DETERMINATION
OF A DOUBLE LIE GROUPOID
BY ITS CORE DIAGRAM[*]

RONALD BROWN
School of Mathematics
University of Wales
Bangor, Gwynedd, LL57 1UT

KIRILL C. H. MACKENZIE[†]
Department of Mathematical Sciences
University of Durham
Durham, DH1 3LE
and
Department of Pure Mathematics
University of Sheffield
Sheffield, S3 7RH

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August 1998

Abstract

In a double groupoid S, we show that there is a canonical groupoid structure on the set of those squares of S for which the two source edges are identities; we call this the *core groupoid* of S. The target maps from the core groupoid to the groupoids of horizontal and vertical edges of S are now base–preserving morphisms whose kernels commute, and we call the diagram consisting of the core groupoid and these two morphisms the *core diagram* of S. If S is a double Lie groupoid, and each groupoid structure on S satisfies a natural double form of local triviality, we show that the core diagram determines S and, conversely, that a locally trivial double Lie groupoid may be constructed from an abstractly given core diagram satisfying some natural additional conditions.

In the algebraic case, the corresponding result includes the known equivalences between crossed modules, special double groupoids with special connection (Brown and Spencer), and cat^1–groups (Loday). These cases correspond to core diagrams for which both target morphisms are (compatibly) split surjections.

A double groupoid is a groupoid object in the category of groupoids; that is, a double groupoid consists of a set S with two groupoid structures upon (generally distinct) bases H and V, which are themselves groupoids on a common base B, all subject to the compatibility condition that the structure maps of each structure on S are morphisms with respect to the other. We call H and V the *side groupoids* of S, and B the *double base*. Elements of S are pictured as squares

$$
\begin{array}{c}
h_2 \\
v_2 \;\boxed{ s }\; v_1 \\
h_1
\end{array}
\tag{1}
$$

in which v_1, $v_2 \in V$ are the source and target of s with respect to the horizontal structure on S, and h_1, $h_2 \in H$ are the source and target with respect to the vertical structure. The compatibility condition ensures that a diagram of four squares of S,
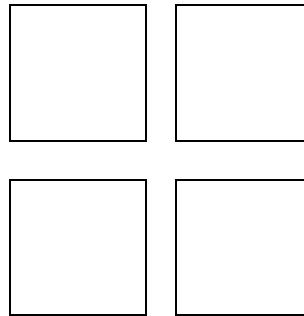
in which each pair of parallel inner edges matches, has a unique composition; one obtains the same answer whether one composes first horizontally and then vertically, or first vertically and then horizontally. This is known as the *interchange law*. Double groupoids (and double categories) should be distinguished from 2-categories, where multiple compositions present complicated "pasting" problems, and from bicategories, where the basic laws (such as associativity, and the existence of identities) hold only up to equivalence.

Double groupoids were introduced by Ehresmann in the early 1960's (see for example [9], [7]), but primarily as instances of double categories, and as a part of a general exploration of categories with structure. Since that time their main use has been in homotopy theory. Brown and Higgins [2] gave the earliest example of a "higher homotopy groupoid", by associating to a pointed pair of spaces (X, A) a *special double groupoid with special connection*, $\rho(X, A)$. Such double groupoids have identical side groupoids, and the two groupoid structures on squares are isomorphic under a rotation operator. In terms of this functor ρ, [2] proved a Generalized Van Kampen Theorem, and deduced from it a Van Kampen Theorem for the second relative homotopy group $\pi_2(X, A)$, viewed as a crossed module over the fundamental group $\pi_1(A)$.

In differential geometry, double Lie groupoids, but usually with one of the structures totally intransitive, have been considered in passing by Pradines ([19], [20] and elsewhere). Very recently *double Lie groups* have been studied by several authors in connection with Poisson Lie groups and related structures (see Lu and Weinstein [13] and references given there); these may be regarded as double Lie groupoids in which the double base is a singleton, so that the side groupoids are in fact groups, and which satisfy a further strong condition. Also quite recently, Weinstein [24] has introduced a notion of symplectic double groupoid.

The double groupoids which arise in homotopy theory are of a particularly special type, and admit several equivalent descriptions. The special double groupoids with special connection, which were shown in [2] to model 2-dimensional homotopy theory, have identical side groupoids $H = V$, and for each $h \in H$ a special kind of "degenerate", or *thin*, square,

$$
\begin{array}{c}
h \\
h \quad \boxed{\Gamma(h)} \quad 1 \\
1
\end{array}
$$

The map $\Gamma \colon H \to S$ has the property that $\Gamma(hh')$ is equal to the double composite of

$$
\begin{array}{cc}
\quad h \qquad\qquad h' \\
h \;\boxed{\Gamma(h)}\; 1 \quad\; \boxed{\widetilde{1}^V_{h'}}\; 1 \\
\quad 1 \qquad\qquad h' \\[4pt]
\quad 1 \qquad\qquad h' \\
h' \;\boxed{\widetilde{1}^H_{h'}}\; h' \;\; \boxed{\Gamma(h')}\; 1 \\
\quad 1 \qquad\qquad 1
\end{array}
\qquad (2)
$$

for all composable $h, h' \in H$ (where $\widetilde{1}^H_{h'}$, $\widetilde{1}^V_{h'}$ denote identities for the horizontal and vertical

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structures on S; see §1). Such maps Γ had already been studied by Brown and Spencer [6] under the name of "special connections"; condition (2) is called the *transport law.* The special connection encodes the basic properties characteristic of the homotopy double groupoid $\rho(X, A)$ [2]; in particular, the rotation operator referred to above may be defined in terms of the special connection [6, §4]. The main result of [6] showed that a special double groupoid with special connection whose double base is singleton is entirely determined by a certain *crossed module*; crossed modules had arisen much earlier in the work of J. H. C. Whitehead on 2-dimensional homotopy. This result is easily extended to give an equivalence between arbitrary special double groupoids with special connection and crossed modules over groupoids; this is included in the results of [3]. We recall these results in more detail in §3 below.

Special double groupoids with special connection and a singleton double base are also equivalent to the cat^1-groups of Loday. Cat^n-groups, for any positive integer n, were introduced in [12] as algebraic models of homotopy $(n+1)$-types; a cat^1-group is a group G together with endomorphisms $s, t \colon G \to G$ such that $st = t$, $ts = s$ and such that $\ker(s)$ and $\ker(t)$ commute elementwise. It was noted in [12] that cat^1-groups are equivalent to crossed modules (over groups) and to group objects in the category of groupoids. That group objects in the category of groupoids are equivalent to crossed modules had been shown much earlier by Brown and Spencer [5], in a result there attributed to Verdier.

There is thus a commuting square of equivalences between the concepts of special double groupoid with special connection and a singleton base, group object in the category of groupoids, cat^1-group, and crossed module. Each of these concepts has its particular advantages, but that of double groupoid is perhaps closest to the underlying intuition that 2-dimensional homotopy refers to the study of squares in a topological space.

The purpose of the present paper is to show that a far wider class of double groupoids is completely determined by a simple diagram of ordinary groupoids and morphisms, the *core diagram.* This result, 2.7 below, is a considerable extension of the equivalence [6] between special double groupoids with special connection and crossed modules. Given a double groupoid S as above, the *core groupoid K* consists of those squares k in S of the form

$$
\begin{array}{c}
h \\
v \;\boxed{\; k \;}\; 1^V \\
1^H
\end{array}
$$

where 1^H, 1^V are suitable identity elements of H and V; the groupoid composition in K combines the horizontal and vertical compositions in S (see §2). The core diagram of S consists of K together with two maps $\partial_H \colon K \to H$, $\partial_V \colon K \to V$ defined for the square above by $\partial_H(k) = h$, $\partial_V(k) = v$; these are groupoid morphisms over B, the double base of S. Under suitable conditions we show that K, H and V, and the morphisms ∂_H, ∂_V, determine S up to isomorphisms preserving H and V.

It may not be clear how this generalizes the equivalence of [6]. For a general double groupoid S satisfying the conditions of 2.7, the morphisms ∂_H, ∂_V in the core diagram are surjective and we will usually display the core diagram as

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$$
\begin{array}{ccc}
M^H & & H \\
& \searrow \quad \nearrow & \\
& K & \\
& \nearrow \quad \searrow & \\
M^V & & V
\end{array}
\qquad\qquad (3)
$$

with ∂_H labelling the arrow $K \to H$ and ∂_V labelling the arrow $K \to V$.

where $M^H = \ker(\partial_V)$, $M^V = \ker(\partial_H)$. Now the composite map $M^V \to V$ is a crossed module (over the groupoid V) with respect to the action of V on M^V obtained by lifting elements of V to K and conjugating. Crucial here is the fact that M^H and M^V commute in K. Similarly $M^H \to H$ is a crossed module. If S is a special double groupoid with special connection then these two crossed modules can be canonically identified, and either may be taken as the crossed module associated by [6] to S. The fact that a special double groupoid with special connection is determined by this crossed module is reflected in the fact that in this case the two exact sequences in the core diagram are split. Conversely, any crossed module over a groupoid determines a core diagram in which both exact sequences are split (3.5).

In fact, for a special double groupoid with special connection and singleton base, S, the core group(oid) K is precisely the cat^1-group corresponding to S. Because the two exact sequences are split, H and V may be regarded as subgroups of K, and ∂_H and ∂_V then correspond to the endomorphisms denoted s and t above. This viewpoint may be easily extended to any special double groupoid with special connection. Thus, while 2.7 generalizes the equivalence between crossed modules and special double groupoids with special connection, the concept of core diagram may also be regarded as generalizing the notion of cat^1-group.

Theorem 2.7 also establishes that any abstractly given diagram of the form (3), where both sequences are short exact and M^H and M^V commute in K, is the core diagram of a double groupoid, unique up to isomorphism. This, and the more general results of §4, delineate additional structure which a pair of crossed modules over groupoids must possess if they are to be contained within a double groupoid.

We have so far described this result in purely algebraic terms. However our chief concern is with a differentiable form of the results, giving an equivalence between locally trivial double Lie groupoids and locally trivial core diagrams. We should first explain that we are using the term "Lie groupoid" in a sense different to that in which it was used in [14] and elsewhere in the work of the second-named author. In [14] a Lie groupoid was taken to be a differentiable groupoid satisfying a local triviality condition, and Lie groupoids were accordingly essentially equivalent to principal bundles. In this paper, by a "Lie groupoid" we mean what in [14]
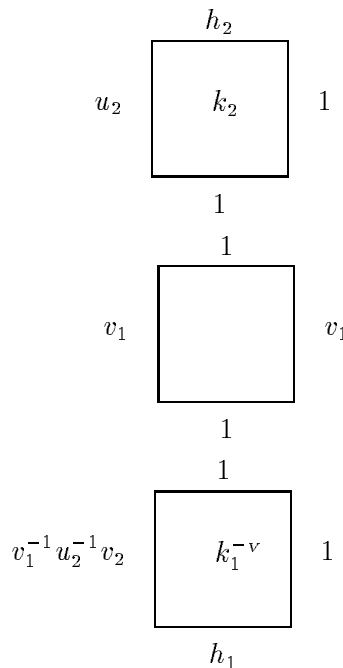
was called a differentiable groupoid, and when local triviality conditions are used they will be explicitly stated. This change in usage seems consonant with the growing importance of symplectic groupoids [24].

For an ordinary Lie groupoid G on base B, local triviality is the condition that the anchor map $G \to B \times B$, which maps arrows in G to their 0-skeleton, is a surjective submersion. A locally trivial Lie groupoid is determined by the principal bundle $G_b(B, G_b^b)$ where G_b consists of all arrows radiating from a fixed $b \in B$, and G_b^b is the group of arrows whose source and target are both b; for different choices of b these principal bundles are isomorphic (a detailed account is given in [14]). In considering analogous conditions for a double Lie groupoid S, one could require the two Lie groupoid structures on S to be locally trivial, and one would then obtain an equivalence with "principal bundles in the category of Lie groupoids". However, a far more interesting result arises if one takes account of the double structure, and imposes the stronger condition that the maps which to each square in S assign three sides of its 1-skeleton, be surjective submersions. This may also be expressed as the requirement that each anchor on S be not only a surjective submersion with respect to its own structure, but a fibration of groupoids with respect to the other (see 2.3). Our main theorem 2.7 then proves that a double Lie groupoid which is locally trivial in this sense is determined by its core diagram, and conversely, any locally trivial core diagram determines a locally trivial Lie groupoid.

