General Topology and its Applications 8 (1978) 165–179.

ON THE APPLICATION OF FIBRED MAPPING SPACES TO EXPONENTIAL LAWS FOR BUNDLES, EX-SPACES AND OTHER CATEGORIES OF MAPS

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Received 7 August 1976

A previous paper constructed exponential laws in the category $\mathbf{Top}_B$ of spaces over B. The present paper relates these laws to constructions known for locally trivial maps, and constructs also new exponential laws for ex-spaces, fibred section spaces and fibred relative lifting spaces. Versions of these laws for homotopy classes of maps are discussed.

AMS/MOS Subj. Class. (1970): Primary: 54C35, 55F10, 55F99
Secondary: 54D50

associated principal fibration	fibred mapping space
associated principal bundle	k-space
exponential law	principal G-bundle
ex-space	vector bundle

0. Introduction

Let $q : Y \to B$, $r : Z \to B$ be maps of topological spaces. In a previous paper [6] we described a *fibred mapping space over B*

$$(qr) : (YZ) \to B$$

and established some basic properties including "fibred exponential laws" (i.e. exponential laws in the category $\mathbf{Top}_B$ of spaces over B).

The aim of this paper is to introduce some new fibred exponential laws, for example in the category of ex-spaces of James [13] and others, and in other categories derived from **Top**. We shall also see that the corresponding fibred mapping spaces include many cases which are scattered in the literature and have been constructed in an *ad hoc* manner, sometimes only for the locally trivial case.

In Section 1 we consider the projection $s^*(qr) : A \sqcap (YZ) \to A$ induced from (qr) by map $s : A \to B$, and also the projection $(s^*q\ s^*r) : (A \sqcap Y\ A \sqcap Z) \to A$; we prove these projections are equivalent if A is locally compact and Y, A, B are Hausdorff, and apply this in Section 2 to relate (qr) to known constructions for locally trivial maps (in particular to the vector bundle $\mathrm{Hom}(q, r)$ when q, r are vector bundles [2, p. 8], and to the functional principal G-bundles of [7, pp. 249–250]).

In Section 3 we consider q to be a projection $F \times B \to B$, and find conditions for $(F \times B\, Z)$ to be imbedded in the space $M(F, Z)$ of maps $F \to Z$ with the compact open topology. This enables us to relate (qr) to the fibration $\mathrm{prin}\, r : \mathrm{Prin}_F Z \to B$ associated with a fibration $r : Z \to B$ with fibre F (see [1, p. 120; 15, p. 241; 9, p. 434]) and to the associated principal bundle from the Ehresmann–Feldbau point of view [16, p. 39].

In Section 4 we develop the ex-exponential law for ex-spaces and relate this to the ex-space projection $(M/x)(\xi_1, \xi_2)$ of [3, p. 372]. In Section 5 we prove an exponential law for "fibred section spaces" and relate this to a map $p_E : E \to C$ of [12, p. 461] (misprinted there as $p_E : E \to B$), and to the fundamental theorem of topoi, which gives in the category of sets a right adjoint to a pull back functor q^* (see [10, p. 24]).

All the above examples are special cases of the "fibred relative lifting space" $Y \pitchfork Z$ of Section 6, which also has its exponential law.

In Section 7 we develop homotopy versions of the fibred exponential laws of our previous paper [6]. The outline of arguments in the convenient category of $\mathfrak{k}$-spaces analogous to the rest of our discussion is given in Section 8. Such versions of Section 7 have already been applied in [4, 5] to problems in homotopy theory.

A sequal to this paper will discuss conditions for our fibred mapping spaces to be fibrations over B.

The terminology and notation of [6] will be used throughout.

1. Induced fibred mapping spaces

Let $s : A \to B$ be a map of topological spaces. Then s induces a functor $s^* : \mathbf{Top}_B \to \mathbf{Top}_A$. In this section we determine the effect of s^* on fibred mapping spaces.

Suppose then given a diagram

$$\begin{array}{ccccccc} A \sqcap Y & & A \sqcap Z & & Y & & Z \\ & {\scriptstyle s^*q} \searrow & \swarrow {\scriptstyle s^*r} & & & {\scriptstyle q} \searrow & \swarrow {\scriptstyle r} \\ & & A & \xrightarrow{\quad s \quad} & B & & \end{array}$$

where A, B are T_1. Then $s^*(qr) : A \sqcap (YZ) \to A$ and $(s^*q\, s^*r) : (A \sqcap Y\, A \sqcap Z) \to A$ are well-defined. Also there is a map

$$\alpha : s^*(qr) \to (s^*q\, s^*r),$$

$\alpha(a,f)(a,y) = (a, f(y))$ if $s(a) = (qr)(f) = q(y)$ (if $Y_{s(a)} = \emptyset$, then f is the empty map with range $Z_{s(a)}$ and $\alpha(a,f)$ is the empty map with range $\{a\} \times Z_{s(a)}$). Clearly α is a well-defined bijection.

Theorem 1.1. *In the above situation, if Y, A, B are Hausdorff, then α is continuous. If A is locally compact and B is Hausdorff, then α^{-1} is continuous.*

The proof depends on the following result.

