X is a CW-complex, and C is a crossed complex, it seems reasonable to define the *cohomology of* X *with coefficients in* C simply as $[\pi\underline{X}, C]$. (A similar idea for chain complexes is developed in [2] and applied to Postnikov invariants of function spaces in [3].) It would be interesting to have applications in homotopy theory of such a non-abelian cohomology.

Remark 4. A *homotopy* f_t of filtered maps $f_0, f_1 : \underline{X} \to \underline{Z}$ is a homotopy $f_t : X \to Z$ such that $f_t(X_n) \subset Z_{n+1}$ for $n \geq 0$. We will prove elsewhere that if X_0 and Z_0 are discrete, then such a homotopy induces a homotopy $\pi f_0 \simeq \pi f_1$ of morphisms of crossed complexes. Consequently, the non-abelian cohomology suggested in Remark 3 above for CW-complexes, is a homotopy invariant.

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* *Note.* Reference [0] continues work of [2].