For ordinary Lie groupoids, and principal bundles, local triviality is the condition which ensures the existence of a connection theory. In the same way it seems reasonable to expect that the connection theory of a locally trivial double Lie groupoid can be studied in terms of connections (in the slightly extended sense of [16]) in the exact sequences of its core diagram. The correspondence given in §3 between special connections in the sense of [6] and splittings in the core diagram may be regarded as the flat case of this result. The general case will be taken up elsewhere.

The concept of core groupoid may also be regarded as a generalization of Pradines' concept of the core (French: *cœur*) of a double vector bundle [20], [21], [22]; this is, of course, the origin of our terminology. The core of a double vector bundle is the intersection of the kernels of the two bundle projections; it inherits a unique vector bundle structure and plays a crucial role in the connection theory of the double vector bundle. Vector bundles, however, are totally intransitive and so the other elements of our core diagram are absent; there is accordingly no possibility of reconstructing a double vector bundle from its core, and there is no antecedent in the vector bundle theory for our main result.

We begin in §1 by considering actions and fibrations of double (Lie) groupoids. These are technical results which facilitate the proof of 2.7 in the differentiable setting. One of the key algebraic ingredients of 2.7 is a representation, in a double groupoid satisfying suitable local triviality conditions, of an arbitrary element, as in (1) above, as a vertical composition

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$$
\begin{array}{c}
h_2 \\
u_2 \;\boxed{\;k_2\;}\; 1 \\
1 \\[4pt]
1 \\
v_1 \;\boxed{}\; v_1 \\
1 \\[4pt]
1 \\
v_1^{-1}u_2^{-1}v_2 \;\boxed{\;k_1^{-v}\;}\; 1 \\
h_1
\end{array}
$$

where the middle element is an identity for the horizontal structure on S, and k_1 and k_2 are elements of the core groupoid; here k_1^{-v} denotes the inverse of k_1 in the vertical structure. This representation generalizes the relationship between thin elements and special connections given in [2]. In the differentiable setting we formulate this representation in terms of a concept of comma double groupoid (1.8).

§2 contains the main results of the paper. In §3 we recover the original Brown-Spencer correspondence in its differentiable form, and in the final section we indicate some generalizations valid in the algebraic case.

Throughout the paper we refer to [14] for basic facts on Lie groupoids. Our conventions follow [14], except as noted above and at the start of §1. Manifolds are C^∞, second-countable and Hausdorff.

1 CLASSES OF MORPHISMS OF DOUBLE GROUPOIDS

This section gives necessary preliminaries on double groupoids, with particular attention to actions and fibrations. The apparatus we develop here allows an efficent treatment of the differentiability aspects of the Lie case, and a reader who is primarily interested in the underlying algebra may prefer to read only to 1.2 and then proceed to §2, referring back to the examples as necessary.

We begin by recalling the main classes of morphisms of (ordinary) Lie groupoids; the terminology which follows is an amalgam of that of [10] with that of Pradines [23].

We will often use the notation $G \rightrightarrows B$ to indicate briefly that G is a groupoid on base B. Here the two arrows should be thought of as the source $\alpha\colon G \to B$ and target $\beta\colon G \to B$ maps. The anchor map $G \to B \times B$, $g \mapsto (\beta g, \alpha g)$ we generically denote by χ. The identity element corresponding to $b \in B$ we denote by 1_b, not, as in [14], by $\tilde{b}$. The multiplication, or

composition, map we denote by κ; it is defined on $G * G = \{(g_2, g_1) \in G \times G \mid \alpha(g_2) = \beta(g_1)\}$. Lastly, the *division map* of G we take to be $\delta\colon G \underset{\alpha}{\times} G \to G$, $(g', g) \mapsto g'g^{-1}$, where $G \underset{\alpha}{\times} G$ denotes the pullback of α over itself. Notice that we are composing from right to left, so that $g_2 g_1$ has source $\alpha(g_1)$ and target $\beta(g_2)$. A groupoid $G \rightrightarrows B$ is a *Lie groupoid* on B if G and B have manifold structures such that $\alpha\colon G \to B$ and $\beta\colon G \to B$ are surjective submersions and $\delta\colon G \underset{\alpha}{\times} G \to G$ is smooth. As noted in the introduction, this is the concept called *differentiable groupoid* in [14].

Consider a morphism of Lie groupoids $\phi\colon G' \dashrightarrow G$, $f\colon B' \dashrightarrow B$, and form the pullback manifold

$$
\begin{array}{ccc}
f^*G & \longrightarrow & G \\
\downarrow & & \downarrow \ \alpha \\
B' & \xrightarrow{\ \ f\ \ } & B.
\end{array}
$$

Let $\phi^*\colon G' \to f^*G$ be the induced map $g' \mapsto (\alpha'g', \phi(g'))$. Then (ϕ, f) is defined to be a *fibration* if ϕ^* is a surjective submersion, and to be an *action morphism* if ϕ^* is a diffeomorphism. If $f\colon B' \to B$ is also a surjective submersion then we speak of an *s-fibration* or an *s-action morphism*. This concept of fibration, introduced by Pradines [23] under the name "exacteur", is a smooth form of the algebraic notion of fibration of groupoids [1]. See also [10], [11]. In the algebraic setting, action morphisms were formerly often called "coverings" (for example [1]).

When f is a surjective submersion one can also form the pullback (which is in fact the pullback groupoid),

$$
\begin{array}{ccc}
f^{**}G & \longrightarrow & G \\
\downarrow & & \downarrow \ \chi \\
B' \times B' & \xrightarrow{\ \ f \times f\ \ } & B \times B.
\end{array}
$$

There is now an induced map $\phi^{**}\colon G' \to f^{**}G$, $g' \mapsto (\beta'g', \phi(g'), \alpha'g')$. In this case, (ϕ, f) is a *regular fibration* if ϕ^{**} is a surjective submersion, and is an *inductor* if ϕ^{**} is a diffeomorphism.

Turning now to double groupoids, we use the following conventions and notations. A *double groupoid* consists of a quadruple of sets $(S; H, V; B)$, together with groupoid structures on H and V, both with base B, and two groupoid structures on S, a *horizontal structure* with base V, amd a *vertical structure* with base H, such that the structure maps (source, p.245 target, division and identity maps) of each groupoid structure on S are morphisms with respect to the other.

Within H and V we use the multiplicative notation of [14]. It will normally be clear from the notation for elements which groupoid is under consideration. The identity elements however we denote by $1_b^H \in H$ and $1_b^V \in V$ for $b \in B$. The source, target, anchor,

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multiplication and division maps of H are denoted $\alpha_H\colon H \to B$, $\beta_H\colon H \to B, \chi_H\colon H \to B \times B$, $\kappa_H\colon H * H \to H$ and $\delta_H\colon H \underset{\alpha}{\times} H \to H$, and similarly for V, but we will omit the subscripts H and V whenever the meaning is clear.

The two groupoid structures on S we will also write multiplicatively. The horizontal structure with base V, denoted S_H, will have source and target maps $\widetilde{\alpha}_H\colon S \to V$, $\widetilde{\beta}_H\colon S \to V$, anchor $\widetilde{\chi}_H\colon S \to V \times V$, composition $\widetilde{\kappa}_H\colon S \underset{V}{*} S \to S$, division $\widetilde{\delta}_H\colon S \underset{V}{\times} S \to S$, and identities $\widetilde{1}_v^H$ for $v \in V$. The multiplication $\widetilde{\kappa}_H(s_2, s_1)$ we denote by $s_2 \circ_H s_1$, and the horizontal inverse of s we denote by s^{-H}. For the vertical structure with base H, denoted S_V, we correspondingly write $\widetilde{\alpha}_V\colon S \to H$, $\widetilde{\beta}_V\colon S \to H$ for the source and target projections, $\widetilde{\chi}_V\colon S \to H \times H$ for the anchor, $\widetilde{\kappa}_V\colon S \underset{H}{*} S \to S$ for the composition, $\widetilde{\delta}_V\colon S \underset{H}{\times} S \to S$ for the division, and $\widetilde{1}_h^V$ for $h \in H$ for the identities. The multiplication $\widetilde{\kappa}_V(s_2, s_1)$ we denote by $s_2 \circ_V s_1$, and the vertical inverse of s we denote by s^{-V}. For $b \in B$, the *double identity* $\widetilde{1}_{1_b^H}^V = \widetilde{1}_{1_b^V}^H$ is denoted 1_b^2.

We have used multiplicative notation for all four groupoid structures here in order to reserve additive notation for the associated Lie algebroids. It may well be that future work will need to consider expressions involving a groupoid multiplication in one structure and a Lie algebroid addition in the other.

Definition 1.1 *A double Lie groupoid is a double groupoid $(S; H, V; B)$ together with differentiable structures on S, H, V and B, such that all four groupoid structures are Lie groupoids and such that the double source map $s \mapsto (\widetilde{\alpha}_V(s), \widetilde{\alpha}_H(s))$, $S \to H \underset{\alpha}{\times} V = \{(h,v)|\alpha_H(h) = \alpha_V(v)\}$ is a surjective submersion.*

A morphism of double Lie groupoids $(\phi; \phi_H, \phi_V; \phi_B)\colon (S'; H', V'; B') \to (S; H, V; B)$ is a quadruple of smooth maps, $\phi\colon S' \to S$, $\phi_H\colon H' \to H$, $\phi_V\colon V' \to V, \phi_B\colon B' \to B$ such that (ϕ, ϕ_H), $(\phi, \phi_V), (\phi_H, \phi_B)$ and (ϕ_V, ϕ_B) are morphisms of their respective groupoids.

We will often indicate the spaces in a double groupoid and a typical element of it by the diagrams

$$
\begin{array}{ccc}
S & \xLongrightarrow{\;\widetilde{\alpha}_H,\widetilde{\beta}_H\;} & V \\[2pt]
\widetilde{\alpha}_V,\widetilde{\beta}_V \Big\Downarrow & & \Big\Downarrow \alpha_V,\beta_V \\[2pt]
H & \xLongrightarrow[\alpha_H,\beta_H]{} & B
\end{array}
\qquad\qquad
\begin{array}{c}
\widetilde{\beta}_V(s) \\
\widetilde{\beta}_H(s)\ \boxed{\ \ s\ \ }\ \widetilde{\alpha}_H(s) \\
\widetilde{\alpha}_V(s)
\end{array}
$$

Throughout the paper, diagrams of the latter type are oriented so that the horizontal sides point leftwards and the vertical sides upwards. The various algebraic conditions on the groupoid structures of S are written out in detail in, for example, [6]. The surjectivity condition on the double source map in 1.1 ensures that given $h \in H$ and $v \in V$ with matching sources, as in

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v

h
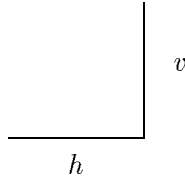
there exists an $s \in S$ having these sides. Conditions of this type are often called "filling conditions" (for example, Brown and Higgins [4]).

The importance of the submersion condition on the double source map is shown by the following proposition. It guarantees that the domain of each division (and multiplication) map in S is a Lie subgroupoid of the Cartesian square $S \times S$ of the other structure, so that one may legitimately speak of S as a "Lie groupoid in the category of Lie groupoids". This condition also ensures, as will be shown elsewhere, that there is a natural construction of a double Lie algebroid associated to S. In the present paper we make use of this condition only in the definition 2.1 of the core groupoid.