Lemma 1.2. *Let $p : X \to B$, $q : Y \to B$, $r : Z \to B$ be maps, let X, Y be Hausdorff and let $\mathbf{U}$ be a sub-basis for the open sets of Z. Then as sub-basis sets of the first kind for $(X \sqcap Y\, Z)$ we may take $W(C \sqcap D, U)$ for C, D compact in X, Y respectively and $U \in \mathbf{U}$.*

Proof of Lemma 1.2. The sets $U^{\sim}$, $U \in \mathbf{U}$, form a sub-basis for the topology of $Z^{\sim}$ and hence by [8, p. 264] the sets $W(C \times D, U^{\sim})$ form a sub-basis for the open sets of $\mathbf{M}(X \times Y, Z^{\sim})$. Hence a sub-basis for the open sets of $\mathbf{P}(X \times Y, Z)$ is formed by the sets $W(C \times D, U)$, which intersect $(X \sqcap Y\, Z)$ in the sets $W(C \sqcap D, U)$.

Proof of Theorem 1.1. To prove the continuity of α, suppose that $(a,f) \in A \sqcap (YZ)$. If $W(N)$ is a sub-basic neighbourhood of $\alpha(a,f)$, where N is an open neighbourhood of a, then $\alpha(N \sqcap (YZ)) \subseteq W(N)$. Let $S = W(C \sqcap D, U \sqcap V)$ be a sub-basic neighbourhood of the other kind of $\alpha(a,f)$, where C, D are compact in A, Y respectively and U, V are open in A, Z respectively. Now $\alpha(a', g)(C \sqcap D) \subseteq U \sqcap V$ is equivalent to $\{a'\} \sqcap C \subseteq U$ and $g(D) \subseteq V$. It follows that $\alpha(U \sqcap W(D, V)) \subseteq S$, and so α is continuous at (a,f).

We now define an inverse to α

$$\beta : (s^*q\, s^*r) \to s^*(qr)$$

by $\beta(f) = (a, f')$ where $a \in A$, $f \in (s^*q\, s^*r)^{-1}(a)$ and $f' : Y_{s(a)} \to Z_{s(a)}$ is the map such that $f(a,y) = (a, f'(y))$ for all $y \in Y_{s(a)}$. It is easily seen that $\beta = \alpha^{-1}$. If we write $\beta = (\beta_1, \beta_2)$ then $\beta_1 = s^*(qr)$ which is continuous, so it is enough for continuity of β to prove β_2 continuous.

Let $W(V)$ be a sub-basic neighbourhood of f', where V is an open neighbourhood of $s(a)$. Then there is a neighbourhood N of a such that $s(N) \subseteq V$ and $\beta_2(W(N)) \subseteq W(V)$. Let $T = W(C, U)$ be a sub-basic neighbourhood of f' of the other kind, so that C is compact, U is open and $f'(C) \subseteq U$. Let D be a compact neighbourhood of a; then $R = W(D \sqcap C, D \sqcap U) \cap W(D)$ is a neighbourhood of f such that $\beta_2(R) \subseteq T$. So β_2 is continuous.

Example 1.3. Let $q : G \times A \to A$, $r : H \times A \to A$, $p : M(G,H) \times A \to A$ denote the projections of the products. (i) If A is T_1, there is a bijection $\alpha : p \to (qr)$ such that $\alpha(f, a) = f \times 1_{\{a\}} : G \times \{a\} \to H \times \{a\}$. (ii) If G, A are Hausdorff, then α is continuous. (iii) If A is locally compact, then α^{-1} is continuous.

Proof. This follows from Theorem 1.1 on taking B to be a point.

Remark 1.4. The locally compact condition in Theorem 1.1 cannot be dropped since by [6, Example 5.5] if G is Hausdorff with more than one point and A is Hausdorff but not locally compact, then $M(G, G) \times A$ is Hausdorff but $(G \times A, G \times A)$ is not.

However, there are other circumstances where α of Theorem 1.1 is a homeomorphism, and a particular useful case for many applications is when $s : A \to B$ is the inclusion of a subspace. We then write $Y \mid A$ for $q^{-1}(A)$ and $q \mid A$ for the restriction of q mapping $Y \mid A \to A$. To within an obvious homeomorphism we can identify $s^*(qr)$ and $(s^*q\ s^*r)$ with $(qr)\mid A : (YZ)\mid A \to A$ and $(q \mid A\ r \mid A) : (Y \mid A\ \ Z \mid A) \to A$ respectively.

Theorem 1.5. *Let $q : Y \to B$, $r : Z \to B$ be maps and let A be a subspace of B. Then the identity bijection*

$$\gamma : (YZ)\mid A \to (Y \mid A\ \ Z \mid A)$$

is continuous, and is a homeomorphism if (i) *A is closed, or* (ii) *A is open and B is regular.*

Proof. We will show that the sub-basic sets in $(Y \mid A\ \ Z \mid A)$ are restrictions of open sets of (YZ). This is clear for sub-basic sets $W(U)$. Let $S = W(K, V)$ be a sub-basic set where K is compact in $Y \mid A$ and V is open in $Z \mid A$. Then $V = V' \cap (Z \mid A)$ where V' is open in Z, and if $T = W(K, V')$, a sub-basic neighbourhood in (YZ), then $T \cap (Y \mid A\ \ Z \mid A) = S$.

