Some Computations of Non-Abelian Tensor Products of Groups*

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INTRODUCTION

A generalised tensor product $G \otimes H$ of groups G, H has been introduced by R. Brown and J.-L. Loday in [3, 4]. It arises in applications in homotopy theory of a generalised Van Kampen theorem. The reason why $G \otimes H$ does not necessarily reduce to $G^{ab} \otimes_{\mathbb{Z}} H^{ab}$, the usual tensor product over $\mathbb{Z}$ of the abelianisations, is that it is assumed that G acts on H (on the left) and H acts on G (on the left), and these actions are taken into account in the definition of the tensor product.

A group G acts on itself by conjugation ($^h g = hgh^{-1}$) and so the tensor square $G \otimes G$ is always defined. Further, the commutator map $G \times G \to G$ induces a homomorphism of groups κ: $G \otimes G \to G$, sending $g \otimes h$ to $[g, h] = ghg^{-1}h^{-1}$. We write $J_2(G)$ for Ker κ; its topological interest is the formula [3, 4]

$$\pi_3 SK(G, 1) = J_2(G).$$

Results in [3, 4] give a commutative diagram with exact rows and central extensions as columns:

$$
\begin{array}{ccc}
& 0 & 0 \\
& \downarrow & \downarrow \\
H_3(G) \longrightarrow \Gamma(G^{ab}) \overset{\psi}{\longrightarrow} J_2(G) \longrightarrow H_2(G) \longrightarrow 0 \\
=\downarrow \qquad =\downarrow \qquad \downarrow \qquad \downarrow \\
H_3(G) \longrightarrow \Gamma(G^{ab}) \overset{\psi}{\longrightarrow} G \otimes G \longrightarrow G \wedge G \longrightarrow 1 \\
\kappa\downarrow \qquad \kappa'\downarrow \\
G' \overset{\cdot}{\longrightarrow} G' \\
\downarrow \qquad \downarrow \\
1 \qquad 1
\end{array}
\tag{1}
$$

Here, G' is the derived group of G, Γ is Whitehead's quadratic functor [15], and $G \wedge G$ is a generalised exterior product (Γ and the map ψ are defined in Section 2).

These relations with well-known constructions suggest the interest in the computation of $G \otimes G$, and this is our chief aim. Note that diagram (1) implies that $G \otimes G$ is finite if G is finite, so that explicit answers can be expected.

1. THE CONSTRUCTION

Let G and H be groups which act on themselves by conjugation, ${}^g g' = gg'g^{-1}$, and each of which acts upon the other in such a way that the following compatibility conditions hold:

$$
{}^{(gh)}g' = {}^{ghg^{-1}}g', \qquad {}^{(hg)}h' = {}^{hgh^{-1}}h'
\tag{2}
$$

for all g, $g' \in G$ and h, $h' \in H$, where ghg^{-1}, hgh^{-1} are here interpreted as elements of the free product $G * H$. Then the *tensor product* $G \otimes H$ is the group generated by the symbols $g \otimes h$ and defined by the relations

$$
gg' \otimes h = ({}^g g' \otimes {}^g h)(g \otimes h),
\tag{3}
$$

$$
g \otimes hh' = (g \otimes h)({}^h g \otimes {}^h h'),
\tag{4}
$$

for all g, $g' \in G$ and h, $h' \in H$.

Remark 1. This definition differs from that given in [3] at two points: in [3], condition (2) is not assumed at the outset, and the right-hand side of (3) reads $^g(g' \otimes h)(g \otimes h)$ (and similarly for (4)), where

$$^p(g \otimes h) = {}^pg \otimes {}^ph \tag{5}$$

for $g \in G$, $h \in H$, and $p \in G*H$. As pointed out by Higgins, this involves an ambiguity: (5) will define a genuine action only if it maps (3) and (4) into relations holding in $G \otimes G$. Condition (2) guarantees this, and so will be assumed throughout.

Remark 2. When G and H act trivially on each other (but by conjugation on themselves, as we always assume), $G \otimes H$ is the ordinary tensor product. This can be shown using (9) and (10) (see Proposition 3 below), or by direct calculation.

Remark 3. Let L be a group. A function $\phi: G \times H \to L$ is called a *crossed pairing* if for all g, $g' \in G$, h, $h' \in H$,

$$\phi(gg', h) = \phi({}^gg', {}^gh) \, \phi(g,h),$$

$$\phi(g, hh') = \phi(g, h) \, \phi({}^hg, {}^hh').$$

A crossed pairing ϕ determines a unique homomorphism of groups ϕ^*: $G \otimes H \to L$ such that $\phi^*(g \otimes h) = \phi(g, h)$ for all $g \in G$, $h \in H$. This fact is used frequently, for example, in the proof of the following proposition, details of which are left to the reader.

PROPOSITION 1. (i) *The groups G and H act on $G \otimes H$ so that*

$$^g(g' \otimes h) = {}^gg' \otimes {}^gh, \qquad ^h(g \otimes h') = {}^hg \otimes {}^hh'$$

for all g, $g' \in G$, h, $h' \in H$. Hence an action of $G*H$ on $G \otimes H$ is obtained.

(ii) *Suppose $\theta: G \to A$, $\phi: H \to B$ are homomorphisms of groups, A, B act compatibly on each other, and θ, ϕ preserve the actions in the sense that*

$$\phi({}^gh) = {}^{\theta g}(\phi h), \qquad \theta({}^hg) = {}^{\phi h}(\theta g)$$

for all $g \in G$, $h \in H$. Then there is a unique homomorphism

$$\theta \otimes \phi: G \otimes H \to A \otimes B$$

such that $(\theta \otimes \phi)(g \otimes h) = \theta g \otimes \phi h$ for all $g \in G$, $h \in H$. Further, if θ, ϕ are onto, so also is $\theta \otimes \phi$.

(iii)   *There is a unique isomorphism*

$$\tau : G \otimes H \to H \otimes G \qquad (6)$$

such that $\tau(g \otimes h) = (h \otimes g)^{-1}$, *for all* $g \in G$, $h \in H$.

In the next proposition, (i) and (ii) are of a familiar type, while (iii) is of a novel kind and is important in calculations.

PROPOSITION 2   [4].   (i)   *There   are   homomorphisms   of   groups* $\lambda : G \otimes H \to G$, $\lambda' : G \otimes H \to H$ *such that* $\lambda(g \otimes h) = g \, {}^h g^{-1}$, $\lambda'(g \otimes h) = {}^g h \, h^{-1}$.

(ii)   *The crossed module rules hold for* λ *and* λ', *that is,*

$$(\text{CM1})   \quad \lambda({}^g t) = g(\lambda(t)) \, g^{-1},$$

$$(\text{CM2})   \quad t t_1 t^{-1} = {}^{\lambda(t)} t_1,$$

for all t, $t_1 \in G \otimes H$, $g \in G$ (*and similarly for* λ').

(iii) $\lambda(t) \otimes h = t \, {}^h t^{-1}$, $g \otimes \lambda'(t) = {}^g t \, t^{-1}$, *and thus* $\lambda(t) \otimes \lambda'(t_1) = [t, t_1]$ *for all* t, $t_1 \in G \otimes H$, $g \in G$, $h \in H$. *Hence,* G *acts trivially on* Ker λ' *and* H *acts trivially on* Ker λ.

It is essentially shown in [4] that these rules are consequences of the following special cases, which are themselves direct consequences of (2)–(5) (cf. [4]).

PROPOSITION 3.   *The following relations hold for all* g, $g' \in G$ *and* h, $h' \in H$:

$$^g(g^{-1} \otimes h) = (g \otimes h)^{-1} = {}^h(g \otimes h^{-1}), \qquad (7)$$

$$(g \otimes h)(g' \otimes h')(g \otimes h)^{-1} = {}^{[g,h]}(g' \otimes h'), \qquad (8)$$

$$(g \, {}^h g^{-1}) \otimes h' = (g \otimes h) \, {}^{h'}(g \otimes h)^{-1}, \qquad (9)$$

$$g' \otimes ({}^g h \, h^{-1}) = {}^{g'}(g \otimes h)(g \otimes h)^{-1}, \qquad (10)$$

$$[g \otimes h, g' \otimes h'] = (g \, {}^h g^{-1}) \otimes ({}^{g'} h' \, h'^{-1}). \qquad (11)$$

2. BASIC RESULTS

We now restrict attention to the tensor square $G \otimes G$, and begin by giving for the kernel $J_2(G)$ of the commutator map κ in the diagram (1) two crucial consequences of Proposition 2(ii), (iii).

