Some problems of Algebraic Topology

A study of function spaces, function complexes and FD-complexes.

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ABSTRACT.

This thesis studies some aspects of the homotopy type of function spaces X^Y where X,Y are topological spaces.

The thesis is in two parts. Part A (Chapters I - IV) contains a discussion of some known facts on the homotopy type of function spaces under the heads of homology (Chapter II), homotopy groups (Chapter III) and Postnikov systems (Chapter IV). Also, in Chapter II, a theorem on duality is given which is useful in determining the low-dimensional homotopy type of $(S^n)^X$ when $X = S^r \cup e^{r+q}$ $(r + q < n)$.

Chapter IV contains the statements of the problems whose solution is the motivation of the theory of Part B. These problems, which occur naturally in attempting to find the Postnikov system of X^Y by induction on the Postnikov system of X, are roughly of the type of determining $k^Y : X^Y \longrightarrow A^Y$ when X,Y are spaces, A is a topological abelian group and $k : X \rightarrow A$ is a map. This problem we call here the "k-invariant problem".

It is a commonplace that the most important property of function spaces is the "exponential law" which states that under certain restrictions the spaces $X^{Z \times Y}$ and $(X^Y)^Z$ are homeomorphic. In fact it is usually the case that the only properties of the function space required are that as a set X^Y is the set of maps $Y \longrightarrow X$, and that the exponential law holds.

In Chapter I, as preparation for the work of Part B, a brief discussion of the exponential law in a general category is given. The rest of the chapter shows how the well-known weak-topological product may be used to obtain an exponential law for all (Hausdorff) spaces. The weak

product is also shown to be convenient in the theory of the identification maps.

The theory of Part B is given in terms of css-complexes (complete semi-simplical complexes) with base point. In Chapter V the well-known css-exponential law is extended to the category of css-M-ads, and the exponential law for complexes with base point obtained. The relation between the topological and css- function spaces is discussed, and it is shown that the singular functor preserves the exponential law.

The further theoretical work of Part B is initially of two kinds. First, the function complex A^Y where A is an FD-complex, is related, by means of maps and functors, with mapping objects in the category of FD-complexes and chain complexes. This is done in such a way to preserve the exponential law. Second, a generalised cohomology of a complex is introduced, with the coefficient group replaced by an arbitrary chain complex (or FD-complex). The theories of cohomology operations and of Eilenberg-Maclane complexes are correspondingly generalised. Using these two sets of constructions, a solution of the k-invariant problem is given in terms of chain complexes (Chapter IX. § 2).

The rest of Part B is concerned with obtaining the cohomological solution of the k-invariant problem, putting the results in a form suitable for computation, and obtaining applications.

In detail Part B proceeds, after Chapter V, as follows. Chapter VI discusses chain complexes and the functor $\pitchfork$ of chain complexes, with particular attention to signs. The exponential map here is, for chain complexes A, B, C an isomorphism $\mu : (A \otimes B) \pitchfork C \to A \pitchfork (B \pitchfork C)$.

In Chapter VII, on FD-complexes, a map product $A \pitchfork B$ of FD-complexes A, B is introduced. The exponential map here is an isomorphism $(A \times B) \pitchfork C \to A \pitchfork (B \pitchfork C)$, where $A \times B$ is the cartesian product of A and B. If Y is a css-complex, and $C(Y)$ is essentially the free FD-complex generated by Y, it is shown that there is an isomorphism $D : A^Y \longrightarrow C(Y) \pitchfork A$ which preserves the exponential law, and by which these complexes may be identified. The well-known properties of the normalisation functor N and Dold-Kan functor R are given, and the generalised cohomology introduced. The fundamental classification theorem is proved, and the theory of operations derived.

Chapter VIII relates the Dold-Kan theory and generalised Eilenberg-Maclane complexes; the exactness properties of the latter are discussed.