The proof of (i), that γ is a homeomorphism if A is closed, follows easily from [6, Proposition 1.2(iii)] and the initial topology description of the open sets of (YZ) and $(Y \mid A\ \ Z \mid A)$.

For the proof of (ii), let $\delta : (Y \mid A\ \ Z \mid A) \to (YZ)$ denote the inclusion. Let $f : Y_a \to Z_a$, $a \in A$. That $\delta^{-1}(W(U))$ is sub-basic for U open in B is clear. Let $W(C, V)$ be the other kind of sub-basic neighbourhood in (YZ) of $\delta(f)$. Since B is regular there is an open set U and a closed set D such that $a \in U \subseteq D \subseteq A \subseteq B$. Let $N = W(C \mid D, V \mid U) \cap W(U)$. Then $\delta(N) \subseteq W(C, U)$. This completes the proof.

The following corollary was stated without proof as part (f) of the fibred exponential law [6, Theorem 3.5].

Corollary 1.6. *Suppose that B is a regular T_1-space, $p : X \to B$ is a map with X*

Hausdorff, $q : Y \to B$ is a locally trivial map with Hausdorff fibre G such that $G \times X$, $G \times B$ are k-spaces. If $r : Z \to B$ is a map, then the exponential function $\theta : M(p \sqcap q, r) \to M(p, (qr))$ is surjective.

Proof. Let $f : p \to (qr)$ be a map; we have to prove $\theta^{-1}(f) : p \sqcap q \to r$ continuous.

Let U be an open set of B such that $q \mid U$ is trivial, and so equivalent to the projection $G \times U \to U$. Since $G \times X$, $G \times B$ are k-spaces, so also are $G \times (X \mid U)$ and $G \times U$ (see [11, 1.5.3, p. 10]). It follows from (d) of [6, Theorem 3.5] that $\theta^{-1}(f) \mid U = \theta^{-1}(f \mid U)$ is continuous. Hence $\theta^{-1}(f)$ is continuous.

2. Fibred mapping spaces as bundles

If $q : Y \to B$, $r : Z \to B$ are locally trivial with fibres G, H respectively (see [6, Section 3]) then there is what we call the *classical topology* on the set (YZ) making $(qr) : (YZ) \to B$ locally trivial with fibre $\mathrm{M}(G, H)$. This topology is the final topology with respect to the injections $\mathrm{M}(G, H) \times U \to (YZ)$; these injections exist whenever U is an open set over which both q, r are trivial and are defined using the locally trivial structures for q, r.

In order to relate this classical topology to the modified compact-open topology we first prove the following result.

Theorem 2.1. *Let $q : Y \to B$, $r : Z \to B$ be locally trivial maps with fibres G, H respectively, and let Y, B be Hausdorff and B locally compact. Then the classical and modified compact open topologies on (YZ) coincide.*

Proof. It is sufficient to prove that the two topologies agree over the open sets of an open cover of B. However, if q, r are both locally trivial over the open set U of B, then by Example 1.3, Theorem 1.1 and the fact that U is locally compact, we have

$$\mathrm{M}(G, H) \times U \cong (G \times U\, H \times U) \cong (Y \mid U \;\; Z \mid U) \cong (YZ) \mid U$$

and the result follows.

Example 2.2. If $q : Y \to B$, $r : Z \to B$ are vector bundles of finite dimension, then the vector bundle $\mathrm{Hom}(q, r) : \mathrm{Hom}(Y, Z) \to B$ is well-defined [2, p. 8], and has fibre over b the vector space $L(Y_b, Z_b)$ of linear mappings $Y_b \to Z_b$. This vector space is of finite dimension and so has a unique Hausdorff topology making it a topological vector space. It follows from Theorem 2.1 that if B is locally compact Hausdorff then the vector bundle $\mathrm{Hom}(q, r)$ is a restriction of our fibred mapping space projection $(qr) : (YZ) \to B$.

Example 2.3. A similar result to Example 2.2 holds for principal bundles $q : Y \to B$, $r : Z \to B$ with group a topological group G. Let us write

$A(q,r): A(Y,Z) \to B$ for the set over B whose fibre over $b \in B$ is the set $A(Y_b, Z_b)$ of admissible maps $Y_b \to Z_b$. Now $A(G,G)$ with the compact-open triviality to give $A(Y,Z)$ the classical topology making it a locally trivial map $A(q,r): A(Y,Z) \to B$. If G is abelian then the action of G on $A(YZ)$ is well defined and $A(q,r)$ is a principle G-bundle. Then $A(q,r)$ is the functional bundle $(q, r, 1_B)$ of [7, p. 250]. It follows from Theorem 2.1 that if G, B are Hausdorff and B is locally compact then $A(q,r)$ is a subspace of our fibred mapping space $(qr): (YZ) \to B$. (On [7, p. 250] a more general bundle (q, r, f) is defined where $q: Y \to B$, $r: Z \to C$ and $f: B \to C$ is a map. However, (q, r, f) can be identified with $(q, f^*(r), 1_B)$ where $f^*(r)$ is the induced bundle.)

