THE HOLONOMY GROUPOID OF A LOCALLY TOPOLOGICAL GROUPOID

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Abstract

The well known holonomy groupoid of a foliation is here generalised to the holonomy groupoid of a locally topological groupoid. This gives an account of an important theorem of J. Pradines (1966) on the globalisation of locally topological groupoids.

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Introduction

The notion of holonomy groupoid in this paper is due to Pradines [P1]. It arose from two sources. One source is the idea of the holonomy groupoid of a foliation, as discussed by Ehresmann and Shih Weishu in 1956 in [Eh-We], and by Ehresmann in [Eh] (for a more recent account, see [W]). The other source is the grand scheme, described in the sequence of notes [P1,P2], of generalising the classical correspondence between Lie groups and Lie algebras to a correspondence between certain Lie groupoids and Lie algebroids. For a recent account of some aspects of this, see [M]. For a general survey of the utility of groupoids, see [B2].

For groups, one key part of this Lie correspondence requires a classical and fairly simple, though not entirely trivial, procedure for extending a topology (see, for example, [Bo]), as follows. From a Lie algebra one obtains a group G and a topology on a subset W of G containing the identity. Suitable conditions are obtained on W for the topology on W to be translated, by the operations of left translation, to a topology on G. The basic reason for this success is that in a topological group, left translation by an element maps open sets to open sets.

Such a procedure fails in the groupoid case, since in a topological groupoid, left translation of an open set by an element usually fails to be open, because the multiplication is only partially defined. It is this failure which, under suitable conditions, gives rise to the holonomy groupoid. These ideas are more fully expressed as follows.

Let G be a groupoid, and suppose that there is given a topology on a subset W of the set of arrows of G such that W contains the identities O_G of G. For certain conditions on W, we call the pair (G, W) a *locally topological groupoid*. We give simple examples, due to Pradines [P3], which show that in general there is no topology on G which restricts to that on W and which makes G a topological groupoid. In other words, a locally topological groupoid is not in general extendible to a topological groupoid.

Instead, there is a topological groupoid H with a morphism $H \to G$ such that H contains W as a subspace and H is in a clear sense universal among such topological groupoids. It is this groupoid H which is called the *holonomy groupoid* of the locally topological groupoid, and the construction of H is called *globalisation*.

The existence of the holonomy groupoid in the smooth case is essentially the main result of [P1], the first of the announcements in [P1,P2]. However no details of the construction have been made generally available. Pradines told R.Brown the main ideas of his construction in the period since 1981, and the completion of the details was the main work of the first author's Doctoral Thesis at Bangor. The importance of the construction clearly warrants a complete account, with the use of each assumption clearly displayed. We have also been able to make some simplification of the proofs in [Ao], with the result that fewer assumptions are required for the construction than those which are given in [Ao], and which are implicit in [P1].

The structure of this paper is as follows. Section 1 contains the definition of a locally topological groupoid and some examples of these which are not extendible. Section 2 contains the statement of the Globalisation Theorem, which asserts that the holonomy groupoid exists, and the construction of this as a groupoid. Section 3 constructs the topology on the holonomy groupoid, and section 4 verifies the universal property. Section 5 contrasts the holonomy construction with the fact that there is an appropriate topology on each α-fibre $\alpha^{-1}x, x \in G$. Section 6 proves that locally trivial locally topological groupoids are extendible. Section 7 gives

some historical comments.

1   Definition of a locally topological groupoid

We first establish some basic notation.

Let G be a groupoid. We write O_G for the set of objects of G, and also identify O_G with the set of identities of G. An element of O_G may be written as x or 1_x as convenient. We write $\alpha, \beta : G \to O_G$ for the source and target maps. The product hg of two elements of G is defined if and only if $\alpha h = \beta g$, and so the product map $\gamma : (h, g) \mapsto hg$ is defined on the pullback $G_\alpha \times_\beta G$ of α and β. The difference map $\delta : G \times_\alpha G \to G$ is given by $\delta(g, h) = gh^{-1}$, and is defined on the double pullback of G by α.

The construction of the holonomy groupoid is intimately bound up with the properties of the admissible local sections of G. We recall their definition due to Ehresmann [Eh1], but following the notation of [M], with some modifications.

Suppose that $X = O_G$ is given the structure of a topological space.

Definition 1.1 An *admissible local section* of G is a function $s : U \to G$ from an open subset U of X such that s satisfies:

(i) $\alpha s x = x$ for all $x \in U$,

(ii) $\beta s(U)$ is open in X, and

(iii) βs maps U homeomorphically to $\beta s(U)$.

The set U is called the domain of s and is written $\mathcal{D}(s)$. If s and t are two admissible local sections, then their *product* ts is defined by

$$(ts) : x \mapsto (t\beta sx)(sx)$$

where the right hand product is the product in G. Thus $\mathcal{D}(ts)$ is an open subset of $\mathcal{D}(s)$, and the product ts is again an admissible local section. It is convenient to say that t and s are *composable* if $\mathcal{D}(ts) = \mathcal{D}(s)$. If s is an admissible local section, then we write s^{-1} for the admissible local section with domain $\beta s\mathcal{D}(s)$ and which is given by $\beta sx \mapsto (sx)$. With this product, the set $\Gamma(G)$ of admissible local sections becomes an inverse semigroup. (Recall that an inverse semigroup S is a semigroup such that for all $s \in S$ there is a unique element $t \in S$ such that $s = sts$, $t = tst$. This element t is called the (generalised) inverse of s.)

Let $s \in \Gamma(G)$. Then s defines a left translation L_s on G by $L_s g = (s\beta g)g$, and a right translation R_s by $R_s g = g(s\alpha g)$, $g \in G$. These are injective partial functions on G. If G is a topological groupoid, and s is also continuous, then L_s and R_s map open sets of G to open sets.