Proposition 1.2 *Let* $\phi_1 \colon G_1 \to G$, $f_1 \colon B_1 \to B$ *and* $\phi_2 \colon G_2 \to G$, $f_2 \colon B_2 \longrightarrow B$ *be fibrations of Lie groupoids such that the pullback manifold* $\overline{B}$ *of* f_1 *and* f_2 *exists. Then the pullback manifold* $\overline{G}$ *of* ϕ_1 *and* ϕ_2 *exists and is an embedded Lie subgroupoid of the product groupoid* $G_1 \times G_2$. *Further, it is the pullback of* ϕ_1 *and* ϕ_2 *in the category of Lie groupoids.*

PROOF: It suffices to show that the source map $\overline{G} \to \overline{B}$ is a surjective submersion. Let d denote the diagonal map $\overline{B} \to B$, $(b_1, b_2) \mapsto f(b_1) = f(b_2)$, and observe that $d^*G \to f_1^*G \times f_2^*G$, $((b_1, b_2), g) \mapsto ((b_1, g), (b_2, g))$ represents d^*G as an embedded submanifold of $f_1^*G \times f_2^*G$. Let ϕ be the map $\phi_1^* \times \phi_2^* \colon G_1 \times G_2 \to f_1^*G \times f_2^*G$. Then $\overline{G}$ is precisely $\phi^{-1}(d^*G)$ and since ϕ is a surjective submersion, it follows that $\overline{G}$ is an embedded submanifold of $G_1 \times G_2$, and that the restriction

$$\overline{G} \to d^*G, \ (g_1, g_2) \mapsto ((\alpha_1 g_1, \alpha_2 g_2), \phi_1(g_1))$$

is still a surjective submersion. Now the obvious map $d^*G \to \overline{B}$ is the left-hand side of the pullback

$d^*G \longrightarrow G$

$\overline{B} \xrightarrow{\ \ d\ \ } B$

α

and is therefore a surjective submersion also. ∎

Example 1.3 (Compare Weinstein [24, 4.5].) For any manifold M, the product manifold $M \times M$ has a natural Lie groupoid structure, where (m_2, m_1) has source m_1, target m_2, and the composition is $(m_3, m_2')(m_2, m_1) = (m_3, m_1)$, defined if $m_2' = m_2$. This is known as the *pair* or *coarse groupoid* on M. If $G \rightrightarrows B$ is a Lie groupoid, then $G \times G$ can be considered

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both as the Cartesian product groupoid on base $B \times B$, and as the pair groupoid on base G. These two structures constitute a double Lie groupoid

$$
\begin{array}{ccc}
G \times G & \Longrightarrow & B \times B \\
\Downarrow & & \Downarrow \\
G & \Longrightarrow & B
\end{array}
\qquad\qquad
\begin{array}{c}
g_2 \\
(\beta g_2, \beta g_1) \quad \boxed{(g_2, g_1)} \quad (\alpha g_2, \alpha g_1) \\
g_1
\end{array}
$$

Given any double Lie groupoid $(S; H, V; B)$, the anchor $\widetilde{\chi}_V : S \to H \times H$ together with $\mathrm{id} : H \to H$, $\chi_V : V \to B \times B$, $\mathrm{id} : B \longrightarrow B$ is a morphism of double groupoids $(S; H, V; B) \to (H \times H; H, B \times B; B)$. Similarly $(\widetilde{\chi}_H ; \chi_H, \mathrm{id}_V ; \mathrm{id}_B)$ is a morphism $(S; H, V; B) \to (V \times V; B \times B, V; B)$.

Example 1.4 Let H and V be Lie groupoids on the same base B, and suppose that the two anchors $\chi_H : H \to B \times B$ and $\chi_V : V \to B \times B$ are transversal as smooth maps; that is, the tangent bundle of $B \times B$ is generated, at each point, by the images of the tangent maps to χ_H and χ_V. (This condition is satisfied, for example, if one or both of H and V are locally trivial.) Then the pullback of

$$
\begin{array}{ccc}
& & V \times V \\
& & \Big\downarrow {\scriptstyle \chi_V \times \chi_V} \\
H \times H & \xrightarrow{\ \chi_H \times \chi_H\ } & B^4
\end{array}
$$

may be regarded as defining either the pullback groupoid $\chi_H^{**}(V \times V)$ on base H or the pullback groupoid $\chi_V^{**}(H \times H)$ on V. These two structures constitute a double Lie groupoid which we denote $\square\,(H, V)$, and whose elements are squares

$$
\begin{array}{c}
h_2 \\
v_2 \quad \boxed{} \quad v_1 \\
h_1
\end{array}
$$

with $h_1, h_2 \in H$, $v_1, v_2 \in V$ and sources and targets matching as shown. If $H = V$ we write $\square H$ for $\square\,(H, H)$. Taking $H = B \times B$, the pair groupoid on B, we obtain the double groupoid $(B^4; B^2, B^2; B)$ in which all four groupoid structures are pair groupoids.

Note that in most of the Ehresmann literature on double groupoids, the notation $\square\ G$
refers to the double groupoid of *commuting* squares in G.

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There are several possible concepts of action for double groupoids. For our purposes the appropriate concept arises from regarding a double groupoid as a groupoid in the category of groupoids. So, since ordinary groupoids act on maps, we are lead to actions of double groupoids on maps in the category of groupoids, that is, to actions on morphisms of ordinary groupoids. For the standard concept of an action of an ordinary groupoid which underlies the following, see, for example, [10, §2].

Definition 1.5 *Let $(S; H, V; B)$ be a double Lie groupoid, and suppose that G is a Lie groupoid on base M and that $p \colon G \to V$, $p_\circ \colon M \to B$ is a morphism. Then a* horizontal action *of S on $(p, p_\circ)$ consists of actions of the horizontal groupoid S_H on $p \colon G \to V$ and of H on $p_\circ \colon M \to B$ in the standard sense (both denoted by juxtaposition), such that*

(i) $\beta_G(sg) = \widetilde{\beta}_V(s)\beta_G(g)$ and $\alpha_G(sg) = \widetilde{\alpha}_V(s)\alpha_G(g)$ for all $s \in S$, $g \in G$ with $\widetilde{\alpha}_H(s) = p(g)$;

(ii)   given $s_2, s_1 \in S$ and $g_2, g_1 \in G$ such that the two vertical compositions and the two actions in

are defined, we have $(s_2 g_2)(s_1 g_1) = (s_2 \circ_v s_1)(g_2 g_1)$, where juxtaposition denotes the product in G, as well as the action;

(iii)   for all $h \in H$ and $m \in M$ with $\alpha_H(h) = p_\circ(m)$ we have $\widetilde{1}^V_h(1_m) = 1_{hm}$.

There is of course a corresponding concept of *vertical action* of S on a morphism $p \colon G \to H$, $p_\circ \colon M \to B$.

Given a horizontal action of S on $p \colon G \to V$, $p_\circ \colon M \to B$, there are action groupoids $S_H \ltimes p$ on base G and $H \ltimes p_\circ$ on base M, and one can check that there is a double Lie groupoid $(S_H \ltimes p; H \ltimes p_\circ, G; M)$ and a morphism of double Lie groupoids p.249

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$$
\begin{array}{ccc}
S_H \ltimes p & \Longrightarrow & G \\
\big\Downarrow & & \big\Downarrow \\
H \ltimes p_\circ & \Longrightarrow & M \\
& c \qquad\qquad p & \\
& c_0 \qquad\qquad p_0 & \\
& S \Longrightarrow V & \\
& \big\Downarrow \qquad \big\Downarrow & \\
& H \Longrightarrow B &
\end{array}
\qquad (4)
$$

where (c, p) and $(c_\circ, p_\circ)$ are action morphisms of ordinary groupoids. We denote the double groupoid $(S_H \ltimes p; H \ltimes p_\circ, G; M)$ by $S \ltimes (p, p_\circ)$ and call it the *action double groupoid corresponding to the horizontal action of S on $(p, p_\circ)$*. Elements of $S \ltimes (p, p_\circ)$ have the form

$$
\begin{array}{c}
(\widetilde{\beta}_V(s), \beta_G(g)) \\[4pt]
sg \;\;\boxed{\;\;(s, g)\;\;}\;\; g \\[4pt]
(\widetilde{\alpha}_V(s), \alpha_G(g))
\end{array}
$$

with compositions

$$
(s_2, g_2) \; \circ_H \; (s_1, g_1) = (s_2 \; \circ_H \; s_1, g_1); \qquad\qquad (s_2, g_2) \; \circ_V \; (s_1, g_1) = (s_2 \; \circ_V \; s_1, g_2 g_1).
$$

Definition 1.6 *A morphism of double Lie groupoids $(\phi; \phi_H, \phi_V; \phi_B)$ from $(S'; H', V'; B')$ to $(S; H, V; B)$ is a horizontal action morphism if (ϕ, ϕ_V) and (ϕ_H, ϕ_B) are action morphisms of ordinary Lie groupoids.*

Theorem 1.7 *Let $(\phi; \phi_H, \phi_V; \phi_B): (S'; H', V'; B') \to (S; H, V; B)$ be a horizontal action morphism of double Lie groupoids. Then the induced actions of S_H on $\phi_V: V' \to V$ and*

of H on $\phi_B \colon B' \to B$ constitute a horizontal action of S on (ϕ_V, ϕ_B). This construction, and the construction of (4) from a horizontal action, yield mutually inverse equivalences between the category of horizontal actions of the double Lie groupoid S, and the category of horizontal action morphisms into S.

The proof is a routine extension of the corresponding result for ordinary groupoids.

Example 1.8 Let $\phi \colon H \to V$ be a morphism of Lie groupoids over a fixed base B. Then there is a horizontal action ρ of the double groupoid $\mathcal{H} = (H \times H; H, B \times B; B)$ on $(\chi_V \colon V \to B \times B, \mathrm{id} \colon B \to B)$ given by

$$\rho(h_2, h_1)(v) = \phi(h_2)v\phi(h_1)^{-1}.$$

The resulting action double groupoid $\mathcal{H} \ltimes \chi_V$ consists of triples $(h_2, v, h_1) \in H \times V \times H$ with sources and targets

$$\phi(h_2)v\phi(h_1)^{-1} \quad \boxed{\begin{array}{c} h_2 \\ (h_2, v, h_1) \\ h_1 \end{array}} \quad v$$

and compositions

$$(h_2', v', h_1') \circ_H (h_2, v, h_1) = (h_2'h_2, v, h_1'h_1) ; \qquad (h_2', v', h_1') \circ_v (h_2, v, h_1) = (h_2', v'v, h_1).$$

We call $\mathcal{H} \ltimes \chi_V$ the *comma double groupoid* of $\phi \colon H \to V$ and denote it $\Theta(H, \phi, V)$. In the case $H = V, \phi = \mathrm{id}$, we write $\Theta(H)$ and call it the *comma double groupoid* of H.

The horizontal structure of $\Theta(H, \phi, V)$ is precisely the comma category, in the sense of Mac Lane [18], arising from the diagram $H \longrightarrow V \longleftarrow H$. In the differentiable setting, this construction arose from the study of extensions of principal bundles [15, §2].

This construction is less special than it may appear: there are in fact no other actions of double groupoids of this type on anchors. The proof of this result is straightforward:

Proposition 1.9 *Let H and V be Lie groupoids on the same base B, and suppose ρ is an action of $(H \times H; H, B \times B; B)$ on $\chi_V \colon V \to B \times B$, $\mathrm{id} \colon B \to B$. Then $\phi \colon H \to V$ defined by $\phi(h) = \rho(h, 1_b)(1_b)$, where $b = \alpha(h)$, is a morphism over B, and*

$$\rho(h_2, h_1)(v) = \phi(h_2)v\phi(h_1)^{-1}$$

for all compatible $(h_2, h_1) \in H \times H$, $v \in V$.

Example 1.10 Consider an exact sequence of locally trivial Lie groupoids over base B,

$$M \rightarrowtail \Phi \overset{\pi}{\longrightarrow\!\!\!\gg} \Omega. \tag{5}$$

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Choose $b \in B$ and write $P = \Omega_b, G = \Omega_b^b, p = \beta_b, Q = \Phi_b, N = M_b$. Let $\Upsilon = \Phi \ltimes p = \frac{Q \times Q}{N}$ be the associated PBG-groupoid in the sense of [13]. This is both a locally trivial Lie groupoid over P and a principal bundle $\Upsilon(\Phi, G)$ over Φ. Form the Lie groupoid $\Theta = \frac{\Upsilon \times \Upsilon}{G}$ on base Φ associated to $\Upsilon(\Phi, G)$. Since Υ is also a Lie groupoid on base P, the Cartesian square $\Upsilon \times \Upsilon$ is a Lie groupoid on $P \times P$, and since G acts on Υ by Lie groupoid automorphisms, it follows that $\frac{\Upsilon \times \Upsilon}{G}$ is a Lie groupoid over $\frac{P \times P}{G} = \Omega$ (compare [12,§2]). These two structures make Θ a double groupoid $(\Theta; \Phi, \Omega; B)$, and one may now easily see that this is precisely the comma double groupoid corresponding to $\pi \colon \Phi \to \Omega$. Further, there is a morphism of double groupoids $(\Upsilon \times \Upsilon; \Upsilon, P \times P; P) \to (\Theta; \Phi, \Omega; B)$ in which each of the four maps is the quotient projection for a principal action of G.

