Erratum on multirelative algebraic K-theory and on Hopf formulae for the higher homology of a group

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Several years ago I omitted a crucial hypothesis from the statement of results in three papers [6, 2, 7]. It seems that this has been causing a bit of confusion. G. Donadze and N. Inassaridze [4] have recently produced a counterexample to the formulae for the higher homology of a group stated in [2, 7]. Previously S. Bloch and S. Lichtenbaum [1] had noted the incorrectness of a result on multirelative algebraic K-theory stated in [6], and a counterexample was provided by J.-L. Loday and C. Weibel. On the positive side, the omission has lead to new and more transparent proofs of the results [5, 8].

The omitted hypothesis concerns a group G with normal subgroups $N_1, \ldots, N_n$. This data gives rise to an n-cubical diagram involving the groups $G_\Delta = G/\prod_{i\in\Delta} N_i$ for each subset $\Delta \subset [n] = \{1, \ldots, n\}$ and a quotient homomorphism $G_\Delta \to G_{\Delta\cup\{i\}}$ for each $i \notin \Delta$. By inductively defining

$$G_{\Delta\cup\{-i\}} = \ker(G_\Delta \to G_{\Delta\cup\{i\}})$$

for $\Delta \subset [\pm n] = \{\pm 1, \pm 2, \ldots, \pm n\}$ and $i \in [n] \setminus \Delta$ we obtain a functor $\{-1 < 0 < 1\}^n \to$ Groups. Let us say that the $(n+1)$-ad of groups $(G; N_1, \ldots, N_n)$ is *connected* if any one of the following three equivalent conditions is satisfied.

1. The induced n-cubical diagram of classifying spaces $\{BG_\Delta\}_{\Delta\subset[n]}$ is connected in the sense of [3].

2. The homomorphisms $G_\Delta \to G_{\Delta\cup\{i\}}$ are surjective for all $\Delta \subset [\pm n]$ and $i \in [n] \setminus \Delta$.

3. For all subsets $\Delta, \Gamma \subset \{1, \cdots, n\}$ with $|\Delta| \geq 2, |\Gamma| \geq 1$ the following equality holds:

$$\left(\bigcap_{i\in\Delta} N_i\right)\left(\prod_{j\in\Gamma} N_j\right) = \bigcap_{i\in\Delta}\left(N_i(\prod_{j\in\Gamma} N_j)\right) .$$

An $(n+1)$-ad of groups is automatically connected when $n = 1$ or 2. However, for $n \geq 3$ this is not so.

The results in [6, 2] are obtained by applying a generalised van Kampen theorem of R. Brown and J.-L. Loday [3] to the n-cube of spaces $\{BG_\Delta\}_{\Delta\subset[n]}$. As the generalised van Kampen theorem applies only to connected n-cubes, it is necessary to hypothesise that this particular n-cube be connected. This hypothesis was omitted. It is also needed in the purely algebraic setup in [7], the point being that Example 3.1 fails for $n \geq 3$ without the hypothesis.

The formula for the integral homology $H_{n+1}(G)$ of a group given in [2, Theorem 1] involves a group F with normal subgroups $R_1, \cdots, R_n$. The statement of the formula is fine for $n = 1, 2$, but for $n \geq 3$ it is necessary to add the hypothesis that the $(n+1)$-ad $(F; R_1, \ldots, R_n)$ be connected. Fortunately, this hypothesis is satisfied for all G when the $(n+1)$-ad is obtained by either of the methods described in [2]. The connectedness hypothesis also needs to be added to [2, Proposition 2] and to the corresponding results for cohomology obtained by purely algebraic means in [7].

The multirelative algebraic K-theory in [6] concerns a ring Λ with ideals $I_1, \ldots, I_n$. An $(n+1)$-ad of groups $(St(\Lambda); S_1, \ldots, S_n)$ is obtained by considering the Steinberg group $St(\Lambda)$ and setting S_i equal to the kernel of the surjective group homomorphism $St(\Lambda) \to St(\Lambda/I_n)$. To all theorems in the paper it is necessary to add the hypothesis that this $(n+1)$-ad be connected. Fortunately, it is connected in the one explicit calculation given in the paper (Proposition 1.4).

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References

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