A COMPACT-OPEN TOPOLOGY ON PARTIAL MAPS WITH OPEN DOMAIN

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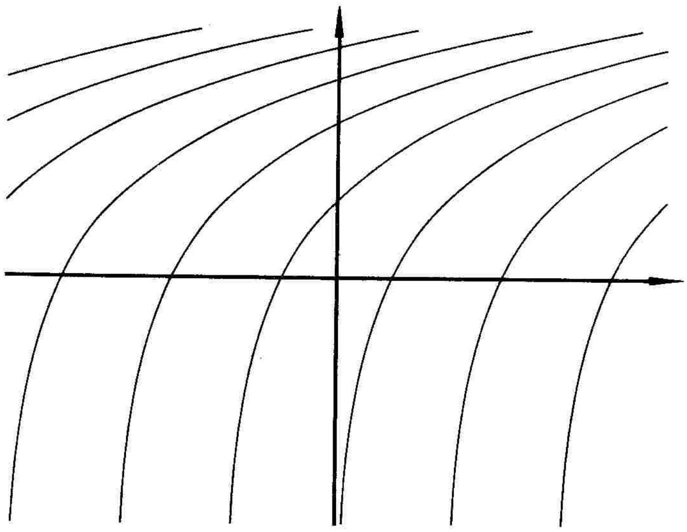
1. *Introduction*

A *partial map* $f: X \rightarrowtail Y$ is defined as a triple (X, Y, G_f) where X, Y are sets and G_f, the *graph* of f, is a subset of $X \times Y$ such that $(x, y), (x, y') \in G_f$ implies $y = y'$. More informally, a partial map consists of X, Y and a function with domain contained in X and range contained in Y.

From the point of view of analysis, partial maps occur more naturally than maps. For example, it is natural to consider functions such as log, $x+1$, $\sin^{-1}$, $\surd$ as partial maps $\mathbb{R} \rightarrowtail \mathbb{R}$, and this allows for a smooth exposition, unclogged by notation, of many aspects of elementary analysis.

It is thus surprising that the algebra and topology of such partial maps has been little studied, and this paper is one of several which attempt the beginning of such a study. Some algebraic aspects of partial maps are considered in [2]. Here our attention is on aspects of their topology.

The following example will indicate some of the motivation for this study. Consider for various $y \in \mathbb{R}$ the real functions $f_y : x \mapsto \log(x+y)$. This family of functions is illustrated below†:



It is natural to suggest that this family varies continuously with y, that is, that $y \mapsto f_y$ is a continuous function in an appropriate topology. However, the domain $\mathscr{D}(f_y)$ of f_y is the interval $(-y, \infty)$ which varies with y. We are in the curious position that such an example, apparently so basic, does not seem to be treated in the enormous literature on

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function spaces, which, as far as we know, is concerned almost entirely with functions with fixed domain.

One attempt to obtain a usable topology for partial maps is indicated in [3]. The basic observation is that if f is a function and C is any set then $f(C)$ can be defined to be $f(C \cap \mathscr{D}(f))$ (this definition occurs in [**10**]) and so the compact-open topology, with its sub-basic open sets of the form $W(C, U) = \{f : f(C) \subseteq U\}$ for C compact and U open, extends immediately to *any* set of partial maps. However it is not clear if this definition gives useful properties of such spaces, in general. For spaces of partial maps with *closed* domain, we do obtain good properties; this was shown in [3], using the idea of representability for such maps (called parc maps in [3]).

In this paper we study partial maps with open domain, which we call *paro maps.* We expand on a hint given in [3] and take the representability of paro maps as the basic idea. This gives a (modified) compact-open topology for spaces of paro maps. As an application we show in §4 how to give a topology for a pseudo-group Γ of transformations of a space X so that if X is locally compact, Γ becomes an example of a *topological pseudo-group.* As far as we know, this concept has not previously been studied.

The compact-open topology on paro maps specialises to a useful topology on the set $\mathscr{O}(X)$ of open sets of a space X. This topology has been considered in [**12**] and elsewhere.

The material of this paper is a part of [1], and the first author is grateful to the Egyptian government for support.

2. *The compact-open topology on paro maps*

For any spaces X, Y, let $M(X, Y)$ be the space of all maps $X \to Y$ (i.e. continuous functions with domain X) with the compact-open topology.

For any space Y, let $Y^\wedge$ be the space $Y \cup \{\omega\}$ (where $\omega \notin Y$) with the topology in which U is open in $Y^\wedge$ if and only if $U = Y^\wedge$ or U is open in Y. Then Y is an open subspace of $Y^\wedge$ and $\{\omega\}$ is closed, but not open, in $Y^\wedge$. (This definition was given in [3; 1.6].)