In Chapter IX products $A \otimes B$, $A \pitchfork B$ of FD-complexes are defined such that $A \otimes B \approx R(NA \otimes NB)$, $A \pitchfork B \approx R(NA \pitchfork NB)$. An exponential map $\mu : (A \otimes B) \pitchfork C \longrightarrow A \pitchfork (B \pitchfork C)$ is defined and proved to be an isomorphism by showing that $N\mu$ is essentially the exponential map for chain complexes. Homotopy equivalences $\Delta : A \otimes B \longrightarrow A \times B$, $\hat{\Delta} : A \pitchfork B \longrightarrow A \pitchfork B$ are defined, and a commutativity relation with the exponential maps established.

Using this amount of structure, in IX § ? theorems are given which determine the compositions

$$k' : X^Y \xrightarrow{k^Y} A^Y \xrightarrow{\hat{\Delta}} C(Y) \pitchfork A ,$$

$$\ell' : C(Y) \pitchfork B \xrightarrow{\hat{\Sigma}} B^Y \xrightarrow{\ell^Y} A^Y \xrightarrow{\hat{\Delta}} C(Y) \pitchfork A ,$$

where X, Y are css-complexes, A, B are FD-complexes, $k : X \rightarrow A$, $\ell : B \rightarrow A$ are maps and $\hat{\xi}$ is a homotopy inverse of $\hat{\Delta}$. The determination of k' is in terms of the evaluation map on X^Y, the determination of ℓ' is in terms of a kind of generalised suspension operation.

Chapter X shows how Künneth isomorphisms may be constructed and computed. A Künneth isomorphism is, in one case, defined for a given css-complex Y and FD-complex A and, for all css-complexes X, maps $H^*(X \times Y, A) \approx H^*(X, H^*(Y, A))$ naturally with respect to maps of X.

In Chapter XI, it is shown that such a Künneth isomorphism has an associated homotopy equivalence $\lambda : C(Y) \pitchfork A \rightarrow RH^*(Y, A)$. The compositions of the maps k', ℓ' with such equivalences $\lambda : C(Y) \pitchfork A \rightarrow R\, H^*(Y, A)$, $\lambda' : RH^*(Y, B) \longrightarrow C(Y) \pitchfork B$ are determined by Theorems A and B of XI. § 1. The theorems, which solve the k-invariant problem, are obtained from those of IX. § 2 by the use of certain "coefficient homomorphisms". It should be emphasised that the theorems of IX. § 2 are natural with respect to maps of the complexes concerned, while the theorems of XI. § 1 are not; this is one of the reasons why the two parts of the solution are kept separate.

In the rest of Chapter XI, examples of computations using Theorem B are given, and the modifications of the theory for the non-base point case discussed.

In Chapter XII a generalisation, due to M.G. Barratt, of the Moore-Postnikov system of a fibre map is given, and this used to describe known techniques for determining homotopy groups of principal bundles with fibre an FD-complex. These techniques, together with those of previous chapters, are applied to recover by algebraic methods most of M.G.Barratt's results on track groups, together with additional information on k-invariants.

The Appendices contain proofs of various theorems in the text, except for Appendix 7; this introduces a new product topology which seems to have considerable advantages over the weak product considered in Chapter I.

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PART A

INTRODUCTION.

If X, Y are topological spaces, the set of all continuous functions $Y \to X$ may be given the compact-open topology [22] to form the function space X^Y. The study of the homotopy type of X^Y and its subspaces has, apart from its obvious intrinsic interest, the additional merit in the light it throws on the topological space X or Y. Thus we have that (for reasonable X, Y) homotopy invariants of X^Y form, as Y varies, homotopy invariants of X (as for example in Hurewicz's original definition of homotopy groups [28]); they also form, as X varies, homotopy invariants of Y (as for example in cohomotopy groups).

We also note that the study of the homotopy type of X^Y includes the very important problem of classifying maps $Y \to X$ into homotopy classes. Again particular information on X^Y has been found of use in other questions of homotopy theory (as for example in Barratt - Paechter's proof that $\pi_6(S^3) = Z_{12}$ [7]).

In Part A of this thesis we shall discuss briefly some of the present information on homotopy invariants of function spaces, under the headings of homology, homotopy groups, and Postnikov systems. Further, in Chapter I, we suggest for the function space a new definition which we believe to be convenient in homotopy theory. In Chapter II we obtain a theorem in duality which has applications to function spaces.