3. The map prin r and associated principal bundles

Let $r: Z \to B$ be a map and let F be a non-empty space. We define $Q_F(Z)$ to be the subspace of $M(F,Z)$ of maps $f: F \to Z$ such that rf is constant. Let $q_F(r): Q_F(Z) \to B$ be the projection $f \mapsto rf(x)$ (for some $x \in F$). Then $q_F(r)$ is continuous.

Proposition 3.1. *Let $F \times B \to B$ be the projection, and let $r: Z \to B$ be a map. The function*

$$\xi: Q_F(Z) \to (F \times B\, Z)$$

$\xi(f)(x,b) = f(x)$, $f \in Q_F(Z)$, $(x,b) \in F \times B$ is a bijection such that (i) *ξ is continuous if F is Hausdorff and* (ii) *ξ^{-1} is continuous if B is locally compact.*

Proof. That ξ is a bijection is clear.

(i) Suppose that $\xi(f) = g$, and $W(U)$ is a sub-basic neighbourhood of g. Let $x \in F$. Then $\xi(W(\{x\}, r^{-1}(U))) \subseteq W(U)$. For the sub-basic sets of the other kind, we note that since B, F are Hausdorff, these can be taken to be of the form $W(C \times D, V)$ for C, D compact in F, B respectively and V open in Z. Then $\xi(W(C, V)) \subseteq W(C \times D, V)$. Thus ξ is continuous.

(ii) Let $\xi^{-1}(g) = f$, and let $W(C, V)$ be a sub-basic neighbourhood of f. Let D be a compact neighbourhood of $b = rf(x)$, $x \in F$. Then $\xi^{-1}(W(C \times D, V) \cap W(D)) \subseteq W(C, V)$.

Example 3.2. Let $\mathrm{Prin}_F Z$ be the subspace of $Q_F(Z)$ of maps which are homotopy equivalences of F to some fibre Z_b of $r: Z \to B$. Let prin $r: \mathrm{Prin}_F Z \to B$ denote the restriction of $q_F(r)$. This map has been used by a number of authors (e.g. [1, p. 120; 15, p. 191]). Proposition 3.1 shows that if B is locally compact and F is Haudorff, then $\mathrm{Prin}_F Z$ can be identified with a subspace of the fibred mapping space $(F \times B\, Z)$.

Example 3.3. Let $r: Z \to B$ be the projection of a fibre bundle with group G and fibre F. One construction of the associated principal bundle [16, p. 39] has total space $r^\wedge : Z^\wedge \to B$ the set of admissible maps $F \to Z$ with the compact-open topology. Again, if B is locally compact and F is Hausdorff then $Z^\wedge$ can be identified with a subspace of $(F \times B\ Z)$.

4. The ex-exponential law

An *ex-space* $\bar{p} = (p, u)$ *over the space* B consists of the pair of maps $p: X \to B$, $u: B \to X$ satisfying $pu = 1_B$. If $\bar{q} = (q, v)$, $\bar{r} = (r, w)$ are ex-spaces over B, where $q: Y \to B$, $r: Z \to B$, then an *ex-map* $\bar{q} \to \bar{r}$ is a map $f: q \to r$ such that $fv = w$. The set of ex-maps $\bar{q} \to \bar{r}$ will be written $\mathrm{Ex}(\bar{q}, \bar{r})$; the space $\mathbf{Ex}(\bar{q}, \bar{r})$ will be $\mathrm{Ex}(\bar{q}, \bar{r})$ with the compact-open topology. So we have a category of ex-spaces.

We now follow [13] in defining the *smash product* of the ex-spaces $\bar{p}, \bar{q}$ as the pair $(p \wedge q, \pi \circ (u, v))$, where

$$X \wedge Y = \bigcup_{b \in B} \{(X_b \times Y_b)/((u(b) \times Y_b) \cup (X_b \times v(b)))\},$$

with the identification topology with respect to the projection $\pi : X \sqcap Y \to X \wedge Y$; the map $p \wedge q : X \wedge Y \to B$ is induced by $X \sqcap Y \to B$; and (u, v) is the map $B \to X \sqcap Y$ defined by u, v.

Let $\bar{q} = (q, v)$, $\bar{r} = (r, w)$ be ex-spaces as above, where B is Hausdorff. The *functional ex-space* $(\bar{q}\bar{r})$ is the pair $(q!r, w')$ defined as follows. The map $q!r: Y!Z \to B$ is the restriction of (qr) to the subspace $Y!Z$ of (YZ) of maps $f: Y_b \to Z_b$ such that $fv(b) = w(b)$, some $b \in B$. To define w' we note that the right-adjoint of $B \sqcap Y \to Z$, $(b, y) \mapsto w(b)$, is a map $B \to (YZ)$ whose image lies in $Y!Z$ and so restricts to a section $w': B \to Y!Z$ of $q!r$.