PROPOSITION 4 [4]. (i) $J_2(G)$ *is a central subgroup of* $G \otimes G$.

(ii) *The elements of* $J_2(G)$ *are fixed under the action of* G.

Another main ingredient in (1) is Whitehead's universal quadratic functor Γ [15], whose properties we now briefly describe.

Given an abelian group A, ΓA is the abelian group with generators γa, $a \in A$, and defining relations

$$\gamma(a^{-1}) = \gamma a,$$
$$\gamma(abc)\, \gamma a\, \gamma b\, \gamma c = \gamma(ab)\, \gamma(bc)\, \gamma(ca), \tag{12}$$

for all a, b, $c \in A$. The following properties are not hard to check, and allow computation of Γ in the finitely generated case [15]:

(a) $\Gamma(A \times B) \cong \Gamma A \times \Gamma B \times (A \otimes B)$

(b) $\Gamma \mathbb{Z}_n \cong \begin{cases} \mathbb{Z}_n, & n \text{ odd} \\ \mathbb{Z}_{2n}, & n \text{ even,} \end{cases}$ \hfill (13)

where $\mathbb{Z}_n = \langle x \,|\, x^n = e \rangle$ for $n \geq 0$ (so that $\mathbb{Z}_0 = \mathbb{Z}$ is the infinite cyclic group).

It is an exercise in the use of Propositions 2 and 3 to check that for every group G there is a well-defined homomorphism

$$\psi : \Gamma(G^{ab}) \to G \otimes G \tag{14}$$

such that $\psi(\gamma g G') = g \otimes g$ [4]. Clearly Im ψ is contained in $J_2(G)$, and is therefore central in $G \otimes G$. The cokernel of ψ is written $G \wedge G$, and is called the *exterior square* of G. The commutator map $\kappa\colon G \otimes G \to G'$ clearly factors through $\kappa'\colon G \wedge G \to G'$; the kernel of κ' is isomorphic to the Schur multiplicator $H_2(G)$ [14, 4, 8]. The remaining parts of the exact rows in (1) derive from Whitehead's Γ-sequence for $SK(G, 1)$ and the generalised Van Kampen theorem of [4].

We state two immediate consequences (see [4]). First, let G be a finite group. Then both $H_2(G)$ (see [13]) and $\Gamma(G^{ab})$ (see (13)) are finite. Hence, $J_2(G)$ is finite, and so is $G \otimes G$. (See also [9].) Similarly, if G is a finite p-group for some prime p, then so is $G \otimes G$.

PROPOSITION 5. *If* G *is a finite group, then so is* $G \otimes G$. *If, in addition,* G *is a* p-*group for some prime* p, *then so is* $G \otimes G$.

Second, let G be a free group, so that $H_2(G)$ and $H_3(G)$ are trivial (see [13]). By (1), ψ is one-to-one. Since G' is free (Nielsen–Schreier), the penultimate column of (1) splits and (using Proposition 4(i)) we have the following result [4].

PROPOSITION 6. *If G is a free group, then*

$$G \otimes G \cong G' \times \Gamma(G^{ab}).$$

In particular, if G is free of finite rank $n \geqslant 2$, G' is then free of countably infinite rank and $\Gamma(G^{ab})$ is free abelian of rank $n(n+1)/2$.

When G is abelian, the value of $G \otimes G$ is given by Remark 2 above. We conclude this section by dealing with the opposite extreme, namely, the case when G is perfect, that is, $G = G'$. The key to this is the following result of [4].

PROPOSITION 7. *Let G be any group and let*

$$1 \longrightarrow A \overset{\iota}{\longrightarrow} K \overset{\pi}{\longrightarrow} G \longrightarrow 1$$

be a central extension. Then there is a homomorphism $\xi: G \otimes G \to K$ such that $\pi\xi$ is the commutator map κ. If G is perfect, then ξ is unique.

Proof. Given $g_i \in G$, pick $k_i \in K$ such that $\pi(k_i) = g_i$ for $i = 1, 2$, and define a map $(g_1, g_2) \mapsto [k_1, k_2]$. This is independent of the choice of k_1, k_2 (since the extension is central) and is a crossed pairing. It therefore induces a homomorphism $\xi: G \otimes G \to K$, which obviously satisfies $\pi\xi = \kappa$. If two homomorphisms ξ, $\xi': G \otimes G \to K$ satisfy $\pi\xi = \pi\xi'$, then $\xi(\xi')^{-1} = \iota\eta$, where $\eta: G \otimes G \to A$ is a homomorphism which factors through the projection $G \otimes G \to G^{ab} \otimes G^{ab}$. The uniqueness of ξ for perfect G follows. ∎

The definition of a covering group $\hat{G}$ of a group G is well known if G is finite [12, Chap. V, Sect. 23]. We adopt a similar definition in the general case. So a *covering group* $\hat{G}$ of a group G is a central extension

$$1 \longrightarrow H_2(G) \overset{\iota}{\longrightarrow} \hat{G} \longrightarrow G \longrightarrow 1,$$

where $H_2(G)$ is the Schur multiplicator and $\operatorname{Im} \iota \subseteq \hat{G}'$. Note that, while $\hat{G}$ is not uniquely determined by G, the commutator subgroup $\hat{G}'$ is so determined. It follows that when G is perfect, $\hat{G}$ is unique (and also perfect). We now list some consequences of Proposition 7, of which the first is in [6, 3, 4].

COROLLARY 1. *When G is perfect, $G \otimes G$ is the (unique) covering group $\hat{G}$ of G.*

Proof. When $G = G'$, G^{ab} is trivial and so is $\operatorname{Im} \psi$. Hence, from diagram (1), $G \otimes G$ is a central extension of G by $H_2(G)$, and it is sufficient to prove that $G \otimes G$ is a perfect group (for then $H_2(G) \subseteq (G \otimes G)'$ automatically). Now, $G \otimes G$ is generated by elements $g \otimes g'$, where g and g' are products of commutators. It follows from (3) and (4) that $G \otimes G$ is generated by the $g \otimes g'$ with g and g' simple commutators. But it follows from (11) that such

an element is a commutator in $G \otimes G$. Hence, $G \otimes G$ is perfect, and thus is isomorphic to $\hat{G}$. ∎

COROLLARY 2. *If $\hat{G}$ is a covering group of G, then there is a map $\eta: G \wedge G \to \hat{G}'$, which is an isomorphism if $H_2(G)$ is finitely generated.*

Proof. A map $\xi: G \otimes G \to \hat{G}$ is given by Proposition 7. Clearly $\xi(g \otimes g) = 1$ for all $g \in G$. So we have a homomorphism $\eta: G \wedge G \to \hat{G}'$ inducing a homomorphism $\eta': H_2(G) \to H_2(G)$. The condition $\mathrm{Im}\, \iota \subseteq \hat{G}'$ implies that η' is surjective. Since $H_2(G)$ is finitely generated, it follows that η' is an isomorphism. The five-lemma implies that η is an isomorphism. ∎

Remark. It would seem reasonable in the general case to define a covering group as above but with the additional requirement that the map η of Corollary 2 is an isomorphism.

The following result was suggested by the computational results of Section 6.

PROPOSITION 8. *If G is a group in which G' has a cyclic complement C, then $G \otimes G \cong (G \wedge G) \times C$.*

Proof. Let $C = \langle x \rangle$; by assumption, the projection $G \to G^{ab}$ maps C isomorphically to G^{ab}, so we can write $C = G^{ab}$. The exact sequence

$$\Gamma(C) \overset{\psi}{\longrightarrow} G \otimes G \overset{p}{\longrightarrow} G \wedge G \longrightarrow 1$$

shows that $\mathrm{Ker}\, p$ is generated by $x \otimes x$. So the canonical map $\xi: G \otimes G \to C \otimes C \cong C$ maps $\mathrm{Ker}\, p$ onto $C \otimes C$. If x has order n, then $x \otimes x$ has order at most n, as $(x \otimes x)^n = x \otimes x^n$. So ξ maps $\mathrm{Ker}\, p$ isomorphically to $C \otimes C$. Hence there is a retraction $G \otimes G \to \mathrm{Ker}\, p$, and so $G \otimes G$ is the direct product $(G \wedge G) \times C$. ∎

EXAMPLE. In the tables in Section 6 it is stated that $A_4 \otimes A_4 \cong \mathbb{Z}_3 \times Q_2$. This can be obtained from the previous two results, where the $\mathbb{Z}_3$ factor is generated by $a \otimes a$ (using the notation of Table I) while the Q_2 factor has generators $a \otimes b$ and $a \otimes a^{-1}ba$ corresponding to the generators ab and ba for $\hat{A}_4'$, where $\hat{A}_4 = \langle a, b \mid a^3 = b^3 = (ab)^2 \rangle$.