The philosophy underlying 1.5 and 1.6 may be extended: given any class of morphisms of ordinary groupoids, one may consider morphisms of double groupoids for which the two "horizontal" morphisms lie in this class. In §3 we will need the following further instance of this process.

Definition 1.11 *[10, 3.2(iii)] A split fibration of Lie groupoids is an s-fibration* $\phi \colon G' \to G$, $f \colon B' \to B$ *together with an action of G on $f \colon B' \to B$ and a morphism $s \colon G \ltimes f \to G'$ of groupoids over B' which is right-inverse to* $\phi^* \colon G \to f^*G$.

See [10, 3.3] for the equivalence between split fibrations and general semi-direct products.

Definition 1.12 *A morphism* $(\phi; \phi_H, \phi_V; \phi_B) \colon (S'; H', V'; B') \to (S; H, V; B)$ *of double Lie groupoids is a* horizontally split fibration of double Lie groupoids *if there is given a horizontal action of S on (ϕ_V, ϕ_B) such that the induced map*

$$(6)$$

has a right-inverse $(\sigma; \sigma_\circ, \mathrm{id}_{V'}; id_{B'}): (S \ltimes \phi_V; H \ltimes \phi_B, V'; B') \to (S'; H', V'; B')$ which is a morphism of double differentiable groupoids.

We leave the reader to work out the equivalence between split fibrations and general semi-direct products in the category of double groupoids.

2   THE CORE DIAGRAM OF A DOUBLE GROUPOID

Until 2.4 we consider a fixed double Lie groupoid $(S; H, V; B)$. Let K be the preimage of $\{(1_b^H, 1_b^V)|b \in B\} \subseteq H \underset{\alpha}{\times} V$ under the double source map; that is, the set of all elements of S of the form

$$
\begin{array}{c}
\quad\overset{\textstyle h}{} \\
c\ \ \boxed{\begin{array}{c} \\ v \quad k \quad 1_b^V \\ \\ \end{array}} \\
1_b^H
\end{array}
\qquad\qquad (7)
$$

Since the double source map is a surjective submersion, it follows that K is a closed embedded submanifold of S. Now we define a groupoid structure on K with base B as follows: in terms of (7), the source and target maps are $\alpha_K(k) = b$, $\beta_K(k) = c = \beta_H(\widetilde{\beta}_V(k))$, and composition, denoted $\circ_K$, is defined by

$$
k_2 \circ_K k_1 = \left(k_2 \ \circ_V \ \widetilde{1}_{v_1}^H\right) \ \circ_H \ k_1 = \left(k_2 \ \circ_H \ \widetilde{1}_{h_1}^V\right) \ \circ_V \ k_1
$$

where $v_1 = \widetilde{\beta}_H(k_1)$, $h_1 = \widetilde{\beta}_V(k_1)$. It is easy to check the algebraic conditions that K is a groupoid on B, and the differentiability properties follow because K is a closed embedded submanifold of S. The identity of K at $b \in B$ is $1_b^K = 1_b^2$ and the inverse of $k \in K$ is

$$
k^{-K} = k^{-H} \ \circ_V \ \widetilde{1}_{v^{-1}}^H = k^{-V} \ \circ_H \ \widetilde{1}_{h^{-1}}^V \, ,
$$

where v and h are as in (7). Further, the restrictions of the two target maps   p.253

$$
\partial_H = \widetilde{\beta}_V : K \to H \, ; \qquad \partial_V = \widetilde{\beta}_H : K \to V \, ,
$$

are morphisms of groupoids over B. Lastly, note that if $m, n \in K$ have $\partial_H(m) = 1_b^H$, $\partial_V(n) = 1_b^V$ for some $b \in B$, then $m\circ_K n = m \ \circ_H \ n = \left(1_b^2 \circ_V m\right) \circ_H \left(n \circ_V 1_b^2\right) = \left(1_b^2 \circ_H n\right) \circ_V \left(m \circ_H 1_b^2\right) = n \ \circ_V \ m = n \circ_K m.$

Definition 2.1 K *is the* core groupoid *of* $(S; H, V; B)$, *and* K *together with* $\partial_H : K \to H$ *and* $\partial_V : K \to V$ *is the* core diagram *of* $(S; H, V; B)$.

Let $(\phi; \phi_H, \phi_V; \phi_B) : (S'; H', V'; B') \to (S; H, V; B)$ be a morphism of double Lie groupoids. Then the restriction of ϕ to the core K' of S' is a morphism of the core groupoids $\phi_K : K' \to K$ over $\phi_B : B' \to B$, and $\partial_V \circ \phi_K = \phi_V \circ \partial'_V$, $\partial_H \circ \phi_K = \phi_H \circ \partial'_H$.

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Example 2.2 Given a Lie groupoid $G \rightrightarrows B$, the core groupoid of $(G \times G; G, B \times B; B)$ is G itself, with $\partial_H : G \to G$ the identity and $\partial_V : G \to B \times B$ the anchor of G.

Given locally trivial Lie groupoids H and V on the same base B, the core groupoid of $\square\;(H, V)$ is the Cartesian product $H \underset{B \times B}{\times} V$ in the category of locally trivial Lie groupoids over B (see, for example, [14, I §3]) with ∂_H and ∂_V the natural projections.

Given a morphism $\phi : H \to V$ of Lie groupoids over B, the comma double groupoid $\Theta(H, \phi, V)$ has core groupoid H, with $\partial_V = \phi : H \to V$ and $\partial_H : H \to H$ the identity.

If a double vector bundle is considered to be a double Lie groupoid (with all four groupoid structures totally intransitive), then the core groupoid is precisely the *cœur* vector bundle in the sense of Pradines [20, C §2]) and the two projections ∂_H and ∂_V are zero morphisms.

Certain interesting double groupoids have core diagrams which give no information at all. For example, the double Lie groups of Lu and Weinstein [13], considered as double groupoids, have core groupoids which consist only of identity elements.

We now show that a large class of double Lie groupoids, defined in terms of local triviality conditions, can be reconstructed from their core diagrams. Recall that an ordinary Lie groupoid $G \rightrightarrows B$ is *locally trivial*, if its anchor $\chi : G \to B \times B$ is a surjective submersion [14]; this is the smooth form of the algebraic condition of transitivity. For a double Lie groupoid $(S; H, V; B)$ one might therefore define S to be "horizontally locally trivial" if both $S_H \rightrightarrows V$ and $H \rightrightarrows B$ are locally trivial. However for our purposes the following stronger definition p.254 is more appropriate.

Definition 2.3 *A double Lie groupoid* $(S; H, V; B)$ *is* horizontally locally trivial *if*

$$
\begin{array}{ccc}
S_V & \xrightarrow{\;\;\widetilde{\chi}_H\;\;} & V \times V \\[2pt]
\Big\Downarrow & & \Big\Downarrow \\[2pt]
H & \xrightarrow[\;\;\chi_H\;\;]{} & B \times B
\end{array}
$$

is an s-fibration; it is vertically locally trivial *if* $\widetilde{\chi}_V : S_H \to H \times H$, $\chi_V : V \to B \times B$ *is an s-fibration; it is a* locally trivial double Lie groupoid *if it is both horizontally and vertically locally trivial.*

So if $(S; H, V; B)$ is horizontally locally trivial, both $H \rightrightarrows B$ and $S_H \rightrightarrows V$ are locally trivial Lie groupoids, but $\widetilde{\chi}_H$ also satisfies the further condition that it be a fibration with respect to the *vertical* structure on S. For a set-theoretic double groupoid $(S; H, V; B)$, the corresponding condition of *horizontal transitivity* is equivalent to the filling condition that every configuration of matching sides

$$\tag{8}$$

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are the sides of some element of S. In the differentiable case, the horizontal and vertical local triviality conditions may be regarded as smooth filling conditions. In the set-theoretic case, horizontal transitivity is also equivalent to the condition that $\partial_V \colon K \to V$ is a surjection. For if all such fillers exist then certainly each

$$
\begin{array}{ccc}
v & \begin{array}{|c|} \\ \\ \hline \end{array} & 1_b^V \\
& 1_b^H &
\end{array}
$$

where $b = \alpha(v)$, has a filler, and so ∂_V is surjective. Conversely, if ∂_V is surjective, then a filler of (8) is given by a horizontal composition

$$
\begin{array}{ccccccccc}
v_2 & \boxed{k_2} & 1 & 1 & \boxed{\widetilde{1}_h^V} & 1 & 1 & \boxed{k_1^{-H}} & v_1 \\
& 1 & & & h & & & 1 &
\end{array}
$$

for appropriate k_1, $k_2 \in K$. In the smooth case, the corresponding result is the following.

Proposition 2.4 *Continuing the above notation, $\partial_V : K \to V$ is a surjective submersion if* *and only if $\widetilde{\chi}_H \colon S_V \to V \times V$, $\chi_H \colon H \to B \times B$ is a fibration of Lie groupoids. In particular, S is horizontally locally trivial iff H is locally trivial and $\partial_V \colon K \to V$ is a surjective submersion.*

PROOF: If $(\widetilde{\chi}_H, \chi_H)$ is a fibration, then $\widetilde{\chi}_H^* : S \to \chi_H^*(V \times V)$ is a surjective submersion. Within $\chi_H^*(V \times V) = \{(h, v_2, v_1) | \chi_H(h) = \alpha_{V \times V}(v_2, v_1)\}$, consider the embedded submanifold $E = \{(1_b^H, v, 1_b^V) | v \in V, b = \alpha(v)\}$. Now K is the complete inverse image of E under $\widetilde{\chi}_H^*$ and so it follows that the restriction of $\widetilde{\chi}_H^*$ to $K \to E$, which may be identified with $\partial_V : K \to V$, is a surjective submersion.

Conversely, if ∂_V is a surjective submersion, let the Cartesian square groupoid $K \times K$ on base $B \times B$ act on $\chi_H : H \to B \times B$ as in 1.8, and let $G = (K \times K) \ltimes \chi_H$ be the (ordinary) action groupoid on base H. Then there is a commutative diagram

$$
\begin{array}{ccccccc}
G & \longrightarrow & \chi_H^*(V \times V), & \quad (h, k_2, k_1) & \longmapsto & (h, \partial_V(k_2), \partial_V(k_1)) \\
\downarrow & \nearrow & & \downarrow & & \\
S & \widetilde{\chi}_H^* & & k_2 \circ_H \widetilde{1}_h^V \circ_H k_1^{-H} & &
\end{array}
$$

and since the horizontal arrow is a surjective submersion, it follows that $\widetilde{\chi}_H^*$ is also. The second statement follows immediately. ∎

The core diagram of a locally trivial double Lie groupoid may therefore be represented in the form

$$
\begin{array}{ccc}
M^H & & H \\
& K & \\
M^V & & V
\end{array}
\qquad \partial_H \qquad \partial_V
\tag{9}
$$

where $M^H = \ker(\partial_V)$ and $M^V = \ker(\partial_H)$. Note that M^H and M^V are Lie group bundles, since ∂_V and ∂_H are surjective submersions, and morphisms of Lie groupoids over the fixed base B. Further, M^H and M^V commute elementwise in K, though neither need itself be commutative.

The following are locally trivial double Lie groupoids: double groupoids of the form $(G \times G; G, B \times B; B)$ where $G \rightrightarrows B$ is a locally trivial Lie groupoid; double groupoids of the form $\square\,(H, V)$ where H and V are locally trivial, and comma double groupoids $\Theta(H, \phi, V)$ where H and V are locally trivial and $\phi : H \to V$ is a surjective submersion.

To express the main result, we need the following terminology.