Theorem 4.1 (The ex-exponential law). *The exponential correspondence determines a natural injection*

$$\theta : \mathbf{Ex}(\bar{p} \wedge \bar{q}, \bar{r}) \to \mathbf{Ex}(\bar{p}, (\bar{q}, \bar{r})).$$

Also (i) *if p, q satisfy any one of the conditions* (a) ... (f) *in* [6, Theorem 3.5], *then θ is surjective.*

(ii) *if X is Hausdorff, then θ is continuous.*

Proof. The map $p \sqcap q \to p \wedge q$ induces a continuous injection $\mathbf{M}(p \wedge q, r) \to \mathbf{M}(p \sqcap q, r)$ which maps the subspace $\mathbf{Ex}(\bar{p} \wedge \bar{q}, \bar{r})$ of $\mathbf{M}(p \wedge q, r)$ to the space P of maps $f: p \sqcap q \to r$ such that

$$f(x, v(b)) = f(u(b), y) = w(b), \quad b \in B,\ x \in X_b,\ y \in Y_b.$$

The inclusion $q!r \to (qr)$ induces a continuous injection $\mathbf{M}(p, q!r) \to \mathbf{M}(p, (qr))$

which maps the subspace $\mathrm{Ex}(\bar{p}, (\bar{q}\bar{r}))$ of $\mathrm{M}(p, q\,!\,r)$ homeomorphically to the space Q of maps $g : p \to (qr)$ such that

$$g(x)(v(b)) = g(u(b))(y) = w(b) \quad b \in B,\ x \in X_b,\ y \in Y_b.$$

The exponential function θ of [6, Theorem 3.1] clearly maps P into Q, and this proves the first part. Also if g lies in Q, then $\theta^{-1}(g)$, if it is continuous, clearly lies in P, and this proves (i).

Finally, (ii) follows from [6, Theorem 3.2(i)].

Example 4.2. If B is a point, then the ex-exponential law reduces to the usual exponential law for pointed spaces. We know of only one circumstance when this latter exponental function is a homeomorphism, namely when X, Y are compact Hausdorff.

Example 4.3. We saw in Section 2 that if q, r are locally trivial then there is a classical topology on (YZ); a similar procedure has been used in [3, p. 372] to topologise $Y\,!\,Z$ when $\bar{q}, \bar{r}$ are locally trivial ex-spaces so that, when the base B is a finite CW-complex and the fibres are locally compact, $Y\,!\,Z$ becomes an ex-space. (The notation $(M/x)(Y, Z)$ is used in [3] for this space.) However, Theorem 2.1 shows that it is enough for B to be locally compact, Hausdorff and Y to be Hausdorff for the topology defined in [3] to give an ex-space $Y\,!\,Z$ whose topology agrees with the modified compact open topology.

5. The exponential law for fibred section spaces

Suppose we have the following situation of spaces and maps

$$\begin{array}{ccc} Z & & X \\ {\scriptstyle t}\downarrow & \overset{r}{\searrow} & \downarrow{\scriptstyle p} \\ Y & \xrightarrow[q]{} & B \end{array}$$

where $r = qt$ and B is a Hausdorff space. It is well-known that there is a canonical bijection

$$M(t, q^*p) \cong M(qt, p).$$

We recall that $\mathbf{Top}_B$ is the category whose objects are maps into B (e.g. p, q above) and morphisms $f : p \to q$ are maps $f : X \to Y$ such that $qf = p$. The existence of the above bijection can also be expressed by saying that the functor $q^* : \mathbf{Top}_B \to \mathbf{Top}_Y$ has a left adjoint, i.e. composition with q (where the two functors have the obvious effect on morphisms). We will obtain a right adjoint to q^*, at least for Y locally compact.

Definition 5.1. We consider Y, Z as spaces over B via q, r and let Y_*Z be the subspace of (YZ) of maps $f: Y_b \to Z_b$ which are partial sections to $t: Z \to Y$. We call Y_*Z the *fibred section space* of q, t and q_*t (= the restriction of (qr) to Y_*Z) the *fibred section projection* for Y_*Z. The reader will notice that in cases where Y_b (and hence Z_b) are empty, the fibre of q_*t over B consists of a single map, i.e. the empty map.

We notice that if $f: t \to u$ is a map in $\mathbf{Top}_Y$, then it induces a function

$$f_\bigcirc : q_*t \to q_*u,\ f_\bigcirc(s) = sf, \quad s \in Y_*Z;$$

this function is continuous because it is a restriction of one of the maps in [6, Proposition 2.2(i)]. It is easily seen that

$$q_*: \mathbf{Top}_Y \to \mathbf{Top}_B, \quad t \mapsto q_*t, \quad f \mapsto f_0$$

is a functor.

Theorem 5.2. *The exponential law for fibred section spaces. The exponential correspondence defines an injection*

$$\theta: \mathbf{M}(q^*p, t) \to \mathbf{M}(p, q_*t),$$

natural in the variables p and t.

Furthermore (i) *if p, q satisfy any one of the conditions* (a) ... (f) *of* [6, Theorem 3.5], *then θ is surjective*;

(ii) *if X is Hausdorff, then θ is continuous*;

(iii) *if X, Y are Hausdorff, then θ is a homeomorphism into.*

Proof. Clearly $\mathbf{M}(q^*p, t)$ is a subspace of $\mathbf{M}(p \sqcap q, r)$, and is mapped by θ of [6, Theorem 3.1] into the subspace $\mathbf{M}(p, q_*t)$ of $\mathbf{M}(p, (qt))$. Hence our present θ is well-defined. Thus (i), (ii) and (iii) follow from [6, Theorem 3.3 and Theorem 3.5]. The precise description of the naturality of θ is left to the reader.