Let W be a subset of G, and suppose that W has the structure of a topological space with X as a subspace. We say that the triple (α, β, W) has *enough continuous admissible local sections* if for each $w \in W$ there is an admissible local section s of G such that (i) $s\alpha w = w$, (ii) the values of s lie in W, and (iii) s is continuous as a function $\mathcal{D}(s) \to W$. We call such an s a *continuous admissible local section through w*.

he holonomy groupoid will be constructed for a locally topological groupoid, a term we now define. This definition is a modification of one due to J. Pradines in [P1] under the name "un morceau differentiable de groupoïde".

Definition 1.2 A *locally topological groupoid* is a pair (G, W) consisting of a groupoid G and a topological space W such that:

G1) $O_G \subseteq W \subseteq G$;

G2) $W = W^{-1}$;

G3) the set $W_\delta = (W \times_\alpha W) \cap \delta^{-1}(W)$ is open in $W \times_\alpha W$ and the restriction to W of the difference map $\delta : G \times_\alpha G \to G, (g, h) \mapsto gh^{-1}$, is continuous;

G4) the restrictions to W of the source and target maps α and β are continuous, and the triple (α, β, W) has enough continuous admissible local sections;

G5) W generates G as a groupoid.

Note that, in this definition, G is a groupoid but does not need to have a topology. The locally topological groupoid (G, W) is said to be *extendible* if there can be found a topology on G making it a topological groupoid and for which W is an open subspace. We give below examples of locally topological groupoids which are not extendible.

Remark 1.3 (i) The condition $W = W^{-1}$ does not appear in Pradines definition of "un morceau differentiable de groupoide" [P1]. It is proved in [Ao], following a suggestion of K. Mackenzie, that if we are given a pair (G, W) satisfying all the conditions for locally topological groupoid except the condition $W = W^{-1}$, then a locally topological groupoid structure can be given to $(G, W \cup W^{-1})$.
(ii) Axiom (G4) is implied in the differentiable case by the condition which is assumed in [P1] that the restrictions of α and β to W are differentiable surmersions.
(iii) In [Ao], the extra assumption is made that there is an open set V of W satisfying the conditions $X \subseteq V$ and $V^2 \subseteq W$. Such a condition is implied by the condition that the space O_G of objects of G is paracompact (as pointed out by Pradines, see the Appendix to [Ao]), and in [P1] the assumption is made that all spaces are to be paracompact. However the analysis given here shows that this condition may be dispensed with.
(iv) The condition (G5) does not appear in standard expositions of the group case. What does appear is a condition that for each $g \in G$ there is an open neighbourhood V of the identity 1

such that $gVg^{-1} \subseteq W$. Pradines argues (private communication) that this is a condition on so to speak 'large' elements of G, and so is unrealistic. As we shall see, the generation condition allows us to dispense with the above conjugation condition, and so we have in effect a new result even in the group case.

(v) The assumption is also made in [P1] and in [Ao] that (G, W) is α-connected, i.e. that for all $x \in X$, the space $\alpha^{-1}(x) \cap W$ is connected. The present proof does not require this condition.

This is a convenient place to make some deductions from the axioms.

1.4 *The inverse map* $\iota : g \mapsto g^{-1}$ *is continuous as a function* $\iota_W : W \to W$.

Proof Since $W = W^{-1}$, the values of ι do lie in W and so continuity for ι_W makes sense. Let $j : W \to W \times_\alpha W$ be the function $w \mapsto (\alpha w, w)$. Then $\iota_W = \delta j$ and $j(W)$ has image contained in $\delta^{-1}(W)$. The result follows. $\qquad\square$

1.5 *The set* $\gamma^{-1}(W) \cap (W_\alpha \times_\beta W)$ *is open in* $W_\alpha \times_\beta W$ *and* γ *is continuous on this set.*

Proof This follows from (G3) and (1.4). $\qquad\square$

Suppose now that (G, W) is a locally topological groupoid. Let $\Gamma^c(W)$ be the subset of $\Gamma(G)$ consisting of admissible local sections which have values in W and are continuous. It is useful to think of an element of $\Gamma^c(W)$ as a 'local procedure'.

Let $\Gamma^c(G, W)$ be the subsemigroup of $\Gamma(G)$ generated by $\Gamma^c(W)$. Then $\Gamma^c(G, W)$ is again an inverse semigroup. It is useful to think of an element of $\Gamma^c(G, W)$ as an 'iteration of local procedures'.

There are two simple results which we shall use later.

1.6 *Let* $r, s, t \in \Gamma^c(W)$, *and suppose* $y \in \mathcal{D}(rst)$ *and* $x = \beta ty$ *satisfy* $(rs)x \in W$ *and* $(rst)y \in W$. *Then there are restrictions* r', s', t' *of* r, s, t *respectively such that* $y \in \mathcal{D}(r's't')$ *and* $r's't' \in \Gamma^c(W)$.

Proof Let $z = \beta sx \in \mathcal{D}(r)$. By assumption, (rz, sx) lies in the open subset $\gamma^{-1}(W) \cap (W_\alpha \times_\beta W)$ of $W_\alpha \times_\beta W$. The existence of the restrictions r', s' now follows from continuity considerations. A similar argument applies to obtain t', possibly further restricting r' and s'. $\qquad\square$

1.7 *Suppose* $s, t \in \Gamma^c(W)$, $x_0 \in X$ *and* $sx_0 = tx_0$. *Then there is a neighbourhood* U *of* x_0 *such that the restriction of* st^{-1} *to* U *lies in* $\Gamma^c(W)$.