In Part B we shall be largely concerned with the solution of two problems which arise naturally in studying the Postnikov system of function spaces. These problems are stated in Chapter IV.

I. FUNCTION SPACES AND WEAK TOPOLOGIES.

1. Mapping objects.

In any category $\mathcal{C}$ there is the function Map assigning to any objects $A,B \in \mathcal{C}$ the set Map (A,B) of maps $A \to B$ in $\mathcal{C}$. Some categories also admit, for each $A,B \in \mathcal{C}$, an object $A \pitchfork B \in \mathcal{C}$ (adopting a notation of E.C. Zeeman) which "models" the set Map (A,B) in the category $\mathcal{C}$. In such case one expects that a product $\times_M$ should be defined in $\mathcal{C}$ so that there is a natural isomorphism

$$\mu : (A \times_M B) \pitchfork C \longrightarrow A \pitchfork (B \pitchfork C) \qquad A,B,C \in \mathcal{C}$$

The existence of such an isomorphism is often called the "exponential law" for $\mathcal{C}$; we shall call μ the "exponential map."

This isomorphism usually determines an isomorphism

$$\mu : \text{Map}(A \times_M B,\ C) \longrightarrow \text{Map}(A,\ B \pitchfork C).$$

If this does happen, then an evaluation map $\varepsilon : (A \pitchfork B) \times_M A \to B$ is defined by the condition $\mu(\varepsilon) = 1 : A \pitchfork B \to A \pitchfork B$. Then for any $Y \in \mathcal{C}$ a function

$$M^Y : \text{Map}(A,B) \longrightarrow \text{Map}(Y \pitchfork A,\ Y \pitchfork B)$$

is defined by the condition

$$M^Y(k) = \mu(k\varepsilon), \qquad k \in \text{Map}(A,B),$$

where μ here is an isomorphism $\text{Map}((Y \pitchfork A) \times_M Y, B) \longrightarrow \text{Map}(Y \pitchfork A,\ Y \pitchfork B)$. $M^Y(k)$ $(k \in \text{Map}(A,B))$ is often written k^Y.

Example 1.1 Let A,B,C be Λ-modules, where Λ is a commutative ring with unit. Then $\text{Hom}_\Lambda(A,B)$, $A \otimes_\Lambda B$ may be given the structure of Λ-modules and

$$\text{Hom}_\Lambda(A \otimes_\Lambda B, C) \approx \text{Hom}_\Lambda(A, \text{Hom}_\Lambda(B,C)).$$

In this case Map (A,B) is the abelian group $\text{Hom}_\Lambda(A,B)$, while $A \pitchfork B$

is the Λ-module $\mathrm{Hom}_{\Lambda}(A,B)$.

Example 1.2 Let A, B, C be FD-complexes (Chapter VII), and let $\mathcal{F}(A,B)$ denote the group of FD-maps $A \to B$. In Chapter VII, in addition to the well known cartesian product $A \underline{\times} B$ of FD-complexes, we define an FD-complex $A \underline{\pitchfork} B$; this satisfies $(A \underline{\pitchfork} B)_0 = \mathcal{F}(A,B)$, and we construct an isomorphism

$$\mu : (A \underline{\times} B) \underline{\pitchfork} C \longrightarrow A \underline{\pitchfork} (B \underline{\pitchfork} C).$$

Examples 1.3 Let A, B, C be FD-complexes. In Chapter IX we construct FD-complexes $A \otimes B$, $A \pitchfork B$ and an isomorphism

$$\mu : (A \otimes B) \pitchfork C \longrightarrow A \pitchfork (B \pitchfork C).$$

These products $\otimes$, $\pitchfork$ of FD-complexes are closely related to the products $\otimes$, $\pitchfork$ of chain complexes (Chapter VI). The relation between these products and those of Example 1.2 plays a vital role in Part B.