3. FUNCTORIAL PROPERTIES

It is clear that any epimorphism $\pi: K \to G$ induces an epimorphism

$$\pi \otimes \pi: K \otimes K \to G \otimes G$$

$$k_1 \otimes k_2 \mapsto \pi(k_1) \otimes \pi(k_2)$$

(see Proposition 1(ii)). In the favourable case when $\operatorname{Ker} \pi \leqslant Z(K)$, we can say something about the kernel of $\pi \otimes \pi$. The following result, which is a special case of 3.1.3 of [7] and is related to Proposition 1 of [8], was also found by J.-L. Loday (cf. [1]).

PROPOSITION 9. *Given a central extension*

$$1 \longrightarrow A \xrightarrow{\text{inc}} K \xrightarrow{\pi} G \longrightarrow 1$$

there is an exact sequence

$$(A \otimes K) \times (K \otimes A) \xrightarrow{\iota} K \otimes K \xrightarrow{\pi \otimes \pi} G \otimes G \longrightarrow 1$$

in which $\operatorname{Im} \iota$ *is central.*

Proof. By Proposition 1(ii) and (8) there is a homomorphism ι which sends

$$(a_1 \otimes k_1, k_2 \otimes a_2) \mapsto (a_1 \otimes k_1)(k_2 \otimes a_2).$$

Putting $C = \operatorname{Im} \iota$, it is clear that $C \leqslant \operatorname{Ker}(\pi \otimes \pi)$ and that $\pi \otimes \pi$ is onto. Furthermore, since $A \leqslant Z(K)$, we have $C \leqslant J_2(K) \leqslant Z(K \otimes K)$, by Proposition 4(ii), and in particular C is normal in $K \otimes K$. It is thus sufficient to prove that the induced map

$$\pi' \colon (K \otimes K)/C \to G \otimes G$$

is an isomorphism, and this is done by constructing an inverse ρ as follows:

For each $g \in G$ choose $g' \in K$ such that $\pi(g') = g$. Let $v \colon K \otimes K \to (K \otimes K)/C$ denote the quotient map. Define $\rho' \colon G \times G \to (K \otimes K)/C$ by $\rho'(g, h) = v(g' \otimes h')$. By (3) and (4), ρ' is well-defined. We prove that ρ' is a crossed pairing. Let g, h, $k \in G$. Then

$$\rho'(^g h, {}^g k)\, \rho'(g, k) = v(^{g'} h' \otimes {}^{g'} k')\, v(g' \otimes k')$$

$$= v((^{g'} h' \otimes {}^{g'} k')(g' \otimes k'))$$

$$= v(g' h' \otimes k')$$

$$= \rho'(gh, k).$$

This verifies the first rule for a crossed pairing, and the other rule is proved similarly. The homomorphism $\rho \colon G \otimes G \to (K \otimes K)/C$ determined by ρ' clearly satisfies $\pi' \rho = 1_{G \otimes G}$. Also, if k_1, $k_2 \in K$, then

$$\rho\pi'((k_1 \otimes k_2) C) = \rho\pi'v(k_1 \otimes k_2)$$
$$= \rho(\pi(k_1) \otimes \pi(k_2))$$
$$= v(k_1 \otimes k_2)$$
$$= (k_1 \otimes k_2) C.$$

Hence, π' is an isomorphism, and the proof is complete. ∎

This result will be used in the next section to derive the value of $G \otimes G$ for dihedral groups from its value on quaternionic groups, following a suggestion of Loday.

Under certain favourable conditions, the non-abelian tensor product distributes over direct products. These conditions obtain for the tensor square of a direct product, and this reduces the work in cataloguing the values of $G \otimes G$ for groups of small order (see Section 6 below).

PROPOSITION 10. *Let A, B, C be groups, with given actions of A on B and C, and of B and C on A. Suppose that the latter actions*

(a) *commute: $^{bc}a = {}^{cb}a$, so that $B \times C$ acts on A,*

(b) *induce the trivial action of B on $A \otimes C$: $^b(a \otimes c) = a \otimes c$, and*

(c) *induce the trivial action of C on $A \otimes B$: $^c(a \otimes b) = a \otimes b$,*

for all $a \in A$, $b \in B$, $c \in C$. Then

$$A \otimes (B \times C) \cong (A \otimes B) \times (A \otimes C).$$

Proof. Define

$$\alpha: A \otimes (B \times C) \to (A \otimes B) \times (A \otimes C)$$
$$a \otimes (b, c) \mapsto (a \otimes b, a \otimes c),$$

and check that (3) and (4) are preserved. For (3)

$$\alpha(aa' \otimes (b, c)) = (aa' \otimes b, aa' \otimes c), \qquad \text{by definition,}$$
$$= (({}^a a' \otimes {}^a b)(a \otimes b), ({}^a a' \otimes {}^a c)(a \otimes c))$$
$$= ({}^a a' \otimes {}^a b, {}^a a' \otimes {}^a c)(a \otimes b, a \otimes c)$$
$$= \alpha({}^a(a' \otimes (b, c))) \alpha(a \otimes (b, c)).$$

For (4)

$$\alpha(a \otimes ((b, c)(b', c'))) = \alpha(a \otimes (bb', cc'))$$
$$= (a \otimes bb', a \otimes cc')$$
$$= ((a \otimes b)({}^b a \otimes {}^b b'), (a \otimes c)({}^c a \otimes {}^c c'))$$
$$= (a \otimes b, a \otimes c)({}^b(a \otimes b'), {}^c(a \otimes c')).$$

The first factor is $\alpha(a \otimes (b, c))$, while the second should be

$$\alpha(^{(b,c)}(a \otimes (b', c'))) = {}^{(b,c)}(a \otimes b', a \otimes c'),$$

which it is, using (b) and (c).

The inverse map must be

$$\beta: (A \otimes B) \times (A \otimes C) \to A \otimes (B \times C)$$

$$(a \otimes b, e) \mapsto a \otimes (b, e)$$

$$(e, a \otimes c) \mapsto a \otimes (e, c),$$

and it must be checked that

   (i)   relations (3) and (4) are preserved,
   (ii)   the images under β of $a \otimes b$ and $a \otimes c$ commute,
   (iii) $\alpha\beta$ and $\beta\alpha$ are identity maps.

First, (ii) follows by applying (4) to

$$a \otimes (b, c) = a \otimes ((b, e)(e, c)) = a \otimes ((e, c)(b,e))$$

and using (b) and (c). This also shows that $\beta\alpha = 1$, while $\alpha\beta = 1$ is obvious. Finally, (3) carries over automatically, while for (4),

$$\beta(a \otimes bb', e) = a \otimes (bb', e) = a \otimes (b, e)(b', e)$$

$$= (a \otimes (b, e))\, {}^{b}(a \otimes (b', e)),$$

whereas

$$\beta((a \otimes b)\, {}^{b}(a \otimes b'), e) = (a \otimes (b, e))({}^{b}a, ({}^{b}b', e)),$$

as required, and similarly for $(e, a \otimes cc')$. ∎

The distributive law for tensor squares is a straightforward consequence of this; in it, groups G and H are understood to act trivially upon each other and on themselves by conjugation, so that the cross terms on the right-hand side are ordinary tensor products.