Proof Note that st^{-1} is the composition of the partial maps

$$ X \xrightarrow{(\beta t)^{-1}} X \xrightarrow{(s,t)} G \times G \xrightarrow{\delta} G $$

Since s and t are continuous as maps into W and $W_\delta = (W \times_\alpha W) \cap \delta^{-1}(W)$ is open in $W \times_\alpha W$, there is an open neighbourhood U' of x_0 such that $(s, t)(U')$ is contained in W_δ. Hence $\delta(s, t)(U')$ is contained in W. So st^{-1} is continuous on $\beta t(U')$. $\qquad\square$

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Let $J = J(G)$ be the sheaf of germs of admissible local sections of G. Thus the elements of J are equivalence classes of pairs (x, s) such that $s \in \Gamma(G)$, $x \in \mathcal{D}(s)$, and (x, s) is equivalent to (y, t) if and only if $x = y$ and s and t agree on a neighbourhood of x. The equivalence class, i.e. the germ, of (x, s) is written $[s]_x$. The product structure on $\Gamma(G)$ induces a groupoid structure on J with X as the set of objects, and source and target maps $[s]_x \mapsto x$, $[s]_x \mapsto \beta s x$. Let $J^c(G)$ be the subsheaf of J of germs of elements of $\Gamma^c(G, W)$. Then $J^c(G)$ is generated as a subgroupoid of J by the sheaf $J^c(W)$ of germs of elements of $\Gamma^c(W)$. Thus an element of $J^c(G)$ is of the form

$$[s]_x = [sn]_{x_n} \dots [s_1]_{x_1}$$

where $s = s_n \dots s_1$, with $[s_i]_{x_i} \in J^c(W)$, $x_{i+1} = \beta s_i x_i$, $i = 1, \dots, n$ and $x_1 = x \in \mathcal{D}(s)$.

The inverse semigroup $\Gamma^c(G, W)$ and its associated groupoid of germs $J^c(G, W)$ are important because of their rôle in codifying the *iteration of local procedures and their germs*, namely those determined by $\Gamma^c(W)$ and $J^c(W)$.

It is easiest to picture locally topological groupoids (G, W) for groupoids G such that $\alpha = \beta$, so that G is just a bundle of groups. Examination of this special case is also useful for understanding the proof of the main Theorem 2.1 below. Here is a specific such example of a locally topological groupoid which is not extendible. The holonomy groupoid of this example will be discussed later.

Example 1.8 (Pradines [P3]) Let F be the bundle of groups $\alpha_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ where α_1 is the first projection. The usual topology on $\mathbb{R} \times \mathbb{R}$ gives F the structure of topological groupoid in which each $\alpha_1^{-1}(x)$ is isomorphic as additive group to $\mathbb{R}$ by $(x, y) \mapsto y$. Let N be the subbundle of F given by the union of the sets $\{(x, 0)\}$ if $x < 0$ and $\{x\} \times \mathbb{Z}$ if $x \geqslant 0$. Let G be the quotient bundle F/N and let $p : F \to G$ be the quotient morphism. Then the source map $\alpha : G \to \mathbb{R}$ has $\alpha^{-1}(x)$ isomorphic to $\mathbb{R}$ for $x < 0$ and to $\mathbb{R}/\mathbb{Z}$ for $x \geqslant 0$. Let W' be the subset $\mathbb{R} \times (-1/4, 1/4)$ of F. (see Fig.1)
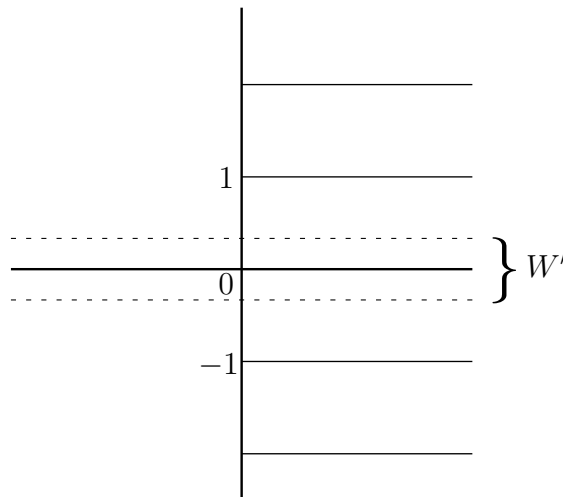

Figure 1

Then p maps W' bijectively to $W = p(W')$; let W have the topology in which this map is a homeomorphism. It is easily checked that (G, W) is a locally topological groupoid. Suppose this locally topological groupoid is extended to a topological groupoid structure on G. Let s' be the section of α_1 in which $x \mapsto (x, 1/8)$, and let $s = ps'$. Then s is an admissible section of α, as is $t = 9s$. However $t(0) = p(0, 1/8)$. Let U be an open neighbourhood of $(1/8, 0)$ in $\mathbb{R}^2$ such that U is contained in W'. Then $p(U)$ is contained in W and is a neighbourhood of $t(0)$. But $t^{-1}p(U)$ is contained in $[0, 8)$, so that t is not continuous. This gives a contradiction, and shows that the locally topological groupoid (G, W) is not extendible. By contrast, if we proceed as before but replace N by N_1 which is the union of the sets $\{(x, 0)\}$ for $x \leqslant 0$ and $\{x\} \times \mathbb{Z}$ for $x > 0$, then the resulting locally topological groupoid (G_1, W_1) is extendible.

Example 1.9 There is a variant of the last example in which F is as before, but this time N is the union of the groups $\{x\} \times (1 + |x|)\mathbb{Z}$ for all $x \in \mathbb{R}$. If one takes W' as before, and W is the image of W' in $G = F/N$, then the locally topological groupoid (G, W) can be extended to give a topological groupoid structure on G. However, now consider W as a differential manifold. The differential structure cannot be extended to make G a differential groupoid with W as submanifold. The reason is analogous to that given in the previous example, namely that such a differential structure would entail the existence of a local differentiable admissible section whose sum with itself is not differentiable, thus giving a contradiction.

Other examples of locally topological and differential groupoids, particularly that defined by a foliation, will be discussed in Section 7. See also [3].