Let $P_0(X, Y)$ be the set of paro maps $X \rightarrowtail Y$.

PROPOSITION 1. *There is a bijection*

$$\mu : P_0(X, Y) \longrightarrow M(X, Y^\wedge)$$

where

$$\mu(f)(x) = \begin{cases} f(x) & \text{if } x \in \mathscr{D}(f) \\ \omega & \text{otherwise.} \end{cases}$$

The topology on $P_0(X, Y)$ which makes μ a homeomorphism has a sub-basis of open sets of the form

$$W(C, U) = \{f \in P_0(X, Y) : C \subseteq \mathscr{D}(f),\ f(C) \subseteq U\}$$

for all compact subsets C of X and open subsets U of Y.

From now on, we give $P_0(X, Y)$ the topology of Proposition 1.

Unlike the usual mapping space, the case of paro maps $X \rightarrowtail Y$ where Y is a singleton is still interesting. Let 1 denote the one-point space with element 1. Let $\mathscr{O}(X)$ denote the set of open sets of X.

PROPOSITION 2. *There is a natural bijection*

$$\chi : \mathscr{O}(X) \longrightarrow P_0(X,1)$$

such that $\chi(U)$ is the constant paro map $X \rightarrowtail 1$ with domain U. The topology on $\mathscr{O}(X)$ which makes χ a homeomorphism has a sub-basis of open sets of the form

$$W(C) = \{U \in \mathscr{O}(X) : C \subseteq U\},$$

for C compact.

An elementary fact about the spaces $P_0(X, Y)$ is the existence of induced maps.

PROPOSITION 3. (i) *Let $h : T \rightarrowtail X$, $g : Y \rightarrowtail Z$ be paro maps. Then h, g induce, by composition, maps*

$$g_* : P_0(X, Y) \longrightarrow P_0(X, Z),$$

$$h^* : P_0(X, Y) \longrightarrow P_0(T, Y).$$

(ii) *Let X', Y' be subsets of X, Y respectively such that X' is open in X. Let the map $i : P_0(X', Y') \to P_0(X, Y)$ send a paro map $X' \rightarrowtail Y'$ to the paro map $X \rightarrowtail Y$ with the same graph. Then i is a homeomorphism into.*

The proof is straightforward.

PROPOSITION 4. *The domain map $\mathscr{D} : P_0(X, Y) \to \mathscr{O}(X), f \mapsto \mathscr{D}(f)$, is continuous.*

Proof. The composite $\chi\mathscr{D} : P_0(X, Y) \to P_0(X, 1)$ is induced by the constant map $Y \to 1$ and so is continuous. Hence $\mathscr{D}$ is continuous.

Remark. The definitions given here suggest a new topology for spaces of germs of which the applications are at present unexplored. Recall that an equivalence relation $\sim_x$ on paro maps $X \rightarrowtail Y$ with domain including a point x of X is obtained by saying $f \sim_x g$ if x has an open neighbourhood U such that $f|U = g|U$. The equivalence classes $\bar{f}_x$ of such paro maps f are called *germs* at x of functions $X \rightarrowtail Y$. The set of all such germs at all $x \in X$ is written $\mathscr{G}(X, Y)$. Let D be the subspace of $X \times P_0(X, Y)$ of points (x,f) such that $x \in \mathscr{D}(f)$. Then the definition of germs gives a surjection

$$G : D \to \mathscr{G}(X, Y), \qquad (x,f) \mapsto \bar{f}_x,$$

and it seems reasonable to consider giving $\mathscr{G}(X, Y)$ the identification topology with respect to G. It is possible that this topology is relevant to singularity theory: A. du Plessis in [7; p. 65] says: "To match more closely the stability theory for mappings, we seem forced to consider representatives of germs: essentially, for 'nearby' germs we need to consider representatives defined on 'nearby' open sets. This looks a very difficult theory to work with...."

The relation between the identification topology and the usual sheaf topology is that the latter topology is the finer, and that the topologies differ if X is non-empty and $Y = \mathbb{R}$. Recall that the sheaf topology has basic open sets $W(f) = \{\bar{f}_y : y \in \mathscr{D}_f\}$ for any paro map $f: X \rightarrowtail Y$. On the other hand if A is open in the identification topology, then $G^{-1}(A)$ is open in D and so is a union of sets of the form $B = D \cap \bigcap_{i=1}^{n} W(C_i, U_i) \times V_i$ where U_i, V_i are open in Y, X respectively, and C_i is compact in X . So if $\bar{f}_x \in A$, then (f, x) belongs to such a B. Now if $g = f \mid \bigcap_{i=1}^{n} V_i$, then $\bar{f}_x \in W(g) \subseteq A$; thus A is open in the sheaf topology.