Definition 2.5 *Let H and V be locally trivial Lie groupoids on B. Then a* locally trivial core diagram *for H and V is a locally trivial Lie groupoid K on B together with surjective submersions $\partial_V : K \relbar\!\!\twoheadrightarrow V$, $\partial_H : K \relbar\!\!\twoheadrightarrow H$ whose kernels $M^H = \ker(\partial_V)$, $M^V = \ker(\partial_H)$ commute elementwise in K. If H' and V' are locally trivial Lie groupoids on B', and $(K', \partial'_H, \partial'_V)$ is a locally trivial core diagram for H' and V', then a* morphism of locally trivial core diagrams *is a triple of Lie groupoid morphisms $\phi_K : K' \to K$, $\phi_H : H' \to H$, $\phi_V : V' \to V$, all over a map $\phi_B : B' \to B$, such that $\partial_V \circ \phi_K = \phi_V \circ \partial'_V$, and $\partial_H \circ \phi_K = \phi_H \circ \partial'_H$. If $B' = B$, $H' = H$ and $V' = V$, and if ϕ_H, ϕ_V and ϕ_B are all identities, then ϕ_K is a* morphism of locally trivial core diagrams over H and V.*

We now begin the proof of our main result, 2.7, which gives an equivalence between locally trivial double Lie groupoids and locally trivial core diagrams. As the first step we construct a locally trivial double Lie groupoid from a locally trivial core diagram as in (9). Let $\partial_h : M^H \to H$ and $\partial_v : M^V \to V$ be the restrictions of ∂_H and ∂_V. Define actions ρ_H, ρ_V of H, V on M^H, M^V by $\rho_H(h)(m) = kmk^{-1}$, where k is any element of K with $\partial_H(k) = h$, and $\rho_V(v)(m) = kmk^{-1}$, where now $\partial_V(k) = v$. That these actions are well-defined follows from the commutativity condition on M^H and M^V, and that they are smooth follows because ∂_H and ∂_V are surjective submersions. It is easy to see that $(M^H, \partial_h, H, \rho_H)$ and $(M^V, \partial_v, V, \rho_V)$ are in fact crossed modules over groupoids (see 3.1 for the definition).

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Form the comma double groupoid $\Theta = \Theta(K, \partial_V, V)$. Elements of Θ are of the form (k_2, v, k_1), where $\alpha(k_2) = \beta(v)$ and $\alpha(v) = \alpha(k_1)$. The double groupoid structure is given by

$$
\begin{array}{ccc}
\Theta & \Longrightarrow & V \\
\big\Downarrow & & \big\Downarrow \\
K & \Longrightarrow & B
\end{array}
\qquad\qquad
\partial_V(k_2)\, v\, \partial_V(k_1)^{-1} \quad
\boxed{\begin{array}{c} k_2 \\ (k_2, v, k_1) \\ k_1 \end{array}} \quad v
$$

with compositions

$$
(k_2', v', k_1') \ \circ_H \ (k_2, v, k_1) = (k_2' k_2, v, k_1' k_1)\,; \qquad (k_2', v', k_1') \ \circ_V \ (k_2, v, k_1) = (k_2', v'v, k_1)
$$

and identities $\widetilde{1}_k^V = (k, 1_b^V, k)\,, \widetilde{1}_v^H = (1_c^K, v, 1_b^K)$ where $b = \alpha(k) = \alpha(v)$, $c = \beta(v)$.

Within the horizontal structure $\Theta_H \Longrightarrow V$ consider the inner group bundle (or gauge group bundle, or union of vertex groups)

$$
I_H\Theta = \{(k_2, v, k_1)\mid \partial_V(k_2)\, v\, \partial_V(k_1)^{-1} = v\},
$$

and within this consider the subgroup bundle   p.257

$$
N = \{(m_2, v, m_1)\mid m_2, m_1 \in M^V,\ \rho_V(v)(m_1) = m_2\}\,;
$$

it is straightforward to check that N is a normal (and totally intransitive) subgroupoid of Θ_H, and that it is a closed embedded submanifold. One can therefore form the quotient Lie groupoid $Q = \Theta/N \Longrightarrow V$, whose elements are orbits $\langle k_2, v, k_1\rangle$, subject to $\langle k_2 m_2, v, k_1 m_1\rangle = \langle k_2, v, k_1\rangle$ whenever $m_2, m_1 \in M^V$ have $\rho_V(v)(m_1) = m_2$.

We claim that Q also has a groupoid structure on base H given by source and target maps

$$
\widetilde{\alpha}_V(\langle k_2, v, k_1\rangle) = \partial_H(k_1)\,, \qquad \widetilde{\beta}_V(\langle k_2, v, k_1\rangle) = \partial_H(k_2)\,,
$$

and composition

$$
\langle k_2', v', k_1'\rangle \ \circ_V \ \langle k_2, v, k_1\rangle = \langle k_2', v'v, k_1\rho_V(v^{-1})(m)\rangle
$$

where $m \in M^V$ is determined uniquely by $k_1' = k_2 m$. It is routine to check that this composition is well-defined and makes Q a locally trivial Lie groupoid on H. For example, suppose that $\langle k_2'', v'', k_1''\rangle = \langle k_2', v', k_1'\rangle$. Then $v'' = v'$ and there exist $m_2, m_1 \in M^V$ such that $k_2'' = k_2' m_2$, $k_1'' = k_1' m_1$ and $\rho_V(v')(m_1) = m_2$. Using $\langle k_2'', v'', k_1''\rangle$ in the formula for the composition produces $\langle k_2'', v'v, k_1\rho_V(v^{-1})(mm_1)\rangle$ and it is only necessary to observe that $\rho_V(v'v)(\rho_V(v^{-1})(m_1)) = \rho_V(v')(m_1) = m_2$.

With respect to these two structures, Q is a double Lie groupoid with sources and targets given by

$$
\begin{array}{ccc}
Q & \Longrightarrow & V \\
\big\Downarrow & & \big\Downarrow \\
H & \Longrightarrow & B
\end{array}
\qquad\qquad
\partial_V(k_2)\,v\,\partial_V(k_1)^{-1}\;
\begin{array}{c}
\partial_H(k_2) \\
\boxed{\;\langle k_2, v, k_1\rangle\;} \\
\partial_H(k_1)
\end{array}
\; v
$$

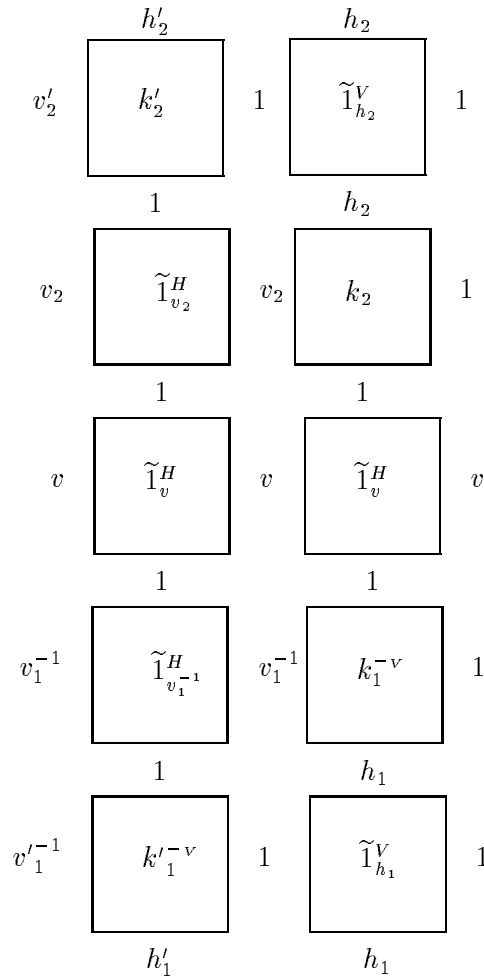The interchange law for Q is no more difficult to verify than that for a comma double groupoid. Indeed, the natural map

$$
\begin{array}{ccc}
\Theta & \Longrightarrow & V \\
\big\Downarrow & & \big\Downarrow \\
K & \Longrightarrow & B \\
& q & \\
& \partial_H & \\
& & Q \Longrightarrow V \\
& & \big\Downarrow \quad\;\; \big\Downarrow \\
& & H \Longrightarrow B
\end{array}
\tag{10}
$$

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where $q : \Theta \to Q$ is $(k_2, v, k_1) \mapsto \langle k_2, v, k_1\rangle$, is a morphism of double groupoids, a base-preserving surjective submersion over V, and a (non-regular) s-fibration over $\partial_H : K \relbar\!\!\gg H$.

Next, observe that the core groupoid of Q consists of all elements of the form $\langle k_2, 1_b^V, m\rangle$, where $m \in M^V$, and that these can be represented uniquely as $\langle k_2', 1_b^V, 1_b^K\rangle$, where $k_2' = k_2 m^{-1}$. There is thus an identification $k \leftrightarrow \langle k, 1_b^V, 1_b^K\rangle$, $b = \alpha(k)$, of K with the core groupoid of Q and this gives an equivalence of the core diagram of $(Q; H, V; B)$ with the given core diagram (9); further, this equivalence preserves H and V. This completes the construction of a locally trivial double Lie groupoid with a preassigned locally trivial core diagram.

Suppose that (9) is the core diagram of a locally trivial double Lie groupoid $(S; H, V; B)$. Then there is a map $\theta: \Theta \to S, (k_2, v, k_1) \mapsto k_2 \;\circ_v\; \widetilde{1}_v^H \;\circ_v\; k_1^{-v}$ which is a morphism of double groupoids with respect to $\partial_H: K \to H$ and $\mathrm{id}: V \to V$. That θ preserves the vertical composition is trivial; that it preserves the horizontal composition follows from interpreting in two ways the following diagram:

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$$
\begin{array}{cccc}
& h'_2 & & h_2 \\
v'_2 \;\boxed{\; k'_2 \;}\; 1 & & \widetilde{1}^V_{h_2} \;}\; 1 \\
\end{array}
$$

h'_2 $\quad$ h_2

$v'_2 \;\; \boxed{k'_2} \;\; 1 \;\;\; \boxed{\widetilde{1}^V_{h_2}} \;\; 1$

$\qquad 1 \qquad\qquad h_2$

$v_2 \;\; \boxed{\widetilde{1}^H_{v_2}} \;\; v_2 \;\; \boxed{k_2} \;\; 1$

$\qquad 1 \qquad\qquad 1$

$v \;\; \boxed{\widetilde{1}^H_{v}} \;\; v \;\; \boxed{\widetilde{1}^H_{v}} \;\; v$

$\qquad 1 \qquad\qquad 1$

$v_1^{-1} \;\; \boxed{\widetilde{1}^H_{v_1^{-1}}} \;\; v_1^{-1} \;\; \boxed{k_1^{-v}} \;\; 1$

$\qquad 1 \qquad\qquad h_1$

$v'^{\,-1}_1 \;\; \boxed{k'^{\,-v}_1} \;\; 1 \;\; \boxed{\widetilde{1}^V_{h_1}} \;\; 1$

$\qquad h'_1 \qquad\qquad h_1$

where (k'_2, v', k'_1), $(k_2, v, k_1) \in \Theta$ and $v' = \partial_V(k_2) v \partial_V(k_1)^{-1} = v_2 v v_1^{-1}$.

$\quad$ To prove that θ is surjective, take any

$\qquad h_2$

$v_2 \;\; \boxed{s} \;\; v_1$

$\qquad h_1$
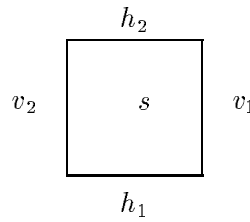
in S. Choose k_1, $k_2 \in K$ such that $\partial_H(k_1) = h_1$, $\partial_H(k_2) = h_2$. Define $m = k_2^{-v} \circ_v s \circ_v k_1 \circ_v \widetilde{1}^H_{v_1^{-1}}$; it is easily verified that $m \in M^V$, and so $k_2 \circ_v m \in K$. Now $(k_2 m, v_1, k_1) \in \Theta$ maps to s.

$\quad$ We now need the following lemma, which is a purely algebraic result.

Lemma 2.6 *The actions associated to the core diagram are also given by* $\rho_V(v)(m) = \widetilde{1}^H_v \circ_v m \circ_v \widetilde{1}^H_{v^{-1}}$, *where* $m \in M^V$, $v \in V$, *and* $\rho_H(h)(m) = \widetilde{1}^V_h \circ_H m \circ_H \widetilde{1}^V_{h^{-1}}$, *where* $m \in M^H$, $h \in H$.