2 Construction of the holonomy groupoid

In this section we state our main Globalisation Theorem, which shows how a locally topological groupoid gives rise to its holonomy groupoid, which is a topological groupoid satisfying a universal property. We also start the proof of the theorem, which then occupies this and the next two sections. As explained earlier, this theorem is the essence of the topological version of Theorem 1 of [P1], which is stated in terms of categories of microdifferential groupoids, i.e. germs of locally differential groupoids. We hope that the current approach will make it easier to relate the theorem to classical work on topological groups, and to understand the beautiful construction.

Theorem 2.1 (Globalisation Theorem) *Let (G, W) be a locally topological groupoid. Then there is a topological groupoid H, a morphism $\phi : H \to G$ of groupoids, and an embedding $i : W \to H$ of W to an open neighbourhood of $\mathrm{Ob}(H)$, such that:*
i) ϕ is the identity on objects, $\phi i = id_W$, $\phi^{-1}(W)$ is open in H, and the restriction $\phi_W : \phi^{-1}(W) \to W$ of ϕ is continuous;
ii) (universal property) if A is a topological groupoid and $\zeta : A \to G$ is a morphism of groupoids such that:

a) ζ is the identity on objects;

b) the restriction $\zeta_W : \zeta(W) \to W$ of ζ is continuous and $\zeta^{-1}(W)$ is open in A and generates A;

c) *the triple (α_A, β_A, A) has enough continuous admissible local sections,*

then there is a unique morphism $\zeta' : A \to H$ of topological groupoids such that $\phi q \zeta' = \zeta$ and $\zeta' a = i\zeta a$ for $a \in \zeta^{-1}(W)$.

The groupoid H is called the holonomy groupoid $\mathrm{Hol}(G, W)$ of the locally topological groupoid (G, W); its essential uniqueness follows from the condition (ii) above.

Here is an outline of the construction of H. We first form the inverse semigroup $\Gamma(G)$ and its subsemigroup $\Gamma^c(G, W)$ generated by $\Gamma^c(W)$ as in section 1. The groupoid $J(G)$ of germs of $\Gamma(G)$ has the subgroupoid $J^c(G)$ of germs of $\Gamma^c(G, W)$, and is generated as a groupoid by the set $J^c(W)$ of germs of $\Gamma^c(W)$. Let $\psi : J(G) \to G$ be the final map defined by $\psi([s]_x) = s(x)$, where s is an admissible local section. Then $\psi(J^c(G)) = G$, by axioms (G4) and (G5).

Let $J_0 = J^c(W) \cap \mathrm{Ker}\ \psi$. We will prove next that J_0 is a normal subgroupoid of $J^c(G)$. Hence we can define H to be the quotient groupoid $J^c(G)/J_0$; in Section 3 we give H a suitable topology, to make it a topological groupoid.

Lemma 2.2 *The set J_0 is a normal subgroupoid of the groupoid $J^c(G)$.*

Proof We write J^c for $J^c(G)$. That J_0 is a subgroupoid of J^c follows easily from (1.4) and (1.6).

Let $[f]_x \in J_0(x, x)$ and $[t]_x \in J^c(x, y)$ for some $x, y \in X$. Then we may assume that f, t are admissible local sections with $y = \beta t x$ and $\alpha f x = \beta f x = \alpha t x = x$. By the definition of J_0, $fx = 1_x$, and we may assume that the image of f is contained in W and f is continuous.

Since J^c is generated by $J^c(W) = J^c(W)^{-1}$, then

$$[t]_x = [tn]_{x_n} \dots [t_1]_{x_1},$$

where $t_i \in \Gamma^c(W)$, $x_1 = x$ and $x_{i+1} = \beta t_i x_i$, $i = 1, \dots, n$. Hence

$$[t]_x[f]_x([t]_x)^{-1} = [tn]_{x_n} \dots [t_1]_x[f]_x([t_1]_x)^{-1} \dots ([tn]_{x_n})^{-1}.$$

Therefore it is sufficient to prove that if $[f]_x \in J_0(x, x)$ and $[t]_x \in J^c(W)(x, y)$, then

$$[t]_x[f]_x([t]_x)^{-1} = [tft^{-1}]_y \in J_0(y, y).$$

But this follows easily from (1.6). □

We now define the *holonomy groupoid* $H = \mathrm{Hol}(G, W)$ to be the quotient groupoid $J^c(G)/J_0$. Let $p : J^c(G) \to H$ be the quotient morphism, and write $\langle s \rangle_x$ for $p[s]_x$. Then the final map $\psi : J^c \to G$ induces a surjective morphism $\phi : H \to G$ such that $\phi \langle s \rangle_x = sx$.

The following lemma will be used later.

Lemma 2.3 *Let $w \in W$, and let s and t be continuous admissible local sections through w. Let $x = \alpha w$. Then $\langle s \rangle_x = \langle t \rangle_x$ in H.*

Proof By assumption $sx = tx = w$. Let $y = \beta w$. Without loss of generality, we may assume that s and t have the same domain U and have image contained in W. By (1.7), $st^{-1} \in \Gamma^c(W)$. So $[st^{-1}]_y \in J_0$. Hence in H

$$\langle t \rangle_x = \langle st \rangle_y \langle t \rangle_x = \langle s \rangle_x.$$

□

3 Topological groupoid structure on H

The aim of this section is to construct a topology on the holonomy groupoid H such that H with this topology is a topological groupoid. In the next section we verify that the universal property of Theorem 2.1 holds. The intuition is that first of all W embeds in H (Lemma 3.1), and second that H has enough local sections for it to obtain a topology by translation of the topology of W.

Let $s \in \Gamma^c(G, W)$. We define a partial function $\sigma_s : W \to H$. The domain of σ_s is the set of $w \in W$ such that $\beta w \in \mathcal{D}(s)$. The value $\sigma_s w$ is obtained as follows. Choose a continuous admissible local section f through w. Then we set

$$\sigma_s w = \langle s \rangle_{\beta w} \langle f \rangle_{\alpha w} = \langle sf \rangle_{\alpha w}.$$

By Lemma 2.3, $\sigma_s w$ is independent of the choice of the local section f.