Corollary 5.3. *If Y is locally compact then the functor* $q^*: \mathbf{Top}_B \to \mathbf{Top}_Y$ *is left adjoint to* $q_*: \mathbf{Top}_Y \to \mathbf{Top}_B$.

Remark 5.4. (i) [12, p. 461] describes the analogue of the argument of this section in the context of the category of compactly generated spaces in which real valued function separate points, though it is not clear if the definition given there ensures either that (a) q_*t is continuous, or (b) Y_*Z is a space in the given category. The corresponding result does however work in the convenient category $\mathbf{K}$ of $\mathfrak{k}$-spaces, assuming only that B is $\mathfrak{k}$-hausdorff (see Section 8 below).

(ii) The analogue of Corollary 5.3 for the category of sets is well-known, and the existence of a right adjoint to q^* in an arbitrary topos is known as the *fundamental theorem of topoi* ([10, p. 24], see also [17, p. 140]).

6. The exponential law for fibred relative lifting spaces

The exponential law of Section 5 involved a projection whose fibres were spaces of sections; there are similar constructions and laws for projections whose fibres are spaces of liftings and spaces of extensions. The concept of relative lifting [14, p. 415] generalises the concepts of section, lifting, extension; hence the exponential law for fibred relative lifting spaces (given below) generalises several other exponential laws.

Suppose that we are given a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\;j\;} & Z \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle t} \\ L & \xrightarrow[\;s\;]{} & W \end{array}$$

of maps of spaces. We define $\mathbf{RL}(s,t;i,j)$ to be the space of maps $f: Y \to Z$ such that $fi = j$, $tf = s$, with the compact-open topology. Suppose further given a map $u: W \to B$; then by composition of the given maps each of L, Y, Z, W can be regarded as a space over B and for each $b \in B$ we have a commutative diagram

$$\begin{array}{ccc} L_b & \xrightarrow{\;j_b\;} & Z_b \\ {\scriptstyle i_b}\downarrow & & \downarrow{\scriptstyle t_b} \\ Y_b & \xrightarrow[\;s_b\;]{} & W_b \end{array}$$

of maps of fibres over b. We define $Y \pitchfork Z$ to be the sub-space of (YZ) of maps $f_b: Y_b \to Z_b$ such that $f_b i_b = j_b$, $t_b f_b = s_b$, $b \in B$.

Let $q = us$, $r = ut$, and let $q \pitchfork r : Y \pitchfork Z \to B$ denote the restriction of (qr). We call $Y \pitchfork Z$ the *fibred relative lifting* space for s, t, i, j, u, and we call $q \pitchfork r$ the *fibred relative lifting projection* for $Y \pitchfork Z$.

If $p: X \to B$ is a map, then there is a commutative diagram

$$\begin{array}{ccc} X \sqcap L & \xrightarrow{\;jp_2\;} & Z \\ {\scriptstyle 1_X \times i}\downarrow & & \downarrow{\scriptstyle t} \\ X \sqcap Y & \xrightarrow[\;sp_2'\;]{} & W \end{array}$$

where $p_2: X \sqcap L \to L$, $p_2': X \sqcap Y \to Y$ denote the projections.

Theorem 6.1 (The exponential law for fibred relative lifting spaces). *If B is Hausdorff then the exponential correspondence restricts to a natural injection*

$$\theta : \mathbf{RL}(sp_2', t; 1_X \times i, jp_2) \to \mathbf{M}(p, q \pitchfork r).$$

Further (i) *if* p, q *satisfy any of the conditions* $(a), \ldots, (f)$ *of* [6, Theorem 3.5], *then* θ *is surjective*;

(ii) *if* X *is Hausdorff, then* θ *is continuous*;

(iii) *if* X, Y *are Hausdorff, then* θ *is a homeomorphism into.*

Proof. The proof is a straightforward check that the exponential correspondence maps the subspace appropriately.

The following examples show how the exponential law of this section includes various exponential laws discussed previously.

Example 6.2. (6.2.1) We recall that $Y \pitchfork Z$ is a subspace of (YZ); on the other hand if $L = \emptyset$, $W = B$, $u = 1_B$ then $Y \pitchfork Z = (YZ)$ and $q \pitchfork r = (qr)$, so (qr) is a particular case of $q \pitchfork r$.

(6.2.2) If $L = \emptyset$, $W = Y$, $s = 1_Y$, then $Y \pitchfork Z = Y_* Z$ and $q \pitchfork r = q_* r$, hence $q_* r$ is a particular case of $q \pitchfork r$.

(6.2.3) If $L = \emptyset$, then the fibres of $q \pitchfork r$ become spaces of liftings and we obtain an *exponential law for fibred lifting spaces* (referred to above).

(6.2.4) If $W = B$, $u = 1_B$ and i is the inclusion of the subspace L in the space Y, then the fibres of $q \pitchfork r$ are spaces of extensions and we obtain an *exponential law for fibred extension spaces* (referred to above).