PROPOSITION 11.

$$(G \times H) \otimes (G \times H) = (G \otimes G) \times (G \otimes H) \times (H \otimes G) \times (H \otimes H).$$

Proof. This will follow from the previous result (together with Proposition 1(iii)) provided we can show that conditions (b) and (c) hold (condition a) is automatic), that is, G acts trivially on $(G \times H) \otimes H$, $G \otimes H$,

$H \otimes H$, and H acts trivially on $(G \times H) \otimes G$, $H \otimes G$, $G \otimes G$. Now G fixes $H \otimes H$ (since it fixes H) and also $G \otimes H$ (since this is an ordinary tensor product). Hence G also fixes their direct product, which is G-isomorphic to $(G \times H) \otimes H$. ∎

4. Dihedral and Quaternionic Groups

Let Q_m be the quaternionic group of order $4m$ with presentation

$$\langle x, y \mid y^m = x^2, xyx^{-1} = y^{-1} \rangle, \tag{15}$$

and let D_m be the dihedral group of order $2m$ with presentation

$$\langle x, y \mid y^m = x^2 = e, xyx^{-1} = y^{-1} \rangle. \tag{16}$$

We compute $Q_m \otimes Q_m$ directly, then use Proposition 9 to find $D_m \otimes D_m$. Immediate consequences of the defining relations in (15) are

$$xy = y^{-1}x, \qquad yx = xy^{-1}, \qquad yxy^{-1} = y^2x,$$
$$[x,y] = y^{-2} = [y, x]^{-1}. \tag{17}$$

We often use these, the formulae in Proposition 3, and the crucial fact that $J_2(Q_m) = \operatorname{Ker} \kappa$ is central and Q_m-trivial (by Proposition 4) without explicit reference. Now we work in $Q_m \otimes Q_m$.

(4.1)

$$
\begin{aligned}
y \otimes y^2 - y \otimes [y, x] \\
&= {}^y(y \otimes x)(y \otimes x)^{-1} \\
&= (y \otimes y^2x)(y \otimes x)^{-1} \\
&= (y \otimes xy^{-2})(y \otimes x)^{-1} \\
&= (y \otimes x)\,{}^x(y \otimes y^{-2})(y \otimes x)^{-1} \\
&= y \otimes y^{-2},
\end{aligned}
$$

since this is central and Q_m-trivial. Hence, using (7),

$$(y \otimes y)^2 = y \otimes y^2 = y \otimes y^{-2} = (y \otimes y)^{-2},$$

so that

$$(y \otimes y)^4 = e.$$

(4.2)   Since $(y \otimes x)(x \otimes y) \in J_2(Q_m)$, it is central, and so $x \otimes y$ and $y \otimes x$ commute. Note also that $x \otimes x$ and $y \otimes y$ are central. As we will see, these four elements generate $Q_m \otimes Q_m$, which is thus abelian.

(4.3)   For $q \geqslant 1$,

$$
\begin{aligned}
x \otimes y^q &= (x \otimes y)^{\,y}(x \otimes y^{q-1}) \\
&= (x \otimes y)(xy^{-2} \otimes y^{q-1}) \\
&= (x \otimes y)^{\,x}(y^{-2} \otimes y^{q-1})(x \otimes y^{q-1}) \\
&= (x \otimes y)(x \otimes y^{q-1})(y \otimes y)^{-2(q-1)} \\
&= (x \otimes y)^q (y \otimes y)^{-q(q-1)} \\
&= (x \otimes y)^q (y \otimes y)^{q(q-1)}, \qquad by \ (4.1).
\end{aligned}
$$

(4.4)   By (4.3) with $q = 2$,

$$
\begin{aligned}
(x \otimes y)^2(y \otimes y)^2 &= x \otimes y^2 \\
&= x \otimes [y, x] \\
&= {}^{x}(y \otimes x)(y \otimes x)^{-1} \\
&= (y^{-1} \otimes x)(y \otimes x)^{-1} \\
&= {}^{y^{-1}}(y \otimes x)^{-1}(y \otimes x)^{-1}, \qquad by \ (7), \\
&= (y \otimes xy^2)^{-1}(y \otimes x)^{-1} \\
&= ((y \otimes x)^{\,x}(y \otimes y^2))^{-1}(y \otimes x)^{-1} \\
&= (y \otimes y)^2(y \otimes x)^{-2}.
\end{aligned}
$$

Hence,

$$
(x \otimes y)^2 = (y \otimes x)^{-2}.
$$

(4.5)

$$
\begin{aligned}
(y \otimes y)^m &= y \otimes y^m \\
&= y \otimes x^2 \\
&= (y \otimes x)^{\,x}(y \otimes x) \\
&= (y \otimes x)(y^{-1} \otimes x) \\
&= (y \otimes x)(y \otimes x)^{-1}(y \otimes y)^2, \qquad as \ in \ (4.4).
\end{aligned}
$$

Hence,

$$(y \otimes y)^{m-2} = e.$$

By (4.1), this means that $y \otimes y = e$ when m is odd, and $(y \otimes y)^2 = e$ when m is divisible by 4.

(4.6) Applying the automorphism τ of (6) to (4.3),

$$y^p \otimes x = (y \otimes x)^p \, (y \otimes y)^{p(p-1)},$$

for $p \geqslant 1$.

(4.7)

$$xy^p \otimes y^q = {}^x(y^p \otimes y^q)(x \otimes y^q)$$

$$= (x \otimes y)^q \, (y \otimes y)^{pq+q(q-1)}.$$

(4.8)   Applying τ to (4.7),

$$y^p \otimes xy^q = (y \otimes x)^p (y \otimes y)^{pq+p(q-1)}.$$

(4.9)

$$xy^p \otimes xy^q = (xy^p \otimes x) \, {}^x(xy^p \otimes y^q)$$

$$= {}^x(y^p \otimes x)(x \otimes x) \, {}^{x^2}(y^p \otimes y^q) \, {}^x(x \otimes y^q)$$

$$= {}^{x^{-1}}(y^p \otimes x)(x \otimes x)(y^p \otimes y^q) \, {}^{x^{-1}}(x \otimes y^q)$$

$$= (y^p \otimes x^{-1})^{-1}(x \otimes x)(y \otimes y)^{pq}(x^{-1} \otimes y^q)^{-1}.$$

But

$$y^p \otimes x^{-1} = y^p \otimes xy^m$$

$$= (y^p \otimes x) \, {}^x(y^p \otimes y^m)$$

$$= (y \otimes x)^p \, (y \otimes y)^{p(p-1)}(y \otimes y)^{pm}, \quad by \ (4.6),$$

$$= (y \otimes x)^p(y \otimes y)^{p(p-1)}, \qquad\qquad by \ (4.5).$$

A similar formula holds for $x^{-1} \otimes y^q$ (using τ), and so $xy^p \otimes xy^q = (y \otimes x)^{-p}(x \otimes x)(x \otimes y)^{-q}(y \otimes y)^{p(p+1)+q(q+1)+pq}$.

This establishes our claim that $Q_m \otimes Q_m$ is generated by $x \otimes x$, $x \otimes y$, $y \otimes x$, and $y \otimes y$, and is thus abelian.

(4.10)

$$x \otimes x^2 = x \otimes y^m$$

$$= (x \otimes y)^m \, (y \otimes y)^{m(m-1)}.$$

So, if m is odd or a multiple of 4,

$$(x \otimes x)^2 = (x \otimes y)^m.$$

(4.11)

$$\begin{aligned}(x \otimes x)^2 &= x^2 \otimes x \\ &= y^m \otimes x \\ &= (y \otimes x)^m (y \otimes y)^{m(m-1)}.\end{aligned}$$

So, in every case

$$(x \otimes y)^m = (y \otimes x)^m,$$

and

$$(x \otimes y)^m (y \otimes x)^m = (y \otimes x)^{2m} = e,$$

by (4.6), with $p = 2m$, and (4.5). By (4.2) and (4.4), it follows that $(x \otimes y)(y \otimes x) = e$ when m is odd.

(4.12)   For any m, the relations (4.7), (4.8), and (4.9) remain valid when p and/or q is negative. This is seen by replacing p (say) by $2m - p$ and using the fact that $(y \otimes y)^{2m} = e = (x \otimes y)^{2m}$.

PROPOSITION 12.   *Let m be odd, and let $\mathbb{Z}_4 \times \mathbb{Z}_m$ have factors generated by a, b, respectively. Then the mapping*

$$\begin{aligned}\theta \colon \mathbb{Z}_4 \times \mathbb{Z}_m &\to Q_m \otimes Q_m \\ a &\mapsto x \otimes x \\ b &\mapsto (x \otimes y)(x \otimes x)^2\end{aligned}$$

is an isomorphism. Also, $J_2(Q_m)$ is isomorphic to $\mathbb{Z}_4$ generated by $x \otimes x$.