Lemma 3.1 σ_s *is injective.*

Proof Suppose $\sigma_s v = \sigma_s w$. Then $\alpha v = \alpha w = x$, say, and $\beta s \beta v = \beta s \beta w$. By admissibility of s, $\beta v = \beta w = y$, say. Let g be a local section through w. Then we now obtain from $\sigma_s v = \sigma_s w$ that $\langle s \rangle_y \langle f \rangle_x = \langle s \rangle_y \langle g \rangle_x$, and hence, since H is a groupoid, that $\langle f \rangle_x = \langle g \rangle_x$. Hence $v = fx = gx = w$. $\qquad\square$

So we have an injective function σ_s from an open subset of W to H. By definition of H, every element of H is in the image of σ_s for some s. These σ_s will form a set of charts and so induce a topology on H. The compatibility of these charts results from the following lemma, which is essential to ensure that W retains its topology in H and is open in H.

Lemma 3.2 *Let* $s, t \in \Gamma^c(G, W)$. *Then* $(\sigma_t)^{-1}(\sigma_s)$ *coincides with* L_h, *left translation by the local section* $h = t^{-1}s$, *and* L_h *maps an open set of* W *homeomorphically to an open set of* W.

Proof Suppose $v, w \in W$ and $\sigma_s v = \sigma_t w$. Choose continuous admissible local sections f and g of α through v and w respectively such that the images of f and g are contained in W. Since $\sigma_s v = \sigma_t w$, then $\alpha v = \alpha w = x$ say. Let $\beta v = y, \beta w = z$.

Since $\sigma_s v = \sigma_t w$, we have

$$\langle sf \rangle_x = \langle tg \rangle_x.$$

Hence there exists a continuous admissible local section r of α with $x \in \mathcal{D}(r)$ such that $[r]_x \in J$ and

$$[sf]_x = [tg]_x [r]_x.$$

Let $h = t^{-1}s$. Then in the semigroup $\Gamma^c(G, W)$ we have from the above that $hf = gr$ locally near x. So $w = (gr)x = (hf)x = (hy)v$. This shows that $(\sigma_t)^{-1}(\sigma_s) = L_h$, left translation by the element $h \in \Gamma(G)$.

However, we also have $h = grf^{-1}$ near βv. Hence $L_h = L_g L_r L_{f^{-1}}$ near v. Now $L_{f^{-1}}$ maps v to 1_x, L_r maps 1_x to 1_x, and L_g maps 1_x to w. So these left translations are defined and continuous on open neighbourhoods of v, 1_x and 1_x respectively. Hence L_h is defined and continuous on an open neighborhood of v. $\qquad\square$

We now impose on H the initial topology with respect to the charts σ_s for all $s \in \Gamma^c(G, W)$. In this topology each element q of H has an open neighbourhood homeomorphic to an open neighbourhood of $1_{\alpha q}$ in W.

Lemma 3.3 *With the above topology, H is a topological groupoid.*

Proof Notice that for $w \in W$, $\alpha_H \sigma_s w = \alpha w$, $\beta_H \sigma_s w = \beta s \beta w$. It follows that α_H and β_H are continuous.

Next we prove continuity of $\delta : H \times_{\alpha_H} H \to H$. Let $\langle s \rangle_x, \langle t \rangle_x \in H$. Then $\sigma_s(1_x) = \langle s \rangle_x$, $\sigma_t(1_x) = \langle t \rangle_x$, and if $h = st^{-1}$, then $\sigma_h(1_y) = \langle st^{-1} \rangle_y$, where $y = \beta t x$. Let $v \in \mathcal{D}(\sigma_s)$, $w \in \mathcal{D}(st)$, with $\alpha v = \alpha w = a$, say, and let f and g be elements of $\Gamma^c(W)$ through v and w respectively. Let $b = \beta(tg)a$. Then

$$
\begin{aligned}
(\sigma_h)^{-1} \delta(\sigma_s \times \sigma_t)(v, w) &= (\sigma_h)^{-1} \delta(\langle sf \rangle_a, \langle tg \rangle_a) \\
&= (\sigma_h) \langle sfg^{-1}t^{-1} \rangle_b \\
&= (L_h)^{-1}(sfg^{-1}t^{-1})b \\
&= (t\beta v)vw^{-1}(t\beta w)^{-1} \\
&= \theta(v, w),
\end{aligned}
$$

say. The continuity of this map θ at $(1_x, 1_x)$ is now easily shown by writing $t = t_n \ldots t_1$ where $t_i \in \Gamma^c(W)$ and using induction and an argument similar to that for (1.6). $\qquad \square$

Note that H is a topological groupoid in which α_H has enough continuous admissible local sections, since if $q = \langle s \rangle_x \in H$, then $y \mapsto \langle s \rangle_y$ is a continuous admissible local section through q.

Note also that if $\mathbf{1}$ is the identity section $x \mapsto 1_x$, then $i = \sigma_{\mathbf{1}}$ is an embedding $W \to H$ such that ϕi is the identity on W. We consider H as a *globalisation of W to a topological groupoid*.

Example 3.4 We can now continue Example 1.8. Let N_1 be the subgroupoid of F given by

$$
N_1(x) = \begin{cases} \{(x, 0)\} & \text{if } x \leqslant 0, \\ \{x\} \times \mathbb{Z} & \text{if } x > 0. \end{cases}
$$

Let W' be as before and let $G_1 = F/N_1$. The image W_1 of W' in G_1 gives a locally topological groupoid (G_1, W_1) which is extendible to a (non-Hausdorff) topological groupoid structure on G_1, and the projection $G_1 \to G$ is isomorphic to the projection of the holonomy groupoid. Thus the kernel of $\psi : H \to G$ is the bundle of groups over $\mathbb{R}$ which is 0 for $x \neq 0$ and is $\mathbb{Z}$ for $x = 0$. The reason is that the element f of $\Gamma^c(G)$ given by

$$
t \mapsto \begin{cases} (t, 1) & \text{if } t < 0, \\ (0, 0) & \text{if } t \geqslant 0, \end{cases}
$$

yields a generator $\langle f \rangle_0$ of $\mathrm{Ker}\, \psi$.