Suppose $Y = \mathbb{R}$ and X is non-empty. Let $W(f)$ be a basic neighbourhood of $\bar{f}_x$ in the sheaf topology, and let A be an open neighbourhood of $\bar{f}_x$ in the identification topology. Assume $(f, x) \in B$ above , and that r is the minimum of the distances $d(f(C_i), \mathbb{R}\setminus U_i)$. Then $r > 0$ and the function $h = f + r$ is such that $W(h)$ is contained in A but does not meet $W(f)$. So the topologies differ in this case.

3. *The exponential law and its applications*

We recall that the exponential function

$$\theta : M(X \times Y, Z) \to M(X, M(Y, Z)) \tag{3.1}$$

$$\theta(f)(x)(y) = f(x, y), \qquad x \in X, y \in Y, \qquad f \in M(X \times Y, Z)$$

is a well-defined injection. The pair (X, Y) is called an *exponential pair* if for all spaces Z the map θ of (3.1) is a surjection. It is standard that (X, Y) is an exponential pair if (i) each point of Y has a fundamental system of compact neighbourhoods, or (ii) $X \times Y$ is a Hausdorff k-space. (We reserve the term *locally compact* to mean the condition (i) of the last sentence.)

The following theorem is immediate from the above, the representability of paro maps, and other standard results on the exponential law.

Theorem 5. (*Exponential law for paro maps.*) *The exponential function*

$$\theta : P_0(X \times Y, Z) \to M(X, P_0(Y, Z))$$

$$\theta(f)(x)(y) = f(x, y)$$

is a well-defined injection. Further:

(i) *if (X, Y) is an exponential pair, then θ is surjective;*

(ii) *if X is Hausdorff, then θ is continuous;*

(iii) *if X, Y are Hausdorff, then θ is a homeomorphism into.*

As usual, the maps f and $\theta(f)$ are called *adjoint*.

Although the proof of the above result is simple, the theorem does have some interesting and important applications. First, we solve the problem posed in the Introduction.

Example 1. The function $\mathbb{R} \to P_0(\mathbb{R}, \mathbb{R})$, $y \mapsto (x \mapsto \log(x+y))$, is continuous.

Proof. This map is adjoint to $f: \mathbb{R} \times \mathbb{R} \rightarrowtail \mathbb{R}$, $(x, y) \mapsto \log(x+y)$, which is clearly a paro map.

The domain of $x \mapsto \log(x+y)$ is $(-y, \infty)$. By Example 1 and Proposition 4, we obtain that the map $\mathbb{R} \to \mathcal{O}(\mathbb{R})$, $y \mapsto (-y, \infty)$, is continuous. Alternatively, this can be proved as a consequence of the following.

For any subset U of $X \times Y$ and $x \in X$, we define

$$U_x = \{y \in Y : (x, y) \in U\},$$

and call U_x the *x-section* of U. If U is open in $X \times Y$, then U_x is open in Y.

PROPOSITION 6. *The section map*

$$\sigma : \mathcal{O}(X \times Y) \longrightarrow M(X, \mathcal{O}(Y))$$

$$\sigma(U)(x) = U_x$$

is a well-defined injection. Further:

(i) *if (X, Y) is an exponential pair, then σ is surjective;*

(ii) *if X is Hausdorff, then σ is continuous;*

(iii) *if X, Y are Hausdorff, then σ is a homeomorphism into.*

Proof. The bijection χ of Proposition 2 transforms σ to θ of Theorem 5, but with $Z = 1$.

The intuitive meaning of Proposition 6 is that if (X, Y) is an exponential pair (and in particular if Y is locally compact) then *an open set in $X \times Y$ corresponds precisely to a continuous (indexed by X) family of open sets of Y.*

Example 2. The set $U = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y + x > 0\}$ is open in $\mathbb{R} \times \mathbb{R}$. Its section map $\sigma(U) : \mathbb{R} \to \mathcal{O}(\mathbb{R})$ is $x \mapsto (-x, \infty)$, and $\sigma(U)$ is continuous by Proposition 6.

Other applications of Theorem 5 are standard applications of exponential laws. They are still interesting results on paro maps, or on spaces of open sets.