(6.2.5) Given ex-maps $\bar{q} = (q, v)$ and $\bar{r} = (r, w)$, taking $W = B$, $u = 1_B$, $L = v(B)$, i as the inclusion and j as the function taking $v(b)$ over to $w(b)$ for all b in B (this is a particular case of (6.2.4)), then $q \pitchfork r : Y \pitchfork Z \to B$ is just the functional ex-space projection $q\,!\,r : Y\,!\,Z \to B$. In this case Theorem 6.1 concerns the exponential map from the appropriate subset of $\mathbf{M}(p \sqcap q, r)$ to $\mathbf{M}(p, q\,!\,r)$, i.e. it is an intermediate step in the proof of Theorem 4.1.

(6.2.6) Finally, if $L = \emptyset$ and B, W are singleton spaces then $Y \pitchfork Z$ is just $\mathbf{M}(Y, Z)$ and Theorem 6.1 reduces to the ordinary exponential law for topological spaces.

7. The homotopy fibred exponential law

This is given in two parts, namely Proposition 7.1 and Theorem 7.2. We assume that B is Hausdorff throughout this section. Given $p : X \to B$, $q : Y \to B$, the set of all classes of maps $p \to q$ under the relation "homotopic over B" will be written $[p, q]$.

Proposition 7.1. *If* $p : X \to B$, $q : Y \to B$, $Z \to B$ *are maps then the natural function*

$$[\theta] : [p \sqcap q, r] \to [p, (qr)], \quad [f] \mapsto [\theta(f)], \quad [f] \in [p \sqcap q, r]$$

is well-defined (*where* θ *is the exponential function of* [6, Theorem 3.1]).

Proof. If $f, g : p \sqcap q \to r$, then a homotopy from f to g over B is a map $(X \sqcap Y) \times I \to Z$ over B, and this is essentially just a map $p \sqcap q \sqcap t \to r$, where t denotes the projection $B \times I \to B$. Now $p \sqcap q \sqcap t = p \sqcap t \sqcap q$ and our map $p \sqcap t \sqcap q \to r$ determines a map $p \sqcap t \to (qr)$ [6, Theorem 3.1], i.e. a map $X \times I \to (YZ)$ over B. It is easily seen that this last map is a homotopy from $\theta(f)$ to $\theta(g)$ over B, and the result follows.

Theorem 7.2. *If $p : X \to B$, $q : Y \to B$, $r : Z \to B$ are maps such that p and q satisfy either*

(a) *(X, Y) is an exponential pair in the sense that either Y is locally compact or $X \times Y$ is a Hausdorff k-space, or one of* (b)–(f) *of* [6, Theorem 3.5] *then* $[\theta] : [p \sqcap q, r] \to [p, (qr)]$ *is a bijection.*

Proof. We have to prove that if $f, g : p \to (qr)$ are such that $f \simeq_B g$ then $\theta^{-1}(f) \simeq_B \theta^{-1}(g)$, for the result then follows from [6, Theorem 3.5]. The homotopy from f to g is a map $p \sqcap t \to (qr)$, where $t : B \times I \to B$ is the projection. In each case (a) ... (f) listed above the pair $(p \sqcap t, q)$ satisfies the same condition as does (p, q) (for $p \sqcap t : X \times I \to B$ is the map $(p \sqcap t)(x, u) = p(x)$, $x \in X$, $u \in I$), and it follows by [6, Theorem 3.3] that the corresponding function $(p \sqcap t) \sqcap q \to r$ is continuous. Now $p \sqcap q \sqcap t = p \sqcap t \sqcap q : (X \sqcap Y) \times I \to B$, $(x, y, u) \mapsto p(x) = q(y)$, $(x, y) \in X \sqcap Y$, $u \in I$ and we have determined the required homotopy $\theta^{-1}(f) \simeq \theta^{-1}(g)$ over B.

Definition 7.3. If $p : X \to B$ is a map then $[\sec p]$ will denote the set $[1_B, p]$ of classes of sections of p, under the relation of homotopy via sections.

Corollay 7.4 (homotopy version of [6, Corollary 3.7]). *If $q : Y \to B$, $r : Z \to B$ are maps, then there is a natural function*

$$[\phi] : [q, r] \to [\sec(qr)],\ [\phi][f] = [\phi(f)],\ [f] \in [q, r]$$

(*where ϕ is the function of* [6, Corollary 3.4]). *If Y is a Hausdorff k-space, then* $[\phi]$ *is a bijection.*

Proof. This follows easily from case (c) of Theorem 7.2, taking p to be the projection $B \times I \to B$.

Lemma 7.5 (homotopy version of [6, Lemma 3.8]). *Given a map $q : Y \to B$ and a space W; if $t = t(w)$ is the projection $W \times B \to B$, then*

$$[\xi] : [Y, W] \to [q, t],\quad [\xi]([f]) = [\xi f)],\quad [f] \in [Y, W]$$

(*where ξ is the function of* [6, Lemma 3.8]) *is a bijection.*

In other words the functor

$$\mathbf{H\,Top}_B \to \mathbf{H\,Top}, \quad (q : Y \to B) \mapsto Y, \ [f]_B \mapsto [f],$$

is left adjoint to the functor

$$\mathbf{H\,Top} \to \mathbf{H\,Top}_B, \quad W \mapsto t, \quad [f] \mapsto [1_B \times f].$$

Proof. This is an easy modification of the proof of [6, Lemma 3.8(i)].