Example 3.5 We now continue Example 1.9. In this case we can construct a *differential holonomy groupoid* H_d by following the previous construction but replacing the word 'continuous' by 'differentiable' and 'homeomorphism' by 'diffeomorphism'. Applying this to Example 1.9, one finds that the kernel of $\psi g : H_d \to G$ is generated by the class of the germ at 0 of the continuous but not differentiable section $t \mapsto (t, |t|)$; this kernel is isomorphic as groupoid to the kernel of ψ as in the previous example. Pradines has pointed out in [P3] that this kind of example may be varied by quotienting the bundle of groups $\alpha_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by subbundles generated by sections with one or another continuity or differentiability properties, and then constructing the holonomy groupoid using only sections with stronger properties.

Remark 3.6 We emphasise, as explained in Remark 1.3(iv), that Theorem 2.1 is a new result even in the case G is a group, although of course the proof would become much simpler if restricted to that case.

4   The universal property

In this section we prove the universal property of the morphism $\phi : H \to G$, namely property (ii) of Theorem 2.1. Suppose then that A and ζ are as in that theorem.

It is clear that $X \subseteq \zeta^{-1}(W) \subseteq A$. Let $a \in A(x,y)$. The aim is to define $\zeta'a \in H$.

Since $\zeta^{-1}(W)$ generates A, we can write $a = a_n \ldots a_1$, where $\zeta a_i \in W$ and $i = 1, \ldots, n$.

Since A has enough continuous admissible local sections, we can choose continuous admissible local sections f_i of α_A through $a_i, i = 1, \ldots, n$, such that they are composable and their images are contained in $\zeta^{-1}(W)$.

By condition (b), the continuity of ζ on $\zeta^{-1}(W)$ implies that ζf_i is a continuous admissible local section of α through $\zeta a_i \in W$ whose image is contained in W. Therefore $\zeta f \in \Gamma^c(G, W)$. Hence we can set

$$\zeta'a = \langle \zeta f \rangle_{\alpha a} \in H.$$

We now prove that ζ' is well defined.

Lemma 4.1 $\zeta'a$ *is independent of the choices which have been made.*

Proof   Let $a = b_m \ldots b_1$, where $\zeta b_j \in W$ and $j = 1, \ldots, m$. Choose a set of continuous admissible local sections g_j of α_A through b_j such that the g_j's are composable and their images are contained in $\zeta^{-1}(W)$.

Let $g = g_m \ldots g_1$. Then $\zeta g \in \Gamma^c(G, W)$, and so $\langle \zeta g \rangle_x \in H$.

Since, by assumption, $fx = gx = a \in A$, then $(fx, gx) \in A \times_\alpha A$ and $\delta_A(fx, gx) = (fx)(gx)^{-1} = 1_x$. Hence $(fx, gx) \in \delta_A^{-1} \zeta^{-1}(W)$ because $1_x \in \zeta^{-1}(W)$.

Because A is a topological groupoid, the groupoid difference map $\delta_A : A \times A \to A$ is continuous. Since $\zeta^{-1}(W)$ is open in A, by condition (b), then $\delta^{-1} \zeta^{-1}(W)$ is open in $A \times_\alpha A$.

But, by the continuity of f and g, the induced map $(f, g) : (\mathcal{D}f) \cap (\mathcal{D}g) \to A \times_\alpha A$ is continuous. Hence there exists an open neighbourhood N of x in X such that $(f, g)(N) \subseteq \delta^{-1} \zeta^{-1}(W)$ which implies that $(fg^{-1})(\beta g N) \subseteq \zeta^{-1}(W)$, and so, after suitably restricting f and g, which we may suppose done without change of notation, we have that fg^{-1} is a continuous admissible local section of α_A through $1_y \in A$ and its image is contained in $\zeta^{-1}(W)$. So $\zeta(fg^{-1})$

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is a continuous admissible local section of α through $1_y \in W$, and its image is contained in W. Therefore $[\zeta(fg^{-1})]_y \in J^c(W)$.

Since $fx = gx$, then $\psi[\zeta f]_x = \psi[\zeta g]_x$. But ψ and ζ are morphisms of groupoids; hence $\psi[\zeta(fg^{-1})]_y = 1_y$, and so $[\zeta(fg^{-1})]_y \in \mathrm{Ker}\,\psi$.

Therefore $[\zeta(fg^{-1})]_y \in J^c(W) \cap \mathrm{Ker}\,\psi = J_0$. Since ζ is a morphism of groupoids, we have $[\zeta(fg^{-1})]_y \in J^c$. Hence $\langle\zeta(fg^{-1})\rangle_y = 1_y \in H$, and so $\langle\zeta g\rangle_x = \langle\zeta(fg)\rangle_y\langle\zeta g\rangle_x = \langle\zeta f\rangle_x$ which shows that $\zeta' a$ is independent of the choices made. $\qquad\square$

Lemma 4.2 ζ' *is a morphism of groupoids.*

Proof Let $c = ab$ be an element of A such that $a = a_n \ldots a_1 and b = b_m \ldots b_1$, where $a_i, bj \in \zeta^{-1}(W), i = 1, \ldots, n$ and $j = 1, \ldots, m$. Then $c = a_n \ldots a_1 b_m \ldots b_1$.

Let f_i, g_j be continuous admissible local sections of α_A through a_i and b_j respectively such that they are composable and their images are contained in $\zeta^{-1}(W)$. Let $f = f_n \ldots f_1$, $g = g_m \ldots g_1$, $s = fg$. Then s is a continuous admissible local section of α_A through $c \in A$, and $\zeta f, \zeta g, \zeta s \in \Gamma^c(G, W)$, and $\zeta s = (\zeta f)(\zeta g)$, since ζ is a morphism of groupoids.