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The lemma implies that for any $(k_2, v, k_1) \in \Theta$ and any $(m_2, v, m_1) \in N$ with which it is compatible, (k_2, v, k_1) and $(k_2 m_2, v, k_1 m_1)$ map under θ to the same element of S. To verify that $\theta \colon Q \to S$ is an isomorphism of double groupoids, we must first determine the kernel of $\Theta \to S$, considered as a morphism of groupoids over V.

Take $(k_2, v, k_1) \in \Theta$ and suppose that it maps to an identity of $S_H \rightrightarrows V$; that is, $k_2 \circ_v \widetilde{1}_v^H \circ_v k_1^{-v} = \widetilde{1}_v^H$. Then $\partial_H(k_2) = \partial_H(\widetilde{1}_v^H)$ and $\partial_H(k_1) = \widetilde{\alpha}_V(\widetilde{1}_v^H)$ are both identities, so $k_1, k_2 \in M^V$ and (again using 2.6), $k_2 = \rho_V(v)(k_1)$. So $(k_2, v, k_1) \in N$. It follows that the induced map $\Theta/N \to S$ is an isomorphism of groupoids over V. Since it is known to be a morphism of groupoids over H, it follows that it is also an isomorphism of groupoids over H, and thus an isomorphism of double groupoids. That it is a diffeomorphism is proved by standard methods.

Finally, let $\phi_K : K' \to K$, $\phi_H : H' \to H$, $\phi_V : V' \to V, \phi_B : B' \to B$ be a morphism of locally trivial core diagrams. Then there is a morphism of the comma double groupoids $\Theta(K', \partial_V', V') \to \Theta(K, \partial_V, V)$ defined by $(k_2', v', k_1') \mapsto (\phi_K(k_2'), \phi_V(v'), \phi_K(k_1'))$ and it is immediate that this maps the corresponding normal subgroup bundle N' to N and therefore induces a morphism of locally trivial double Lie groupoids $\phi : Q' \to Q$ over ϕ_H, ϕ_V, ϕ_B. Further, if $B' = B$, $H' = H$, $V' = V$ and ϕ_K is a morphism of locally trivial core diagrams over H and V, then $\phi : Q' \to Q$ is a morphism of locally trivial double Lie groupoids preserving H and V (and B).

To summarize, we now have a category $\mathcal{DLG}$ of locally trivial double Lie groupoids and a category $\mathcal{LCD}$ of locally trivial core diagrams, and we have functors $\mathcal{K} : \mathcal{DLG} \to \mathcal{LCD}$ and $\mathcal{D} : \mathcal{LCD} \to \mathcal{DLG}$, where $\mathcal{K}$ assigns to a locally trivial double Lie groupoid its core diagram and $\mathcal{D}$ assigns to a locally trivial core diagram $(K, \partial_H, \partial_V)$ the locally trivial double Lie groupoid Q just constructed. We have proved the following theorem.

Theorem 2.7 *The functors $\mathcal{K} : \mathcal{DLG} \to \mathcal{LCD}$ and $\mathcal{D} : \mathcal{LCD} \to \mathcal{DLG}$ defined above are mutually inverse natural equivalences. Further, the adjunctions $\mathcal{KD} \cong \mathrm{id}$ and $\mathcal{DK} \cong \mathrm{id}$ given above preserve the side groupoids H and V.* ∎

Remarks: (i) As was said in the introduction, this correspondence may usefully be thought of as a double groupoid analogue of the correspondence between locally trivial groupoids and principal bundles. Indeed, if one prefers, one may replace a locally trivial core diagram by the corresponding diagram of principal bundles, by taking the vertex bundles of K, H, and V at any chosen point of B.

It should be noted however that there are two other descriptions of classes of locally trivial double Lie groupoids of which this could be said: there is a literal transcription of the locally trivial groupoid—principal bundle correspondence which gives a correspondence between locally trivial double Lie groupoids and principal bundle objects in the category of locally trivial groupoids, and for double Lie groupoids in which both anchors are regular fibrations, there is a description in terms of an exact sequence $M^H \cap M^V \rightarrowtail S \twoheadrightarrow \square(H, V)$. The description given here seems the most powerful in that, unlike the first alternative, we describe locally trivial double Lie groupoids in terms of ordinary groupoids, rather than in terms of merely a different type of double object; and compared to the class of double groupoids described by the second alternative, the locally trivial double Lie groupoids are considerably more general.

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(ii) It is worth emphasizing the importance of the commutativity condition in 2.5. This is crucial to 2.7, and corresponds to the interchange law in the resulting double groupoid. Such commutativity conditions seem to have first been explicitly noted in Loday's [12] definition of a cat^1-group.

(iii) There is a three-fold version of 2.7 in which a suitable triple groupoid is determined by a core groupoid and three morphisms from it to the core groupoids of the three face groupoids: the kernels of these morphisms fit together into three crossed squares [12] over groupoids, subject to several natural compatibility conditions. Presumably there is also an n-fold version.

(iv) We have given a definition of an abstract core diagram only in the locally trivial case, because the kernels of arbitrary morphisms $\partial_H : K \to H, \partial_V : K \to V$ may not be Lie group bundles of locally constant rank, and consideration of such cases would introduce inapposite complications. However in §4 we will indicate how more general adjunctions than those in 2.7 can be obtained in the set-theoretic case.

(v) There is an alternative proof of 2.7, in which the comma double groupoid $\Theta(K, \partial_V, V)$ is replaced by the set of all quadruples of elements from K with common bottom right vertex. One then uses two quotienting processes to obtain Q, corresponding to passing elements of M^H or M^V horizontally or vertically between the constituent elements of K. This proof is considerably more complicated, but has the virtue of preserving symmetry between the horizontal and vertical structures.

(vi) This representation is in some sense dual to the result of A. and C. Ehresmann [7, §D.3] that every double category is canonically embedded as a double subcategory of the double category of squares of its 2-category of strings. There are many differences between the Ehresmanns' theorem and the one presented here, but the most important is probably the fact that, since the 2-category of strings is an algebraic "free object" construction, it is unlikely to admit a differentiable formulation. We are grateful to Madame Ehresmann for this reference.

(vii) Consider again the abstract locally trivial core diagram (9). It is clear that the two associated crossed modules $(M^H, \partial_h, H, \rho_H)$ and $(M^V, \partial_v, V, \rho_V)$ have the same kernel, namely $M^H \cap M^V$. It is also true that the two cokernels are naturally isomorphic. Let C_H and C_V be the quotients of H and V over the normal and totally intransitive subgroupoids $im(\partial_h)$ and $im(\partial_v)$. (Note that C_H and C_V need not be Hausdorff.) Given a coset $\langle h \rangle \in C_H$, choose $k \in K$ with $\partial_H(k) = h$ and define $\psi(\langle h \rangle) = \langle \partial_V(k) \rangle$. Then $\psi : C_H \to C_V$ is an isomorphism of (nonHausdorff) Lie groupoids over B, and preserves the actions of C_H and C_V on $M^H \cap M^V$.

Given locally trivial groupoids H and V on base B, and crossed modules $(M^H, \partial_h, H, \rho_H)$ and $(M^V, \partial_v, V, \rho_V)$ which satisfy these kernel and cokernel conditions, it would be very interesting to know what further conditions are necessary to ensure that there exists an abstract locally trivial core diagram which induces them, and to have an effective classification of such core diagrams. This is the analogue for double groupoids of the problem of describing all locally trivial Lie groupoids with prescribed base and prescribed gauge group bundle (compare [17]). In principal bundle terms, it is the problem solved by the concept of transition function. We treat a very special case of the first part of this problem in the next section.

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3 SPLIT DOUBLE GROUPOIDS

By "split double groupoids" we mean the differentiable analogue of the "special double groupoids with special connection" which were introduced by Brown and Spencer [6], and whose generalization to arbitrary dimensions has been extensively developed by Brown and Higgins ([3], [4] and elsewhere) for proving Generalized Van Kampen Theorems. Special double groupoids with special connection differ from general double groupoids in that their side groupoids are identical and in that they admit "special connections". These special connections encode aspects of the homotopy-theoretical examples with which these authors were at the time chiefly concerned; algebraically, they provide a rotation operation in the double groupoid [6, §4] under which the horizontal and vertical structures are isomorphic.

Special double groupoids with special connection correspond to crossed modules over groupoids ([6] for the case where the double base of the double groupoid is singleton; the general case is an easy extension and included in [3, 6.2]). In this section we deduce a differentiable version of this result from our 2.7, and give a simple characterization of "special double Lie groupoids with special connection" as those locally trivial double Lie groupoids $(S; H, V; B)$ for which $H = V$ and for which both anchors are split fibrations, these splittings being compatible in a natural way. We also consider locally trivial double Lie groupoids in which the side groupoids are distinct and in which only one anchor is split, and show that such double groupoids are characterized by the presence of a "one-sided" version of a special connection. This frees the notion of special connection from the assumption that the side groupoids are identical.

At the time when [6] was written, the notion of special connection was also motivated by the differential-geometric concept of path-lifting. As part of our analysis here we show that the one-sided version of a special connection in a locally trivial double Lie groupoid $(S; H, V; B)$ corresponds precisely to a right-splitting of the exact sequence of locally trivial Lie groupoids $M^V >\!\!-\!\!-\!\!-\!\!> K \overset{\partial_V}{-\!\!-\!\!\gg} V$, or its companion. They may thus be regarded as genuine differential-geometric connections which are subject to the additional requirements that they be flat and without holonomy or, rather, as transverse connections in the sense of [16] which are flat and without holonomy. The question as to whether flat connections with holonomy, or general, not necessarily flat, transverse connections in $M^V >\!\!-\!\!-\!\!> K -\!\!-\!\!\gg V$ induce connections in $S_H \rightrightarrows V$ will be taken up elsewhere.

We begin by recalling the notion of crossed module.

Definition 3.1 *(Compare [3], [17].) Let G be a locally trivial Lie groupoid on base B. A crossed module over G is a quadruple (M, ∂, G, ρ), where $p : M \to B$ is a Lie group bundle on B, where $\partial : M \to G$ is a morphism of Lie groupoids over B, and where ρ is an action of G on M by Lie group isomorphisms such that*

1. *$\partial(\rho(\xi)(m)) = \xi\partial(m)\xi^{-1}$ for all $\xi \in G$, $m \in M$ with $\alpha(\xi) = p(m)$;*

2. *$\rho(\partial(m))(n) = mnm^{-1}$ for all $m, n \in M$ with $p(m) = p(n)$.*

We denote elements of G by $\xi, \eta, \zeta, \ldots$ in order to avoid confusion with earlier conventions. We usually write $\rho(\xi)(m)$ briefly as ξm, and refer to M as a crossed module over G. Condition (i) forces $\partial \colon M \to G$ to be of locally constant rank and it therefore has a well-defined image

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and kernel, both of which are Lie group bundles. However, the image of ∂ need not be closed in G.