Proof.   That θ is a well-defined epimorphism follows from (4.9), (4.10), and (4.11). To complete the proof, we must construct the inverse θ' to θ. According to (4.5), (4.7), (4.8), (4.9), θ' should have the following effect:

$$\begin{aligned}y^p \otimes y^q &\mapsto e, \\ xy^p \otimes y^q &\mapsto a^{2q}b^q, \\ y^p \otimes xy^q &\mapsto a^{2p}b^{-p}, \\ xy^p \otimes xy^q &\mapsto a^{1+2(p+q)}b^{p-q},\end{aligned}$$

that is, for $\delta, \varepsilon \in \{0, 1\}$, and p, q arbitrary (by 4.12),

$$x^\delta y^p \otimes x^\varepsilon y^q \mapsto a^{\delta\varepsilon + 2\varepsilon p + 2\delta q} b^{(-1)^\varepsilon \delta q - (-1)^\delta \varepsilon q}. \qquad (18)$$

It remains to check that this preserves the defining relations (3) and (4) of $Q_m \otimes Q_m$. By Proposition 1(iii), it is enough to check (3), since, if $u \otimes v$ maps to $a^i b^j$ under (18), then $v \otimes u$ maps to $a^i b^{-j}$. To do this, take

$$g = x^\delta y^p, \qquad g' = x^\varepsilon y^q, \qquad h = x^\eta y^r,$$

with $\delta, \varepsilon, \eta \in \{0, 1\}$ and $0 \leqslant p, q, r \leqslant 2m - 1$, and use (17) to reduce $gg' \otimes h$ and $^g(g' \otimes h)$ to standard form. It is then routine to check that the application of (18) to both sides yields the same result. ∎

In the case when m is even, the same method leads to the following result; the proof is omitted.

PROPOSITION 13. *Let* $m = 4r + k$, *where* $k = 0$ *or* 2. *Then*

$$Q_m \otimes Q_m = \mathbb{Z}_2 \times \mathbb{Z}_{2m} \times \mathbb{Z}_{2+k} \times \mathbb{Z}_2,$$

generated by $(x \otimes x)(x \otimes y)^{m/2}(y \otimes y)^{k/2}$, $x \otimes y$, $y \otimes y$, *and* $(x \otimes y)(y \otimes x)$, *respectively. Furthermore,* $J_2(Q_m)$ *is isomorphic to* $\mathbb{Z}_4 \times \mathbb{Z}_{k+2} \times \mathbb{Z}_2$, *generated by* $(x \otimes x)(x \otimes y)^m(y \otimes y)^{k/2}$, $y \otimes y$, *and* $(x \otimes y)(y \otimes x)$, *respectively.*

To calculate the tensor square for dihedral groups, put $z = x^2 = y^m \in Z(Q_m)$, so that $D_m = Q_m/\langle z \rangle$. It follows from Proposition 9 that $D_m \otimes D_m$ is just the factor group of $Q_m \otimes Q_m$ by the subgroup $N = \langle z \otimes Q_m, Q_m \otimes z \rangle$. Now $z \otimes y^p = (y \otimes y)^{mp}$, $z \otimes xy^p = (z \otimes x)^x(z \otimes y^p)$ $= (x \otimes x)^2(y \otimes y)^{-mp}$, so that $N = \langle (x \otimes x)^2, (y \otimes y)^m \rangle$. This leads at once to the following result.

PROPOSITION 14.

$$D_m \otimes D_m = \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_m, & m \text{ odd}, \\ \mathbb{Z}_2 \times \mathbb{Z}_m \times \mathbb{Z}_2 \times \mathbb{Z}_2, & m \text{ even}. \end{cases}$$

where the factors are generated respectively by $x \otimes x$, $x \otimes y$ *and (for m even) by* $y \otimes y$ *and* $(x \otimes y)(y \otimes x)$.

Remark. This result also covers the case $m = 0$, since we interpret $\mathbb{Z}_0$ as $\mathbb{Z}$. However, this does not follow from the calculation for Q_m and has to be proved separately.

5. METACYCLIC GROUPS

Let G be the metacyclic group generated by x and y subject to the defining relations

$$y^n = e, \ x^m = e, \qquad xyx^{-1} = y^l, \tag{19}$$

where l, m, $n \in \mathbb{N}$, and assume that

$$l^m \equiv 1 \qquad (\text{mod } n) \tag{20}$$

to ensure that

$$G = \{ y^i x^j \,|\, 0 \leqslant i \leqslant n - 1, 0 \leqslant j \leqslant m - 1 \} \tag{21}$$

has order mn. Also, for ease of computation, we take the favourable special case when n is odd. The following immediate consequences of (19) will be used without explicit reference:

$$x^p y^q x^{-p} = y^{q l^p}, \qquad y^q x^p y^{-q} = y^{q(1 - l^p)} x^p, \tag{22}$$

for any $p, q \in \mathbb{N}$.

Before beginning the calculation, note that $G \otimes G$ is abelian, since G' is cyclic and $J_2(G) = \text{Ker } \kappa$ is central, by Proposition 4(i). Since $J_2(G)$ is also G-trivial (Proposition 4(ii)),

$$\text{both } x \text{ and } y \text{ fix } x \otimes x, \ y \otimes y, \ (x \otimes y)(y \otimes x). \tag{23}$$

Now it is clear from (3), (4), (7) that these last three elements, together with $x \otimes y$, generate $G \otimes G$ qua G-module; our first main objective is to show that they generate $G \otimes G$ *as a group*. As in Section 4, we proceed in a series of steps.

(5.1) $(y \otimes y)^{l-1} = e$.

Proof. First, it follows from (11) that

$$(y \otimes y)^{(l-1)^2} = y^{l-1} \otimes y^{l-1} = [x, y] \otimes [x, y]$$
$$= [x \otimes y, x \otimes y] = e.$$

Next, by (23)

$$y \otimes y = {}^x(y \otimes y) = y^l \otimes y^l = (y \otimes y)^{l^2}.$$

Finally, by bilinearity,

$$(y \otimes y)^n = y \otimes y^n = y \otimes e = e.$$

(5.1) now follows from the last three formulae, since n is odd. ∎

(5.2) $^y(x \otimes y) = (x \otimes y)$.

Proof.

$$^y(x \otimes y) = (^yx) \otimes y = (y^{1-l}x \otimes y) = {}^{y^{1-l}}(x \otimes y)(y^{1-l} \otimes y)$$
$$= {}^{y^{1-l}}(x \otimes y)(y \otimes y)^{1-l} = {}^{y^{1-l}}(x \otimes y),$$

by (5.1). Hence, $x \otimes y$ is fixed by y^l, and (5.2) follows since l is coprime to $|y| = n$. ∎

(5.3) $^x(x \otimes y) = (x \otimes y)^l$.

Proof.

$$^x(x \otimes y) = x \otimes (^xy) = x \otimes y^l$$
$$= (x \otimes y) \, ^y(x \otimes y^{l-1})$$
$$= \prod_{k=0}^{l-1} {}^{y^k}(x \otimes y)$$
$$= (x \otimes y)^l, \qquad \text{by (5.2)}.$$

It now follows that $G \otimes G$ is a 4-generator group, as claimed above. ∎

(5.4) $x^p \otimes y^q = (x \otimes y)^{q \cdot p}$, $y^q \otimes x^p = (y \otimes x)^{q \cdot p}$ *for* $p, q \in \mathbb{N}$, *where the exponent on the right is defined by*

$$p \cdot q = p(1 + l + \cdots + l^{q-1}).$$

Proof. Using (2), (3) $p - 1$, $q - 1$ times, respectively, we have

$$x^p \otimes y^q = {}^x(x^{p-1} \otimes y^q)(x \otimes y^q)$$
$$= {}^{x^{p-1}}(x \otimes y^q) \cdots {}^x(x \otimes y^q)(x \otimes y^q),$$

and

$$x \otimes y^q = (x \otimes y) \, ^y(x \otimes y^{q-1})$$
$$= (x \otimes y) \, ^y(x \otimes y) \cdots {}^{y^{q-1}}(x \otimes y)$$
$$= (x \otimes y)^q, \qquad \text{by (5.2)}.$$

The first equation now follows from (5.3), and the second by applying the automorphism τ of Proposition 1. ∎

(5.5) $y^p x^q \otimes y^r x^s = (x \otimes x)^{qs}(y \otimes y)^{pr}(x \otimes y)^{r \cdot q}(y \otimes x)^{p \cdot s}$.