Corollary 7.6 (homotopy version of [6, Corollary 3.9]). *Given maps $p : X \to B$, $q : Y \to B$ and a space W; t will denote the projection $t(W) : W \times B \to B$. There is a natural function*

$$[\psi] : [X \sqcap Y, W] \to [p, (qt)], \quad [\psi]([f]) = [\psi(f)], \quad [f] \in [X \sqcap Y, W]$$

(ψ is the function of [6, Corollary 3.9]). *If p, q satisfy any one of the conditions* (a)... (f) *of Theorem* 7.2 *then $[\psi]$ is a bijection.*

Example 7.7. If $X \sqcap Y$ has the homotopy type of a CW-complex and $W = K(\pi, n)$ then Corollary 7.6 determines a bijection $H^n(X \sqcap Y, \pi) \to [p, (qr)]$, a fact that will be used elsewhere in cohomology calculations.

Proof. $[\psi]$ is the composite of $[\xi]$ of Lemma 7.5 and $[\theta]$ of Theorem 7.2.

Corollary 7.8 (homotopy version of [6, Corollary 3.10]). *If $q : Y \to B$ is a map, W is a space and t denotes the projection $t(W) : W \times B \to B$ then there is a natural function*

$$[\eta] : [Y, W] \to [\sec(qt)], \quad [\eta]([f]) = [\eta(f)], \quad [f] \in [Y, W],$$

(η denotes the function of [6, Corollary 3.10]). *If Y is a Hausdorff k-space then $[\eta]$ is a bijection.*

Example 7.9. If Y has the homotopy of a CW-complex and $W = K(\pi, n)$ then there is a bijection $H^n(X, \pi) \to [\sec(qr)]$. This fact that will be used elsewhere in cohomology calculations.

Proof of Corollary 7.8. The result is simply a combination of Corollary 7.4 and Lemma 7.5.

8. Convenient categories

(8.1) If we attempt to modify the above arguments to the category of Hausdorff k-spaces we face the difficulty that some of the mapping spaces constructed may be non-Hausdorff (compare with [6, Section 7]).

(8.2) If our arguments are removed to the category $\mathbf{K}$ of k-spaces then this difficulty goes. In this category the map α of Theorem 1.1 is easily shown to be a homeomorphism, the $\mathfrak{k}$-spaces involved being subject only to the restriction that A and B are $\mathfrak{k}$-Hausdorff. The easiest method of proof here involves applying the fibred exponential law (see [6, Theorem 7.3]) to maps over A into $A \sqcap YA \sqcap Z)$ and $A \sqcap (YZ)$. It follows that the $\mathbf{K}$ analogues of the results of Sections 2, 3 hold and that the exponential functions of the ex-fibred section-, fibred relative lifting- and homotopy-exponential laws are homeomorphisms (in the last case bijections), the $\mathfrak{k}$-spaces involved being subject only to the restriction that B is $\mathfrak{k}$-Hausdorff. In particular, the category of ex-$\mathfrak{k}$-spaces) over a given $\mathfrak{k}$-Hausdorff $\mathfrak{k}$-space B is a symmetric monoidal closed (but not cartesian closed) category. Another advantage of the category of $\mathfrak{k}$-spaces is that the θ of the ex-exponential law of Theorem 4.1 becomes a homeomorphism (in the case of the category of all topological spaces we found no condition sufficient to make θ^{-1} continuous). An additional related difficulty in dealing with ex-spaces in the ordinary category is that we have been unable to prove that for given ex-space X, Y, suitable conditions imply that $X \wedge Y$ is Hausdorff. However, the following holds in the category of $\mathfrak{k}$-spaces.

Proposition 8.3. *Let $p: X \to B$, $q: Y \to B$ be $\mathfrak{k}$-Hausdorff spaces over the $\mathfrak{k}$-Hausdorff space B. Let u, v be sections of p, q respectively, so that (p, u), (q, v) are ex-spaces. Then the corresponding $\mathfrak{k}$-ex-smash product space $X \wedge Y$ is $\mathfrak{k}$-Hausdorff.*

Proof. There is an identification map $\phi: X \sqcap Y \to X \wedge Y$ (where $X \sqcap Y$ is the k-product). To prove $X \wedge Y$ $\mathfrak{k}$-Hausdorff it is sufficient to prove the equivalence relation R defined by ϕ is closed (in the $\mathfrak{k}$-product). But R is the union of the diagonal in $(X \sqcap Y) \times (X \sqcap Y)$, the set $A = \{(x, y, x', y') \in (X \sqcap Y) \times (X \sqcap Y)$: $q(y) = q(y')$ and $x = x' = up(x)\}$ and

$$A' = \{(x, y, x', y') \in (X \sqcap Y) \times (X \sqcap Y): p(x) = p(x') \text{ and } y = y' (= vq(y)\}.$$

The diagonal is closed (since $X \sqcap Y$ is $\mathfrak{k}$-Hausdorff) and A, A' are closed because they are the intersections of sets on which various maps into $\mathfrak{k}$-Hausdorff spaces agree. It follows that R is closed.

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