We recall the usual construction of a double groupoid from a crossed module [3], at the same time giving a differentiable version of it. Until 3.2, we consider a fixed crossed module (M, ∂, G, ρ) over a locally trivial Lie groupoid G. Let $IG = \bigcup_{b \in B} G_b^b$ be the inner group bundle of G (sometimes called the gauge group bundle). Form the semi-direct product groupoid $G \ltimes IG$ on base B; this is not the semi-direct product used in §1, but rather consists of all pairs (ξ, λ) with $\alpha(\xi) = \alpha(\lambda)$, and composition
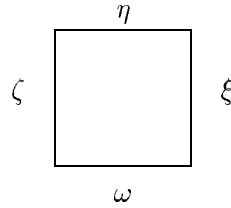
$$(\xi_2, \lambda_2)(\xi_1, \lambda_1) = (\xi_2 \xi_1, I_{\xi_1^{-1}}(\lambda_2)\lambda_1)$$

defined if $\alpha(\xi_2) = \beta(\xi_1)$. Here $I_\xi(\lambda)$ is the groupoid conjugation $\xi \lambda \xi^{-1}$. Next, form the pullback Lie groupoid $\chi^{**}(G \times G)$ of the Cartesian square groupoid over its own anchor; this admits the double groupoid structure $\square\, G$ but we are here considering it merely as an ordinary groupoid. Define a map

$$\delta : \chi^{**}(G \times G) \to G \ltimes IG, \qquad (\zeta, \eta, \omega, \xi) \mapsto (\omega, \omega^{-1}\zeta^{-1}\eta\xi)$$

where ζ, η, ω, ξ are arranged as

with our usual orientation; in particular, ξ is the source and ζ the target. Now δ is a regular fibration over $\alpha : G \to B$, and ∂ is base-preserving, so we can take the pullback in the category of differentiable groupoids of the diagram

$$\chi^{**}(G \times G)$$

$$\Big\downarrow \quad \delta$$

$$G \ltimes M \quad \xrightarrow{\quad id \ltimes \partial \quad} \quad G \ltimes IG.$$

We obtain a groupoid $C = C(M, \partial, G, \rho)$ whose elements are 5-tuples $(m, \zeta, \eta, \omega, \xi,)$ such that $(\zeta, \eta, \omega, \xi) \in \chi^{**}(G \times G)$ and $m \in M$ with $p(m) = \alpha(\xi)$ and $\partial(m) = \omega^{-1}\zeta^{-1}\eta\xi$. To keep the notation clear, we rewrite $(m, \zeta, \eta, \omega, \xi,)$ as $(m; \zeta \; {}^{\eta}_{\omega} \; \xi)$. The source and target of this element are ξ and ζ, respectively, and the composition is

$$(m_2; \zeta_2 \; {}^{\eta_2}_{\omega_2} \; \xi_2) \quad \circ_H \quad (m_1; \zeta_1 \; {}^{\eta_1}_{\omega_1} \; \xi_1) \quad = \quad ((\omega_1^{-1}m_2)m_1; \zeta_2 \; {}^{\eta_2\eta_1}_{\omega_2\omega_1} \; \xi_1),$$

defined if $\xi_2 = \zeta_1$. Now C becomes a double groupoid by defining a vertical structure, whose source and target maps are the displayed horizontal edges and whose composition is

$$(m_2; \zeta_2 \; {}^{\eta_2}_{\omega_2} \; \xi_2) \quad \circ_V \quad (m_1; \zeta_1 \; {}^{\eta_1}_{\omega_1} \; \xi_1) \quad = \quad (m_1(\xi_1^{-1}m_2); \zeta_2\zeta_1 \; {}^{\eta_2}_{\omega_1} \; \xi_2\xi_1),$$

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defined if $\omega_2 = \eta_1$. The check that $(C; G, G; B)$ is a double groupoid is straightforward. In the terminology of [3] it is the *special double groupoid with special connection* corresponding to the crossed module (M, ∂, G, ρ). The elements of the core groupoid of C are of the form $(m; \zeta \; {}_1^{\eta} \; 1)$, where $\zeta \partial(m) = \eta$, and the core composition is

$$
\begin{aligned}
(m_2; \zeta_2 \; {}_1^{\eta_2} \; 1) \quad \circ_K \quad (m_1; \zeta_1 \; {}_1^{\eta_1} \; 1) \quad &= \quad ((m_2; \zeta_2 \; {}_1^{\eta_2} \; 1) \quad \circ_H \quad (1; 1 \; {}_{\eta_1}^{\eta_1} \; 1)) \quad \circ_V \quad (m_1; \zeta_1 \; {}_1^{\eta_1} \; 1) \\
&= \quad (\eta_1^{-1}(m_2); \zeta_2 \; {}_{\eta_1}^{\eta_2 \eta_1} \; 1) \quad \circ_V \quad (m_1; \zeta_1 \; {}_1^{\eta_1} \; 1) \\
&= \quad (m_1 \, \eta_1^{-1}(m_2); \zeta_2 \zeta_1 \; {}_1^{\eta_2 \eta_1} 1).
\end{aligned}
$$

If we now denote $(m; \zeta \; {}_1^{\eta} \; 1)$ by (ζ, m) this becomes

$$
(\zeta_2, m_2) \; \circ_K \; (\zeta_1, m_1) = (\zeta_2 \zeta_1, \zeta_1^{-1}(m_2) m_1)
$$

and so the core groupoid is the semi-direct product $G \ltimes M$. In this notation $\partial_V(\zeta, m) = \zeta$ and $\partial_H(\zeta, m) = \zeta \partial(m)$. Thus M^H can be canonically identified with M via $m \leftrightarrow (1, m)$, and M^V can be canonically identified with M^{opp}, the opposite Lie group bundle to M, via $m \leftrightarrow (\partial(m)^{-1}, m)$. Thus the core diagram of $(C; G, G; B)$ is

$$(11)$$

Since both projections are surjective submersions, it follows that C is a locally trivial double Lie groupoid. Note that the horizontal crossed module $(M, \partial_h, G, \rho_h)$ is the given (M, ∂, G, ρ), whilst the vertical crossed module $(M^{opp}, \partial_v, G, \rho_v)$ has $\partial_v(m) = \partial(m)^{-1}$ but $\rho_v = \rho$.

The most distinctive feature of (11) is that both the exact sequences are split, and by the same map, $\zeta \to (\zeta, 1)$.

Definition 3.2 *Let H and V be locally trivial Lie groupoids on base B, and let $(K, \partial_H, \partial_V)$ be a locally trivial core diagram for H and V. Then a* horizontal splitting *for $(K, \partial_H, \partial_V)$ is a morphism $\sigma \colon V \to K$ of Lie groupoids over B such that $\partial_V \circ \sigma = \mathrm{id}$, and $(K, \partial_H, \partial_V)$ is* horizontally split *if it admits a horizontal splitting. Similarly, a* vertical splitting *for $(K, \partial_H, \partial_V)$ is a morphism $\sigma \colon H \to K$ of Lie groupoids over B such that $\partial_H \circ \sigma = \mathrm{id}$, and $(K, \partial_H, \partial_V)$ is* vertically split *if it admits a vertical splitting.*

Concerning splittings we have the following general result.

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Theorem 3.3 *Let $(S; H, V; B)$ be a locally trivial double Lie groupoid with Lie core diagram $(K, \partial_H, \partial_V)$. Then $(K, \partial_H, \partial_V)$ is vertically split if and only if $(\widetilde{\chi}_V; \mathrm{id}, \chi_V; \mathrm{id}) : (S; H, V; B) \to (H \times H; H, B \times B; B)$ is a horizontally split fibration of double Lie groupoids, and $(K, \partial_H, \partial_V)$ is horizontally split if and only if $(\widetilde{\chi}_H; \chi_H, \mathrm{id}; \mathrm{id}) : (S; H, V; B) \to (V \times V; B \times B, V; B)$ is a vertically split fibration of double Lie groupoids.*

PROOF: Suppose $M^V \rightarrowtail K \xrightarrow{\partial_H} \!\!\!\!\!\gg H$ is split by $\sigma : H \to K$, and write $\phi = \partial_V \circ \sigma : H \to V$. Referring back to 1.12, we use ϕ to define a horizontal action of $\mathcal{H} = (H \times H; H, B \times B; B)$ on (χ_V, id_B) as in 1.8, and the resulting double groupoid is $\Theta = \Theta(H, \phi, V)$. So we have to define a morphism of double groupoids $\Theta \to S$ which is right-inverse to the map $S \to \Theta$ induced by $(\widetilde{\chi}_V; \mathrm{id}_B, \chi_V; \mathrm{id}_B)$. We know from 2.7 that it is suffcient to define a morphism of the corresponding core diagrams. The core diagram of Θ is (H, id_H, ϕ) (see 2.2) and we are denoting the core diagram of S by the usual $(K, \partial_H, \partial_V)$. Now it is easy to see that the equations $\partial_H \circ \sigma = \mathrm{id}_H$, $\partial_V \circ \sigma = \phi$ express precisely the fact that $(\sigma, \mathrm{id}_H, \mathrm{id}_V)$ is a morphism of abstract core diagrams $(H, \mathrm{id}_H, \phi) \to (K, \partial_H, \partial_V)$ with the required property.

Conversely, suppose that $(\widetilde{\chi}_V; \mathrm{id}_H, \chi_V; \mathrm{id}_B)$ is a horizontally split fibration of double groupoids. Then $\mathcal{H}$ acts on (χ_V, id_B) and, by 1.9, this must be the action induced by a morphism $\phi : H \to V$. Forming the action double groupoid, namely $\Theta = \Theta(H, \phi, V)$, there is also, by assumption, a morphism of double groupoids $s : \Theta \to S$ (with respect to $\mathrm{id}_H, \mathrm{id}_V$ and id_B) which is right-inverse to the map $S \to \Theta$ induced by $\widetilde{\chi}_V$. The values of s have the form

$$
\phi(h_2)v\phi(h_1)^{-1} \quad
\begin{array}{c}
h_2 \\
\boxed{s(h_2, v, h_1)} \\
h_1
\end{array}
\quad v
$$

Now s is a morphism of double groupoids, so it induces a morphism of the core groupoids $\sigma = s_K : H \to K$, which assigns to $h \in H$ the element

$$
\phi(h) \quad
\begin{array}{c}
h \\
\boxed{\begin{array}{c}\sigma(h) = \\ s(h, 1_b^V, 1_b^H)\end{array}} \\
1_b^H
\end{array}
\quad 1_b^V
$$

This $\sigma : H \to K$ is a morphism of core diagrams $(H, \mathrm{id}_H, \phi) \to (K, \partial_H, \partial_V)$ preserving H and V and so $\partial_H \circ \sigma = \mathrm{id}_H$ and $\partial_V \circ \sigma = \phi$. In particular, $M^V \rightarrowtail K \xrightarrow{\partial_H} \!\!\!\!\!\gg H$ is split. p.266
The second statement is proved in the same way. ∎

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Definition 3.4 *A locally trivial double Lie groupoid* $(S; H, V; B)$ *is* vertically split *if the anchor* $(\widetilde{\chi}_V; \mathrm{id}, \chi_V; \mathrm{id}) : (S; H, V; B) \to (H \times H; H, B \times B; B)$ *is a horizontally split fibration of double Lie groupoids, and it is* horizontally split *if* $(\widetilde{\chi}_H; \chi_H, \mathrm{id}; \mathrm{id}) : (S; H, V; B) \to (V \times V; B \times B, V; B)$ *is a vertically split fibration of double Lie groupoids.*

Comma double groupoids $\Theta(H, \phi, V)$ with ϕ a surjective submersion, are vertically split, and of course special double groupoids with special connection $C(M, \partial, G, \rho)$ are both horizontally and vertically split.

Theorem 3.3 gives a specific correspondence between splittings of $M^V \rightarrowtail K \xrightarrow{\partial_H} H$ and splittings of $(\widetilde{\chi}_V; \mathrm{id}_H, \chi_V; \mathrm{id}_B)$. Given $\sigma : H \to K$ with $\partial_H \circ \sigma = \mathrm{id}$ and writing $\phi = \partial_V \circ \sigma$ as before, the right-inverse to $\widetilde{\chi}_V^* : S \to \chi_V^*(H \times H)$ is

$$(v, h_2, h_1) \mapsto \ \sigma(h_2) \ \circ_v \ \widetilde{1}_v^H \ \circ_v \ \sigma(h_1)^{-v} = \left(\begin{array}{ccc} & h_2 & \\ \phi(h_2)v\phi(h_1)^{-1} & \boxed{} & v \\ & h_1 & \end{array} \right)$$

This includes the formulas given in [2, §1] which, for a special double groupoid with special connection, express squares with commuting boundary in terms of the special connection and identity elements (the *thin elements*).

We now give a complete description of horizontally split double Lie groupoids. Let $(S; H, V; B)$ be a locally trivial double Lie groupoid which is horizontally split by $\sigma : V \to K$. Write $\phi = \partial_H \circ \sigma : V \to H$ and $M = M^H$, $\rho = \rho_H$, $\partial = \partial_h : M \to H$. Then the core diagram of S is

$$
\begin{array}{ccccc}
M & & & & H \\
& \searrow & \xrightarrow{\;\;\;\;\;} & \nearrow{\scriptstyle \partial_H} & \\
& & V \ltimes M & & \\
& \nearrow & {\scriptstyle \pi} & \searrow & \\
M^V & & & & V
\end{array}
\tag{12}
$$

where π is the canonical projection, $\partial_H(v, m) = \phi(v)\partial(m)$, and the semi-direct product is with respect to the action $vm = \rho(\phi(v))(m)$ for $v \in V$, $m \in M$ such that $\alpha(v) = p(m)$. The following result is a converse to this description.