Proof.

$$y^p x^q \otimes y^r x^s = {}^{y^p}(x^q \otimes y^r x^s)(y^p \otimes y^r x^s), \qquad \text{by (2)}$$

$$= {}^{y^p}((x^q \otimes y^r) {}^{y^r}(x^q \otimes x^s))(y^p \otimes y^r) {}^{y^r}(y^p \otimes x^s), \qquad \text{by (4)}$$

$$= (x^q \otimes y^r)(x^q \otimes x^s)(y^p \otimes y^r)(y^p \otimes x^s),$$

since y acts trivially on $G \otimes G$, and the result now follows by (5.4) and bilinearity. ∎

(5.6)   The right-hand side of (5.5) must be independent of the choices of $p, r \bmod n$, and of $q, s \bmod m$. Adding n to each of p, r and m to each of q, s in turn, while keeping the others fixed, we obtain relations equivalent to

$$(y \otimes y)^n = e = (x \otimes x)^m,$$

$$(y \otimes x)^n = e = (x \otimes y)^n,$$

$$(y \otimes x)^{1 + l + \cdots + l^{m-1}} = e = (x \otimes y)^{1 + l + \cdots + l^{m-1}}.$$

(5.7)   Subject only to the conditions ((5.1) and (23), (5.2))

(a) $(y \otimes y)^{l-1} = e,$

(b) y acts trivially on $G \otimes G$,

the original defining relations (3) and (4) pass (via (5.5)) to trivial relations between the four new generators. That y acts trivially on $y \otimes y$, $x \otimes y$, $y \otimes x$, and $x \otimes x$ means respectively (using (5.5)) nothing, (a), (a), and

$$x \otimes x = {}^y(x \otimes x) = {}^y x \otimes {}^y x$$

$$= (y^{1-l}x) \otimes (y^{1-l}x)$$

$$= (y \otimes y)^{(1-l)^2}(x \otimes x)(x \otimes y)^{1-l}(y \otimes x)^{1-l}.$$

That is, (a) and (b) are equivalent to

$$(y \otimes y)^{l-1} = e = ((x \otimes y)(y \otimes x))^{l-1}.$$

It follows that (5.6) and (5.7) constitute defining relations for $G \otimes G$, and we have proved the following result.

PROPOSITION 15.   *Let G be the metacyclic group*

$$\langle x, y \mid y^n = e = x^m, xyx^{-1} = y^l \rangle,$$

where $l^m \equiv 1 \pmod n$ and n is odd. Then $G \otimes G$ is the direct product of four cyclic groups with generators

$$x \otimes x, \qquad y \otimes y, \qquad (x \otimes y)(y \otimes x), \qquad (x \otimes y)$$

of orders

$$m, (n, l-1), (n, l-1, 1 + l \cdots + l^{m-1}), (n, 1 + l + \cdots + l^{m-1})$$

respectively.

Referring to diagram (1), the image of ψ is generated by the set $g \otimes g$, $g \in G$. Using (5.5), it follows that

$$\operatorname{Im} \psi = \langle x \otimes x, y \otimes y, (x \otimes y)(y \otimes x) \rangle,$$

so that $H_2(G)$ is cyclic of order

$$\frac{|x \otimes y|}{|G'|} = \frac{(n, l-1)(n, 1 + l + \cdots + l^{m-1})}{n}$$

(an integer, because of (20)), which agrees with the results of [2].

6. GROUPS OF SMALL ORDER

If $|G| = n$ then $G \otimes G$ is presented with n^2 generators and $2n^3$ relations. For small values of n, say $n \leqslant 12$, these can be input directly to the Tietze transformation program [11] and the structure of $G \otimes G$ deduced. In order to study tensor squares of groups of larger order we wrote a program to simplify the presentation before it is input to the Tietze program. The n elements of G are denoted by integers 1 to n, 1 being the identity, and the generator $a \otimes b \in G \otimes G$ is denoted by the integer $na + b$.

These n^2 generators for $G \otimes G$ are stored in an array and simplifications are made during a scan of the relations. Whenever a relation shows that

(i)   a generator is trivial, or

(ii)   one generator is equal to another generator (or its inverse)

the array of generators is modified accordingly.

When no further simplification of type (i) or (ii) is possible by considering the relations one at a time then a small subset of the relations is input to the Tietze program. After substring searching (see [11] for a description) more deductions of type (i) and (ii) may be made and the generator array simplified again. It is fairly typical that the number of generators of $G \otimes G$ is reduced to about n by this process and although the number of

TABLE I

No.	Name	$|G|$	Defining relations	$G \otimes G$	$|G \otimes G|$	$H_2(G)$
1	D_3	6	$a^2 = b^3 = (ab)^2 = 1$	Z_6	6	1
2	D_4	8	$a^2 = b^4 = (ab)^2 = 1$	$(Z_2)^3 \times Z_4$	32	Z_2
3	Q_2	8	$a^2 = b^2 = (ab)^2$	$(Z_2)^2 \times (Z_4)^2$	64	1
4	D_5	10	$a^2 = b^5 = (ab)^2 = 1$	Z_{10}	10	1
5	D_6	12	$a^2 = b^6 = (ab)^2 = 1$	$(Z_2)^3 \times Z_6$	48	Z_2
6	A_4	12	$a^3 = b^2 = (ab)^3 = 1$	$Z_3 \times Q_2$	24	Z_2
7	Q_3	12	$a^3 = b^2 = (ab)^2$	Z_{12}	12	1
8	D_7	14	$a^2 = b^7 = (ab)^2 = 1$	Z_{14}	14	1
9	$Z_2 \times D_4$	16	$a^2 = b^2 = c^2 = (ca)^2 = (ab)^2 = (bc)^4 = 1$	$(Z_2)^8 \times Z_4$	1024	$Z_2 \times Z_2 \times Z_2$
10	$Z_2 \times Q_2$	16	$a^2 = b^2 = (ab)^2, c^2 = (ac)^2 = (bc)^2 = 1$	$(Z_2)^7 \times (Z_4)^2$	2048	$Z_2 \times Z_2$
11	D_8	16	$a^2 = b^8 = (ab)^2 = 1$	$(Z_2)^3 \times Z_8$	64	Z_2
12		16	$a^2 = 1, aba = b^3$	$(Z_2)^3 \times Z_8$	64	1
13		16	$a^2 = 1, aba = b^{-3}$	$(Z_2)^3 \times Z_8$	64	1
14		16	$a^4 = b^4 = 1, a^{-1}ba = b^{-1}$	$Z_2 \times (Z_4)^3$	128	Z_2
15		16	$a^4 = b^4 = (ab)^2 = (a^{-1}b)^2 = 1$	$(Z_2)^3 \times (Z_4)^2$	128	$Z_2 \times Z_2$
16		16	$a^2 = b^2 = c^2 = 1, abc = bca = cab$	$(Z_2)^9$	512	$Z_2 \times Z_2$
17	Q_4	16	$a^4 = b^2 = (ab)^2$	$(Z_2)^3 \times Z_8$	64	1
18	$Z_3 \times D_3$	18	$a^2 = b^3 = 1, (ab)^2 = (ba)^2$	$Z_3 \times Z_6$	18	1
19	D_9	18	$a^2 = b^9 = (ab)^2 = 1$	Z_{18}	18	1
20		18	$a^2 = b^2 = c^2 = (abc)^2 = (ab)^3 = (ac)^3 = 1$	$Z_2 \times B(2, 3)$	54	Z_3
21	D_{10}	20	$a^2 = b^{10} = (ab)^2 = 1$	$(Z_2)^3 \times Z_{10}$	80	Z_2