Let $x = \alpha a, y = \alpha b$. Then $\langle\zeta s\rangle_y = \langle\zeta f\rangle_x\langle\zeta g\rangle_y$, and so ζ' is a morphism. $\qquad\square$

Lemma 4.3 *The morphism ζ' is continuous, and is the only morphism of groupoids such that $\phi\zeta' = \zeta$ and $\zeta' a = i\zeta a$ for all $a \in \zeta^{-1}(W)$.*

Proof Since α_A has enough continuous admissible local sections, it is enough to prove that ζ' is continuous at 1_x for any $x \in X$. Let $\mathbf{1}$ denote the section $x \mapsto 1_x$ of the source map of a groupoid.

Let $x \in X$. If $b \in \zeta^{-1}(W)$ and s is a continuous admissible local section through b, then $\zeta' b = \langle\zeta s\rangle = s\zeta b$. Since ζ is continuous, it follows that ζ' is continuous.

The uniqueness of ζ' follows from the fact that ζ' is determined on $\zeta^{-1}(W)$ and that this set generates A. $\qquad\square$

This completes the proof of our main result, Theorem 2.1.

5 α-structured locally topological groupoids

The projection $\phi : H \to G$ of the holonomy groupoid of (G, W) to G maps iW homeomorphically to W. We can also obtain a useful relation between the topology of H and the topology of suitable translates of the topology of W around G. This is done by giving a topology to each α-fibre $\alpha^{-1}x, x \in G_0$, of G. Of course this does not in general give a topological groupoid structure on G. What can be said is as follows. (These results are in Section 1 of [P1].)

Definition 5.1 A locally α-topological groupoid is a pair (G, W) consisting of a groupoid G and a topological space W such that:
(i) $O_G \subseteq W \subseteq G$;

(ii) W generates G as a groupoid;

(iii) W is the topological sum of the subspaces $W_x = W \cap \alpha^{-1}x, x \in O_G$;

(iv) if $g \in G$, then the sets $R_g^{-1}W = \{w \in W : wg \in W\}$ and $W \cap Wg$ are open in W and the right translation $R_g : R_g^{-1}W \to W \cap Wg, w \mapsto wg$, is a homeomorphism.

In particular, an α-topological groupoid is a locally α-topological groupoid (G, G). Note that an α-topological group is not necessarily a topological group.

The following simple proposition shows that any locally α-topological groupoid is extendible to an α-topological groupoid. This is essentially Proposition 1 of section 1 of [P1]. It makes a useful contrast to the Globalisation Theorem of Section 2.

Proposition 5.2 *(α-extendibility) Let (G, W) be a locally α-topological groupoid. Then (G, G) may be given the structure of α-topological groupoid such that for all $x \in O_G$, W_x is an open subset of G_x.*

Proof We define charts for G to be the right translations

$$R_g : W_x \to W_y g$$

for $g \in G(x, y)$ and $x, y \in O_G$. Suppose that $h, g \in G_x$ and $W_y g$ meets $W_z h$. Then there are elements $u \in W_y$ and $v \in W_z$ such that $ug = vh$. So $R_h^{-1}R_g$ maps the open neighbourhood $R_{u^{-1}v}^{-1}W$ of u in W_y to the open neighbourhood $W \cap Wu^{-1}v$ of v in W_z. So these charts define a topology as required. $\square$

We now apply this result to the case of a locally topological groupoid.

Proposition 5.3 *Let (G, W) be a locally topological groupoid. Let $\phi : H \to G$ be the projection of the holonomy groupoid of (G, W). Then there is a topology on each α-fibre $G_x = \alpha_G^{-1}x, x \in O_G$, such that the restriction of ϕ to $H_x \to G_x$ is étale.*

Proof First of all, by redefining the topology on W to make it the topological sum W' of the W_x, we obtain a locally α-topological groupoid (G, W'), and so, by Proposition 5.2, a topology on each G_x. A direct check against the topology defined on H now gives the result. $\square$

6 The locally trivial case

Let G be a groupoid and let W be a subset of G such that $O_G \subseteq W$, and W also has the structure of topological space. Then (G, W) is said to be *locally trivial* if for all $x \in O_G$ there is a neighbourhood U of x and a continuous section $s : U \to W$ of β such that $\alpha sy = x$ for all $y \in U$. This is equivalent to saying that the anchor map $A_W : W \to O_G \times O_G, w \mapsto (\alpha w, \beta w)$, is a surmersion, i.e. for each $w \in W$ there is a local section of A_W through w. This definition goes back to Ehresmann. For more information, see [M]. The main result of this section is the following, which is also given in [P1]. We are grateful for J. Pradines for supplying the proof (private communication) as an improvement to his original (unpublished) proof.

Proposition 6.1 *Locally trivial locally topological groupoids are extendible.*

Proof Let (G, W) be a locally trivial locally topological groupoid. Set $X = O_G$. We use the local triviality to define an inverse θ to the projection $H \to G$ from the holonomy groupoid H of (G, W).

For each $x \in X$, choose a section s_x as in the definition of local triviality. We may and do assume that $s_x(x) = 1_x$.

Let $g \in G$ have source x and target y. Choose a local homeomorphism h of X with $hx = y$. This is possible by the existence of enough continuous admissible local sections and the fact that W generates G. Define an admissible local section t through g by

$$t(z) = s_y(hz).g.(s_x(z))^{-1}$$

for suitable $z \in W$. Let $\theta(g)$ denote the class mod J_0 of the germ $[t]_x$ in $J(G)$.

Then we find the following:

(i) $\theta(g)$ does not depend on the choice of h, so that θ is a well defined function from G to H. Indeed, if u is a local section through g defined in a similar way by a local homeomorphism k, then the local section $v = ut^{-1}$ satisfies

$$v(z) = s_y(kz)s_y(hz)^{-1},$$

so that $v(x) = 1_y$ and for z sufficiently near to x, $v(z) \in W$. Hence $[v]_x \in J_0$, and so $[t]_x \equiv [u]_x$ mod J_0.