PROPOSITION 7. *If Y is locally compact, then the evaluation map $\varepsilon : P_0(Y, Z) \times Y \rightarrowtail Z$, $(f, y) \mapsto f(y)$, is a paro map, and hence has open domain.*

Proof. In Theorem 5, set $X = P_0(Y, Z)$; then $\varepsilon = \theta^{-1}(1)$.

PROPOSITION 8. *If Y is locally compact, then the membership relation*

$$M = \{(U, y) : U \ni y\} \subseteq \mathcal{O}(Y) \times Y$$

is an open subset of $\mathcal{O}(Y) \times Y$.

Proof. Take $Z = 1$ in Proposition 7. If χ is as in Proposition 2, then $(\chi \times 1)(M) = \mathcal{D}(\varepsilon)$. Hence M is open.

PROPOSITION 9. *If X and Y are locally compact, then the composition mapping*

$$\gamma : P_0(X, Y) \times P_0(Y, Z) \longrightarrow P_0(X, Z), (f, g) \longrightarrow gf,$$

is continuous.

Proof. The mapping γ is adjoint to the composite

$$P_0(X, Y) \times P_0(Y, Z) \times X \xrightarrow{\alpha} P_0(Y, Z) \times Y \xrightarrow{\varepsilon} Z$$

where $\alpha(f, g, x) = (g, f(x))$. Both α and ε are continuous, by Proposition 7.

Remark. Topologies on the set $\mathcal{O}(X)$ of open sets of X have also been considered in [**6, 9, 11, 12**]. In particular [**12**] considers a compact-open topology as in Proposition 1, while Proposition 6 is clearly related to [**12**; Theorem 1, p. 271]. However none of these papers consider the set of paro maps from X to Y.

4. *Topological pseudo-groups of transformations*

Let X be a topological space. A *pseudo-group of transformations of* X is a set Γ of paro maps $X \rightarrowtail X$ such that:

(i) each $f \in \Gamma$ is a homeomorphism from an open subset of X to an open subset of X;

(ii) $f \in \Gamma$ implies $f^{-1} \in \Gamma$;

(iii) $f, g \in \Gamma$ implies $fg \in \Gamma$;

(iv) if $f \in \Gamma$ with f having domain and range U, V respectively, then $1_U, 1_V$ (the identities on U, V) also belong to Γ.

Such pseudo-groups are used in many areas of topology and differential geometry (see the Bibliographies in [**4, 5**]). In [**2**] we show the relation of the notion of pseudo-group to that of inverse semi-group. Here we show how Γ may be topologised to become, if X is locally compact, a *topological pseudo-group.*

Let $\Gamma(X)$ be the pseudo-group of all homeomorphisms of open subsets of X. Then each $f \in \Gamma$ is a paro map. Let $\iota : \Gamma(X) \to P_0(X, X)$ be the inclusion and let $\eta : \Gamma(X) \to \Gamma(X)$ be the inverse map $f \mapsto f^{-1}$. We give $\Gamma(X)$ the initial topology with respect to ι and $\iota\eta$.

Proposition 10. *If X is locally compact then the composition map $\gamma' : \Gamma(X) \times \Gamma(X) \to \Gamma(X)$, $(f, g) \mapsto gf$, and the inverse map $\eta : \Gamma(X) \to \Gamma(X)$ are continuous.*

Proof. The following diagram is commutative

$$\begin{array}{ccc} \Gamma(X) \times \Gamma(X) & \xrightarrow{\quad \gamma' \quad} & \Gamma(X) \\ \Big\downarrow \iota \times \iota & & \Big\downarrow \iota \\ P_0(X, X) \times P_0(X, X) & \xrightarrow{\quad \gamma \quad} & P_0(X, X). \end{array}$$

Since X is locally compact, γ is continuous. So $\iota\gamma'$ is continuous. Similarly, $\iota\eta\gamma' = \gamma T(\iota\eta \times \iota\eta)$, where T is a twist map $(f, g) \mapsto (g, f)$. Hence $\iota\eta\gamma'$ is continuous. Hence γ' is continuous.

Also $\iota\eta$ is continuous, as is $\iota\eta\eta = \iota$. So η is continuous.

We can now give any sub-pseudo-group Γ of $\Gamma(X)$ the subspace topology so that it also becomes, if X is locally compact, a topological pseudo-group, in the sense that the composition and inverse mappings are continuous.

In a subsequent paper we show how to extend these definitions to a convenient category of topological spaces.

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