22		20	$a^5 = b^4 = 1, b^{-1}ab = a^2$	Z_{20}	20	1
23	Q_5	20	$a^5 = b^2 = (ab)^2$	Z_{20}	20	1
24		21	$a^3 = 1, a^{-1}ba = b^2$	Z_{21}	21	1
25	D_{11}	22	$a^2 = b^{11} = (ab)^2 = 1$	Z_{22}	22	1
26	$Z_2 \times A_4$	24	$a^3 = b^2 = (a^{-1}bab)^2 = 1$	$Z_6 \times Q_2$	48	Z_2
27	$Z_2 \times D_6$	24	$a^2 = b^2 = c^2 = (bc)^6 = (ca)^2 = (ab)^2 = 1$	$(Z_2)^8 \times Z_6$	1536	$Z_2 \times Z_2 \times Z_2$
28	$Z_3 \times D_4$	24	$a^{12} = b^2 = 1, bab = a^{-5}$	$(Z_2)^3 \times Z_{12}$	96	Z_2
29	$Z_3 \times Q_2$	24	$a^{12} = 1, b^2 = a^6, b^{-1}ab = a^7$	$(Z_2)^2 \times Z_4 \times Z_{12}$	192	1
30	$Z_4 \times D_3$	24	$a^{12} = b^2 = 1, bab = a^5$	$(Z_2)^2 \times Z_4 \times Z_6$	96	Z_2
31	$Z_2 \times Q_3$	24	$a^6 = b^4 = 1, b^{-1}ab = a^{-1}$	$(Z_2)^3 \times Z_4 \times Z_6$	96	Z_2
32	D_{12}	24	$a^2 = b^{12} = (ab)^2 = 1$	$(Z_2)^3 \times Z_{12}$	96	Z_2
33	S_4	24	$a^4 = b^2 = (ab)^3 = 1$	$Z_2 \times A_4$	48	Z_2
34	$\hat{A}_4$	24	$a^3 = b^3 = (ab)^2$	$Z_3 \times Q_2$	24	1
35		24	$a^4 = b^6 = (ab)^2 = (a^{-1}b)^2 = 1$	$(Z_2)^3 \times Z_{12}$	96	Z_2
36		24	$a^2 = b^2 = (ab)^3$	Z_{24}	24	1
37	Q_6	24	$a^6 = b^2 = (ab)^2$	$(Z_2)^2 \times Z_4 \times Z_{12}$	192	1
38	D_{13}	26	$a^2 = b^{13} = (ab)^2 = 1$	Z_{26}	26	1
39	$B(2,3)$	27	$a^3 = b^3 = (ab)^3 = (a^{-1}b)^3 = 1$	$(Z_3)^6$	729	$Z_3 \times Z_3$
40		27	$a^3 = 1, a^{-1}ba = b^{-2}$	$(Z_3)^4$	81	1
41	D_{14}	28	$a^2 = b^{14} = (ab)^2 = 1$	$(Z_2)^3 \times Z_{14}$	112	Z_2
42	Q_7	28	$a^7 = b^2 = (ab)^2$	Z_{28}	28	1
43	$Z_3 \times D_5$	30	$a^2 = 1, aba = b^4$	Z_{30}	30	1
44	$Z_5 \times D_3$	30	$a^2 = 1, aba = b^{-4}$	Z_{30}	30	1
45	D_{15}	30	$a^2 = b^{15} = (ab)^2 = 1$	Z_{30}	30	1

relations is not greatly reduced, the number of distinct relations is reduced to the order of n^2. This technique allows $G \otimes G$ to be computed for $|G|$ up to 48 but certain groups of larger order can also be handled; see Section 7. Table I below gives $G \otimes G$ for all non-abelian G of order $\leqslant 30$. We have

TABLE II

No. 6	Generators	$x_1 = a \otimes a$, $x_2 = a \otimes b$, $x_3 = a \otimes a^{-1}ba$
	Relations	$x_1^3 = 1$, $[x_1, x_2] = [x_1, x_3] = 1$, $x_2^2 = x_3^2$,
		$x_2 x_3 x_2 = x_3$
No. 12	Generators	$x_1 = a \otimes a$, $x_2 = a \otimes b$, $x_3 = b \otimes a$,
		$x_4 = b \otimes b$
	Relations	$x_1^2 = x_4^4 = x_2^2 x_3^2 = x_3^4 x_4^2 = 1$,
		$[x_i, x_j] = 1$, $1 \leqslant i < j \leqslant 4$
No. 13	Generators	$x_1 = a \otimes a$, $x_2 = a \otimes b$, $x_3 = b \otimes a$,
		$x_4 = b \otimes b$
	Relations	$x_1^2 = x_2^2 = x_3^2 = x_4^8 = 1$, $[x_i, x_j] = 1$,
		$1 \leqslant i < j \leqslant 4$
No. 14	Generators	$x_1 = a \otimes a$, $x_2 = a \otimes b$, $x_3 = b \otimes a$,
		$x_4 = b \otimes b$
	Relations	$x_1^4 = x_2^4 = x_4^4 = 1$, $x_2^2 = x_3^2$,
		$[x_i, x_j] = 1$, $1 \leqslant i < j \leqslant 4$
No. 15	Generators	$x_1 = a \otimes a$, $x_2 = a \otimes b$, $x_3 = b \otimes a$,
		$x_4 = b \otimes b$, $x_5 = a \otimes b^2$
	Relations	$x_1^4 = x_2^4 = x_5^2 = 1$, $x_1^2 = x_4^2$, $x_2^2 = x_3^2$,
		$[x_i, x_j] = 1$, $1 \leqslant i < j \leqslant 5$
No. 16	Generators	$x_1 = a \otimes a$, $x_2 = a \otimes b$, $x_3 = b \otimes a$,
		$x_4 = b \otimes b$, $x_5 = a \otimes c$, $x_6 = c \otimes a$,
		$x_7 = c \otimes c$, $x_8 = b \otimes c$, $x_9 = c \otimes b$
	Relations	$x_i^2 = 1$, $1 \leqslant i \leqslant 9$, $[x_i, x_j] = 1$, $1 \leqslant i < j \leqslant 9$
No. 20	Generators	$x_1 = a \otimes b$, $x_2 = a \otimes bc$
	Relations	$x_1^6 = x_2^3 = 1$, $x_1^2(x_2 x_1^{-1})^2 x_2 = 1$,
		$(x_1 x_2)^2 x_1^{-2} x_2 = 1$
No. 33	Generators	$x_1 = a \otimes a$, $x_2 = a \otimes ab$, $x_3 = a \otimes ba^{-1}$
	Relations	$x_1^2 = 1$, $[x_1, x_2] = [x_1, x_3] = 1$,
		$x_2^2 = x_3 x_2 x_3$, $x_3^2 = x_2 x_3 x_2$
No. 34	Generators	$x_1 = a \otimes b$, $x_2 = a \otimes b^2$
	Relations	$x_1^{12} = 1$, $x_1^2 x_2^2 = 1$, $(x_1 x_2)^2 = x_2^6$
No. 35	Generators	$x_1 = a \otimes a$, $x_2 = a \otimes b$, $x_3 = b \otimes a$,
		$x_4 = b \otimes b$
	Relations	$x_1^2 = x_4^2 = x_2^{12} = 1$, $x_2^2 x_3^2 = 1$,
		$[x_i, x_j] = 1$, $1 \leqslant i < j \leqslant 4$
No. 39	Generators	$x_1 = a \otimes a$, $x_2 = a \otimes b$, $x_3 = b \otimes a$,
		$x_4 = b \otimes b$, $x_5 = a \otimes b^{-1}$, $x_6 = a \otimes bab$
	Relations	$x_i^3 = 1$, $1 \leqslant i \leqslant 6$, $[x_i, x_j] = 1$, $1 \leqslant i < j \leqslant 6$

followed the format of Table I in [5]. We should note that the structure of $G \otimes G$ was determined, after simplification by the Tietze program, by using the suite of group theory programs set up on the St. Andrews VAX, in particular a Todd–Coxeter coset enumeration program and an Integer Matrix Diagonalization program. The notation follows that of previous sections and, in addition, A_4, $\hat{A}_4$, $B(2, 3)$ denote the alternating group of degree 4, its (unique) covering group, and the Burnside group of exponent 3 on 2 generators, respectively.

All but 11 of these groups $G \otimes G$ can be deduced from Propositions 11 to 15. (Note that the groups 22, 24, 36, and 40 are metacyclic and covered by Proposition 15.) In Table II we give for each of these 11 groups explicit generators for $G \otimes G$ together with defining relations. Note that the map $\hat{A}_4 \to A_4$ induces a map $\hat{A}_4 \otimes \hat{A}_4 \to A_4 \otimes A_4$ which is an isomorphism since, by the machine computations, the groups have the same orders. Note also that the groups 12, 13, and 14 are all split metacyclic with even kernel, so there is hope of generalising these computations, and that 6, 20, 33, and 34, as well as examples 2 and 3 in Section 7, are covered implicitly by Proposition 8 (see also Proposition 16).

7. MISCELLANEOUS EXAMPLES

For the sake of completeness, we include a list of all cases not mentioned explicitly in the foregoing for which $G \otimes H$ is known to us.

1. (P. J. Higgins) Take two copies of $\mathbb{Z}$ each acting on the other by the non-identity automorphism. These actions are clearly compatible, and $\mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z} \times \mathbb{Z}$, with factors generated by $x \otimes y$ and $x \otimes y^{-1}$, where x, y are generators of the two copies of $\mathbb{Z}$.