The next results are now easy to check.

(ii) θ is a morphism of groupoids.

(iii) The restriction of θ to W coincides with the canonical embedding $i : W \to H \subseteq J(G)/J_0(G)$.

(iv) Therefore θ induces an isomorphism of G onto H, inverse to ϕ. $\qquad\square$

7 Historical remarks

The concept of holonomy groupoid was introduced by C.Ehresmann and Weishu Shih in 1956 [Eh-We] and C.Ehresmann in 1961 [Eh3], for a locally simple topological foliation on a topological space X (this means that X has two comparable topologies, and with respect to the finer topology on X, a cover by open sets, in each of which the two topologies coincide). Such a holonomy groupoid is considered as a topological groupoid H on X. It is constructed as a groupoid of local germs of the groupoid H' of holonomy isomorphisms between the transverse spaces U_i of simple open subsets U_i of X such that (U_i, U_{i+1}) is a 'pure chain'. The holonomy group at $x \in X$ is the vertex group $H(x)$ of H. This holonomy group is isomorphic to the holonomy group $H(y)$ for each y on the same leaf of the foliation as x.

J. Pradines in 1966 [P1] considered this holonomy groupoid H, in a wider context, with its differential structure. He took the point of view that a foliation determines an equivalence relation R by xRy if and only if x and y are on the same leaf of the foliation, and that this equivalence relation should be regarded as a groupoid in the standard way, with multiplication

$(x, y)(y, z) = (x, z)$ for $(x, y), (y, z) \in R$. This groupoid is also written R. In the paracompact case, the locally differential structure which gives the foliation determines a differential structure, not on R itself, but 'locally' on R, that is, on a subset W of R containing the diagonal ΔX of X. That is, the foliation determines a locally topological groupoid. The full details of this are given in [3].

This led Pradines to a definition of "un morceau differentiable de groupoide" G, for which [M], p.161, uses the term "locally differential groupoid". Pradines' note [P1] asserts essentially that such a (G, W) determines a differential groupoid $Q_0(G, W)$ and a homomorphism P : $Q_0(G, W) \to G$ such that the "germ" of W extends to a differential structure on G if and only if P is an isomorphism. However his statement of results assumes that the base $X = O_G$ is paracompact and that (G, W) is α-connected, i.e. $\alpha^{-1}(x) \cap W$ is connected for each $x \in X$. These assumptions seem to be necessary to extend the Globalisation Theorem 2.1 to the case of germs.

The groupoid $Q_0(G, W)$ is called by Pradines the holonomy groupoid of (G, W).

One of the key motivations for the construction of the holonomy groupoid in [P1] is the construction of the monodromy groupoid MG of a differential groupoid G, in which the stars $\alpha_{MG}^{-1}(x)$ of MG are the universal covers of the stars $\alpha_G^{-1}(x)$ of G. See also [M], p.68, for a specific construction of this kind in the locally trivial case. A related work is [D-L], which gives some results for the cases of a group, and of a bundle of groups. An outline of Pradines' construction is given in [B1], and the full details have been given in [Mu] and [2]. This paper also makes clear the monodromy principle available in this general situation.

The monodromy topological groupoid of a foliation is also considered in [Ph] and in [K-M]. The latter paper also develops work of [R1,2].

A construction of the holonomy groupoid in the differential case is attempted by Almeida in [Al], using properties of integration of vector fields. However this construction has not been published elsewhere, and of course does not extend to the topological case.

Following Ehresmann's work, there has long been interest in the holonomy group of a leaf of a smooth foliation, see for example [L1,2]. For the locally differential groupoid corresponding to a smooth foliation, the vertex groups of the Ehresmann-Pradines holonomy groupoid are the holonomy groups in the standard sense.

The holonomy groupoid H of a smooth foliation on a manifold X was rediscovered (using a different, but equivalent, description) by Winkelnkemper [W], as the "graph of the foliation". This was defined as the set S of all triples $(x, y, [\gamma])$, where $x, y \in X$ are on the same leaf L of the foliation, γ is a continuous path on L and $[\gamma]$ is the equivalence class of γ under the equivalence relation $\sim$ which is given by: for the two paths γ_1, γ_2 in L starting at x and ending at y, $\gamma_1 \sim \gamma_2$ if and only if the holonomy of L at x along $\gamma_1^{-1}\gamma_2$ is zero. As pointed out above, these ideas are a special case of the general construction considered here. The way in which the holonomy and monodromy are related in the general case is discussed in [2].

Connes [Co] has considered this differential holonomy groupoid H of the foliation and applied to it his general theory of integration based on transverse measures on a measurable groupoid. More recently, in [4], he has applied this and other groupoids in the theory of non commutative C^*-algebras.

Pradines in [P1] also defines what he calls a germ of a locally differential groupoid, by saying two locally differential groupoids (G, W) and (G, W') are *equivalent* if there is a third locally

differential groupoid (G, W'') such that W'' is an open submanifold of both W and W'. Such a germ is called a microdifferential groupoid. His aim is then to define the holonomy groupoid as a functor on the category of such microdifferential groupoids. One of the problems of this theory is that if (G, W) and (G, W') are locally differential groupoids, then $W \cap W'$ may no longer generate G. This difficulty does not occur if the locally differential groupoids are α-connected, since in this case if W generates G then so also does any open subset of W containing O_G. Thus there is still work to be done in investigating examples of these constructions and the relations between and consequences of various possible definitions.

Three principal examples of groupoids are bundles of groups, equivalence relations, symmetry groupoids, and action groupoids associated with an action of a group (or more generally groupoid) on a set (see for example [B2]). At present, it seems that only the holonomy of an equivalence relation has been extensively studied, namely in the form of the holonomy groups and holonomy groupoid of a smooth foliation (but see also [R1,2], [K-M], [1]). There is presumably considerable potential value in the other cases.

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