Theorem 3.5 *Let H and V be locally trivial Lie groupoids on base B, let (M, ∂, H, ρ) be a crossed module over H, and let $\phi : V \to H$ be a surjective submersion and a morphism over* p.267 *B. Let V act on M by $vm = \rho(\phi(v))(m)$, and define $M^V = \{(v, m) \in V \ltimes M \mid \phi(v)\partial(m) = 1_b, \ \exists b \in B\}$, and $\partial_H : V \ltimes M \to H$ by $(v, m) \mapsto \phi(v)\partial(m)$. Then there is a locally trivial double Lie groupoid S, necessarily horizontally split, whose core diagram is precisely (12) above, and which possesses a horizontal splitting $\sigma : V \to V \ltimes M$ such that $\partial_H \circ \sigma = \phi$. This S is unique, up to isomorphisms which preserve H and V.*

PROOF: It suffices to verify that the given conditions on H, V, M, ∂, ρ and ϕ permit an abstract locally trivial core diagram (12). The main point is to verify that the kernels of $\pi : V \ltimes M \to M$ and $\partial_H : V \ltimes M \to H$ commute in $V \ltimes M$. Take $(1_b^V, m) \in ker(\pi)$, where $b = p(m)$, and $(v, n) \in ker(\partial_H)$, with $p(n) = p(m)$. Now $(v, n)(1_b, m) = (v, nm)$ and $(1_b, m)(v, n) = (v, \rho(\phi(v)^{-1})(m)n) = (v, \rho(\partial(n))(m)n) = (v, nm)$, using $\phi(v)^{-1} = \partial(n)$ and (ii) of 3.1. Thus (12) defines a locally trivial core diagram, and the rest follows from 2.7 and 3.3. ∎

Clearly by taking $H = V$ and $\phi = \mathrm{id}$ in 3.5, one recovers the construction given at the start of this section. If one further assumes that B is a singleton, then (12)—or (11)—is precisely the cat^1-group (Loday [12]) corresponding to the given crossed module.

Definition 3.6 *Let G be a locally trivial Lie groupoid on B, and let*

$$(13)$$

be a locally trivial core diagram. Then (13) is a split core diagram if there exists $\sigma : G \to K$ such that $\partial_H \circ \sigma = \partial_V \circ \sigma = \mathrm{id}$.

Now (11) is a split core diagram, and it is easy to see that every split core diagram is of the form (11). Thus there is a bijective correspondence between crossed modules and split core diagrams, and this correspondence can be elaborated to an equivalence of categories in a straightforward way. In the case where B is singleton, this is the equivalence between cat^1-groups and crossed modules [12].

Since locally trivial double Lie groupoids are determined by their core diagrams, we have effectively characterized those locally trivial double Lie groupoids which arise from crossed

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modules; they are the split double groupoids in the following sense. For most purposes it is easier to work in terms of the core diagram.

Definition 3.7 *Let $(S; G, G; B)$ be a locally trivial double Lie groupoid with both side group-*
oids G. Then S is a split double groupoid *if there exist a horizontal splitting $\sigma_H \colon \chi^*(G \times G) \to S_V$ and a vertical splitting $\sigma_V \colon \chi^*(G \times G) \to S_H$ such that $\widetilde{\chi}_V \circ \sigma_H$ and $\widetilde{\chi}_H \circ \sigma_V$ are both equal to $\chi^*(G \times G) \to G \times G$, $(\omega, (\zeta, \xi)) \mapsto (\zeta \omega \xi^{-1}, \omega)$.*

In a general locally trivial core diagram, splittings of $M^V \rightarrowtail K \xrightarrow{\partial_H} H$ may be regarded as a very special kind of transverse connection in the sense of [16]. The question, to what extent general transverse connections in the core diagram of a general locally trivial double Lie groupoid S induce connections in the groupoid structures on S, will be taken up elsewhere.

4   GENERALIZATIONS

The set-theoretic result underlying 2.7 gives an equivalence of categories between transitive double groupoids and transitive core diagrams (the terminology should need no explanation). In this section we consider generalizations of this result in which the transitivity conditions are weakened.

Definition 4.1 *An* abstract core diagram *consists of a commutative diagram of morphisms of groupoids, each the identity on objects,*

$$
\begin{array}{ccc}
M^H & & H \\
& \stackrel{i_H}{\searrow} \quad \stackrel{\partial_H}{\nearrow} & \\
& K & \qquad\qquad (14) \\
& \stackrel{i_V}{\nearrow} \quad \stackrel{\partial_V}{\searrow} & \\
M^V & & V
\end{array}
$$

where i_H and i_V are injections, and may be regarded as inclusions, together with actions ρ_H of H on M^H and ρ_V of V on M^V, such that
  (i) M^H is the kernel of ∂_V and M^V is the kernel of ∂_H;
  (ii) if $m \in M^V$ and $k \in K$ are such that kmk^{-1} is defined, then $kmk^{-1} = \rho_V(\partial_V(k))(m)$;
  (iii) if $m \in M^H$ and $k \in K$ are such that kmk^{-1} is defined, then $kmk^{-1} = \rho_H(\partial_H(k))(m)$;
  (iv) the morphism $\partial_v = \partial_V \circ i_V$ and the action ρ_V form a crossed module $(M^V, \partial_v, V, \rho_V)$;
  (v) the morphism $\partial_h = \partial_H \circ i_H$ and the action ρ_H form a crossed module $(M^H, \partial_h, H, \rho_H)$

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Because ∂_H and ∂_V are not assumed to be surjective in this definition, the actions ρ_H and ρ_V cannot be deduced from conjugation by K, as in §2, but must be built into the structure. Note, however, that each of (ii) and (iii) imply one of the crossed module conditions included in (iv) and (v), namely that corresponding to (ii) in 3.1. That the images of M^V and M^H commute in K is now not explicitly required, but follows from (ii) (or (iii)) and (i).

A *semicore diagram* is the part of (14) consisting of M^V, K, V, the action ρ_V, and the associated morphisms, i_V, ∂_V, and $\partial_v = \partial_V \circ i_V$, subject to (ii) and (iv). Thus M^V is a totally intransitive subgroupoid of K, and is normal in K by virtue of (ii). It follows that any semicore diagram can be completed to an abstract core diagram in which $H = K/M^V$, and $M^H = \ker(\partial_V)$.

Clearly there are categories $\mathcal{ACD}$ and $\mathcal{SCD}$ of abstract core diagrams and semicore diagrams, respectively. Let $\mathcal{U}: \mathcal{ACD} \to \mathcal{SCD}$ be the forgetful functor. From the algebra underlying 2.1 we have a functor $\mathcal{K}: \mathcal{DG} \to \mathcal{ACD}$, where $\mathcal{DG}$ is the category of (set-theoretic) double groupoids.

We now extract the algebraic heart (*"le cœur algébrique"*) of the arguments in §2.

Theorem 4.2 *There is a functor $\mathcal{D}: \mathcal{SCD} \to \mathcal{DG}$ such that $\mathcal{D}$ is left adjoint to $\mathcal{UK}$.*

Most of the proof has been carried out in the course of §2. We restrict ourselves to commenting on the features which are new.

Let $(S; H, V; B)$ be a double groupoid and let K, ∂_V and M^V be as in 2.1. The action ρ_V cannot be defined in terms of conjugation by K, since ∂_V may not be surjective. Instead we define (compare 2.6)

$$\rho_V(v)(m) = \widetilde{1}_v^H \circ_v m \circ_v \widetilde{1}_{v^{-1}}^H$$

for $v \in V$, $m \in M^V$ compatible. A suitable modification of 2.6 then establishes (ii) of 4.1, but (iv) requires a new proof.

Take compatible $m, n \in M^V$, and let $v = \partial_V(m)$, $w = \partial_V(n)$. It has to be proved that $\widetilde{1}_v^H \circ_v n \circ_v \widetilde{1}_{v^{-1}}^H = mnm^{-1}$, where the right hand side is calculated in K. In fact it is easy to see that $m^{-K} = m^{-V}$ and $m \circ_K n \circ_K m^{-K} = m \circ_v n \circ_v m^{-V}$. It now suffices to prove that the composite of

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$$
\begin{array}{cc}
\overset{\displaystyle 1}{\boxed{\;m^{-H}\;}}\;v & \overset{\displaystyle 1}{\boxed{\;\widetilde{1}^H_v\;}}\;v \\[2pt]
\overset{\displaystyle 1}{\boxed{\;n^{-H}\;}}\;w & \overset{\displaystyle 1}{\boxed{\;n\;}} \\[2pt]
\overset{\displaystyle 1}{\boxed{\;(m^{-v})^{-H}\;}}\;v^{-1} & \overset{\displaystyle 1}{\boxed{\;\widetilde{1}^H_{v^{-1}}\;}}\;v^{-1} \\[2pt]
1 & 1
\end{array}
$$

reduces to a double identity. This follows because the middle terms cancel horizontally, and the two columns then collapse.

The construction of the functor $\mathcal{D}$ follows precisely the steps given in the proof of 2.7, except that the action of V on M^V is now built into the basic structure.

The adjointness rule,

$$
\mathcal{D}\mathcal{G}(\mathcal{D}(K, M^V, V), S) \cong \mathcal{S}\mathcal{C}\mathcal{D}((K, M^V, V), \mathcal{U}\mathcal{K}(S)),
$$

where (K, M^V, V) is a semicore diagram and S is a double groupoid, is obtained as follows. First, it is clear that $\mathcal{U}\mathcal{K}\mathcal{D}$ is naturally equivalent to the identity. Thus the forward map of the adjunction is essentially restriction. For the reverse map, the crux is that every element of $\mathcal{D}(K, M^V, V)$ can be written as a vertical composition $k_2 \circ_v \widetilde{1}^H_v \circ_v k_1^{-v}$, where $k_1, k_2 \in K$ and $v \in V$. (Compare the argument immediately preceding 2.6.) It follows that a morphism of semicore diagrams defined on (K, M^V, V) induces a map defined on $\mathcal{D}(K, M^V, V)$, and the verification that this is a morphism of double groupoids with the required properties is straightforward. This completes our comments on the proof of 4.2.

Remarks 4.3 (i) A double groupoid in the image of $\mathcal{D}$, or isomorphic to one such, has the two properties:

(4.3.1) $\partial_H \colon K \to H$ is surjective;

(4.3.2) Every element of S can be represented in the form $k_2 \circ_v \widetilde{1}^H_v \circ_v k_1^{-v}$, where $k_1, k_2 \in K$ and $v \in V$.

Thus we find that the category of semicore diagrams is equivalent to that of double groupoids with these two properties. Indeed the latter category is a reflective subcategory of the category $\mathcal{D}\mathcal{G}$.

(ii) There do exist interesting double groupoids which do not satisfy (4.3.1) or (4.3.2). The double Lie groups of [13], considered as double Lie groupoids, have already been mentioned. Here is an example in which both ∂_V and ∂_H may fail to be surjective while the core groupoid does contain nonidentity elements. Let $\phi \colon H \to P$ and $\psi \colon V \to P$ be morphisms of groups,

where H and V share a common normal subgroup M, and assume that ϕ and ψ agree on M. Define a double groupoid $(S; H, V; *)$ where S is the set of quintuples

$$(m; v_2 \begin{smallmatrix} h_2 \\ h_1 \end{smallmatrix} v_1)$$

with $\phi(m) = \phi(h_1)^{-1}\psi(v_2)^{-1}\phi(h_2)\psi(v_1)$ and double groupoid structure defined by the evident modification of the construction in §3. Then the core groupoid K will usually be nontrivial, but $\partial_H : K \to H$ will be surjective if and only if for each $h \in H$ there exists $v \in V$ such that $\psi(v)\phi(h) \in \phi(M)$. By taking H and V to be subgroups of P, with $M = \{1\}$ but $H \cap V \neq \{1\}$, an S with the specified properties can be found. ∎

In conclusion, it seems reasonable to expect that the classification of double groupoids will p.271 be more difficult, and exhibit a wider range of possibilities, than that for ordinary groupoids. Double groupoids are intrinsically complicated objects, as is already shown by the fact that certain classes of double groupoids include complete information on all homotopy 2-types. Perhaps we should not even expect there to be descriptions of *all* double groupoids in terms of other more familiar structures, but rather regard double groupoids themselves as basic objects in mathematics. On the other hand, where such descriptions are available, they can be of considerable use.

Acknowledgements

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