2. By the machine method of the last section, it can be shown that when $G = GL(2, 3)$,

$$G \otimes G = SL(2, 3) \times \mathbb{Z}_2.$$

Explicitly, taking $G = \langle a, b \mid a^8 = b^3 = (ab)^2 = [a^4, b] = e \rangle$, we obtain the presentation

$$G \otimes G = \langle x_1, x_2, x_3 \mid x_1^2 = [x_1, x_2] = [x_1, x_3] = e,$$
$$x_2^2 = x_2 x_3 x_2, \, x_3^2 = x_2 x_3 x_2 \rangle,$$

where $x_1 = a \otimes a$, $x_2 = a \otimes b$, $x_3 = a \otimes a^4 b^{-1}$.

3. The biggest group handled so far by the machine is

$$S_5 = \langle a, b \mid a^4 = b^6 = (ab)^2 = (a^{-1}b)^3 = e \rangle,$$

which gives

$$S_5 \otimes S_5 = \langle x_1, x_2 \mid x_1^{10} = e, \ x_1^5 = x_2^3, \ x_1 x_2 x_1 = x_2 x_1^3 x_2 \rangle$$

$$\cong SL(2,5) \times \mathbb{Z}_2,$$

where $x_1 = a \otimes b$ and $x_2 = a \otimes b^3$.

4. In the last example, $SL(2,5)$ is the covering group $\hat{A}_5$ of A_5 (see Corollary 2 in Section 2). Comparison of this with line 33 of Table I suggests a more general result. This turns out to be true in all but the exceptional cases $n = 6$ and $n = 7$, where it needs to be restated for the following reason. While $H_2(S_n) = \mathbb{Z}_2$ for all $n \geqslant 4$ (see [12]), the same is true for A_n except for $n = 6$ and 7, when the multiplicator is $\mathbb{Z}_6$. We can thus state the following consequence of Proposition 8 of Section 2.

PROPOSITION 16. *For the symmetric group S_n, $n \geqslant 4$,*

$$S_n \otimes S_n \cong \hat{S}'_n \times \mathbb{Z}_2,$$

a group of order $2 \cdot n!$.

5. (R. Aboughazi [1]) If H is the Heisenberg group with generators x, y, z and relations $[x,y] = z^{-1}$, $[z,x] = [z,y] = e$, then $H \otimes H = (\mathbb{Z})^6$ is the free abelian group with generators $x \otimes x$, $y \otimes y$, $x \otimes y$, $y \otimes x$, $x \otimes z$, and $y \otimes z$. The non-trivial actions of H on $H \otimes H$ are determined by

$$^{x^m y^n}(x \otimes y) = (x \otimes y)(x \otimes z)^{-m}(y \otimes z)^{-n},$$

$$^{x^m y^n}(y \otimes x) = (y \otimes x)(x \otimes z)^{-m}(y \otimes z)^{n}.$$

6. (D. Guin) If A is a G-module considered as acting trivially on G, then there is an isomorphism $G \otimes A \to (IG) \otimes_{\mathbb{Z}G} A$ which sends $g \otimes a \mapsto (g-1) \otimes a$ [10, Prop. 3.2].

8. OPEN PROBLEMS

1. Let G and H be finite groups acting compatibly on each other. Then is it true that $G \otimes H$ is finite? In addition to the foregoing the values of $\mathbb{Z}_m \otimes \mathbb{Z}_n$ have been computed by R. J. Sanders (Nottingham) for various compatible actions. In every case, the result has been finite and even cyclic (it is always abelian). (This question is settled affirmatively in [9] but no purely algebraic proof is known.)

2. Let $d(G)$ be the minimal number of generators for a group G. Can any general estimate of $d(G \otimes G)$ be found when G is finite? Note that,

when G is free of rank 2, $d(G \otimes G)$ is countably infinite (Proposition 6). On the other hand, for all the groups in Sections 4 and 5, $G \otimes G$ is generated by $\{x \otimes x,\ x \otimes y,\ y \otimes x,\ y \otimes y\} = X \otimes X$, where $X = \{x, y\}$ generates G. Note that, for $G \otimes G = \langle X \otimes X \rangle$ to hold for $G = \langle X \rangle$ when $|X| = 2$, we must have $G' = (G \otimes G)/J_2(G)$ cyclic, since $x \otimes x$, $y \otimes y$, $(x \otimes y)(y \otimes x) \in J_2(G)$.

  3. If G is soluble of derived length $l(G)$, then $G \otimes G$ is soluble and

$$l(G \otimes G) = l(G) \qquad \text{or} \qquad l(G) - 1.$$

Examples of both types appear in the foregoing. Is there any intrinsic characterisation of soluble groups of either type?

  4. If G is nilpotent of class $\mathrm{cl}(G)$, then $G \otimes G$ is nilpotent and

$$\mathrm{cl}(G \otimes G) = \mathrm{cl}(G') \qquad \text{or} \qquad \mathrm{cl}(G') + 1.$$

Can either of these types be characterised internally? Note that, when G is a finite nilpotent group, it follows from Proposition 11 that $G \otimes G$ is just the direct product of the tensor squares of the Sylow subgroups of G, so this is really a problem about p-groups.

  5. Examine the behaviour of $G \otimes G$ under the formation of free products.

  6. Complete the evaluation of $G \otimes G$ for all metacyclic G.

  7. Use the ideas in the proof of Proposition 8 to compute the tensor square of $GL(2, p)$ and other linear groups.

  8. Give an algebraic proof of the exactness of the top row

$$H_3(G) \longrightarrow \Gamma(G^{ab}) \xrightarrow{\psi} J_2(G) \longrightarrow H_2(G) \longrightarrow 0$$

of diagram (1). We suspect that this is an edge exact sequence of a spectral sequence of algebraic origin.

REFERENCES

1. R. ABOUGHAZI, Produit tensoriel du groupe d'Heisenberg, *Bull. Soc. Math. France* **115** (1987), 95–106.

2. R. BEYL, The Schur multiplicator of metacyclic groups, *Proc. Amer. Math. Soc.* **40** (1973), 413–418.

3. R. BROWN AND J.-L. LODAY, Excision homotopique en basse dimension, *C. R. Acad. Sci. Paris Sér. I Math.* **298**, No. 15 (1984), 353–356.

4. R. BROWN AND J.-L. LODAY, Van Kampen theorems for diagrams of spaces, *Topology*, in press.

5. H. S. M. COXETER AND W. O. J. MOSER, "Generators and Relations for Discrete Groups," 4th ed., Springer-Verlag, Berlin, 1980.

6. R. K. DENNIS, In search of new "homology" functors having a close relationship to *K*-theory, preprint, Cornell University, Ithaca, NY, 1976.

7. G. J. ELLIS, "Crossed Modules and Their Higher Dimensional Analogues," Ph.D. thesis, University of Wales, Bangor, 1984.

8. G. J. ELLIS, Non-abelian exterior products of groups and exact sequences in the homology of groups, *Glasgow Math. J.* **29** (1987), 13–19.

9. G. J. ELLIS, The non-abelian tensor product of finite groups is finite, *J. Algebra*, in press.

10. D. GUIN, Cohomologie et homologie non-abélienne des groupes, preprint, Strasbourg, 1985.

11. G. HAVAS, P. E. KENNE, J. S. RICHARDSON, AND E. F. ROBERTSON, A Tietze transformation program, *in* "Computational Group Theory," pp. 69–73, Academic Press, New York/London, 1984.

12. B. HUPPERT, "Endliche Gruppen, I," Springer-Verlag, Berlin, 1967.

13. S. MACLANE, "Homology," Springer-Verlag, Berlin, 1963.

14. C. MILLER, The second homology of a group, *Proc. Amer. Math. Soc.* **3** (1952), 588–595.

15. J. H. C. WHITEHEAD, A certain exact sequence, *Ann. of Math.* **52** (1950), 51–110.

16. N. D. GILBERT, The non-abelian tensor square of a free product of groups, *Arch. Math.* **48** (1987), 369–375.

17. N. D. GILBERT AND P. J. HIGGINS, The non-abelian tensor product of groups and related constructions, *U.C.N.W. Pure Maths*, Preprint 87.11.

18. D. L. JOHNSON, The non-abelian tensor square of a finite split metacyclic group, *Proc. Edinburgh Math. Soc.* **30** (1987), 91–96.

19. A. S.-T. LUE, The Ganea map for nilpotent groups, *J. London Math. Soc.* **14** (1976), 309–312.