Example 1.4 Let $\underline{X}, \underline{Y}, \underline{Z}$ be M-ads of css-complexes. In Chapter V we define M-ads $\underline{X} \ast \underline{Y}$, $\underline{X} \pitchfork \underline{Y}$ and prove the exponential law

$$\mu : (\underline{X} \ast \underline{Y}) \pitchfork \underline{Z} \approx \underline{X} \pitchfork (\underline{Y} \pitchfork \underline{Z}) .$$

Example 1.5 Let X, Y, Z be Hausdorff spaces. Let $X \times_W Y$ be the cartesian product of X and Y with the weak topology with respect to its compact subse [13,46]. Let $Y \pitchfork X$ denote the set of functions $Y \to X$ which are continuous on compact subsets of Y, and let $Y \pitchfork X$ have the weak topology (2.33). In the following section we show that there is a homeomorphism

$$\mu : (Z \times_W Y) \pitchfork X \to Z \pitchfork (Y \pitchfork X).$$

§2. FUNCTION SPACES AND WEAK FUNCTION SPACES.

2.1. Let X, Y be topological spaces, and let X^Y be the set of all continuous functions $Y \to X$. Let the evaluation map $\varepsilon : X^Y \times Y \to X$ be defined by $\varepsilon(f,x) = f(x)$. A topology on X^Y is called admissible if, with this topology on X^Y and with the product topology on $X^Y \times Y$, ε is continuous.

Theorem 2.11 (Fox: [22]) Any admissible topology on X^Y contains the compact-open topology. If Y is locally compact and Hausdorff, then the compact-open topology is admissible.

Theorem 2.12 (Arens; [2]). Let Y be completely regular. If I^Y (I the unit interval) has a smallest admissible topology, then Y is locally compact.

An immediate corollary of 2.11, 2.12 is that, with the compact-open topology on X^Y, $(X^Y)^Z$ and $X^{Z\times Y}$ are not homeomorphic in general; they are homeomorphic if Y and Z are Hausdorff and Y is locally compact [27; Theorem III. 9.9], or if Y and Z satisfy the first axiom of countability, [27].

These restrictive conditions are disagreeable in homotopy theory, and we show how they may be avoided by using a different function space . All spaces are assumed to be Hausdorff.

Definition 2.21 [13,46]. Let X be a space. The space $\langle X \rangle$ is X re-topologized by the weak topology with respect to its compact subsets, i.e. a set $A \subset X$ is closed in X iff $A \cap C$ is closed in X for every compact set $C \subset X$.

Definition 2.22 [29] A space X is a k-space iff $\langle X \rangle = X$. Examples of k-spaces are CW-complex [57], locally compact spaces [29; Theorem 7.13] and spaces satisfying the first axiom of countability [ibid.] The k-spaces are in fact exactly the identification of locally compact spaces [14;1.8]; so any identification of a k-space is again a k-space [14;1.81].

Definition 2.23 [46] For any X, the identity $i_X: \langle X\rangle \to X$ is continuous. For any continuous map $f : X \to Y$ there is a unique continuous map $\langle f\rangle : \langle X\rangle \to \langle Y\rangle$ characterised by commutativity in the diagram.

$$\begin{array}{ccc} \langle X\rangle & \xrightarrow{\langle f\rangle} & \langle Y\rangle \\ {\scriptstyle i_X}\downarrow & & \downarrow{\scriptstyle i_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

We say $f: X \to Y$ is weakly continuous if $fi_X : \langle X\rangle \to Y$ is continuous.

Lemma 2.24 (Spanier, [46]) A map $f : \langle X\rangle \to \langle Y\rangle$ is continuous if and only if $i_Y f : \langle X\rangle \to Y$ is continuous.

Corollary 2.25 A map $f : X \to \langle Y\rangle$ is weakly continuous if and only if $i_Y f : X \to Y$ is weakly continuous.

Definition 2.26 [46] Let X,Y be spaces. The weak product of X and Y is the topological space $X \times_W Y = \langle X \times Y\rangle$, where $X \times Y$ has the usual product topology. Clearly for all X,Y,Z there is a natural homeomorphism $(X \times_W Y) \times_W Z \to X \times_W (Y \times_W Z)$.

Proposition 2.27 (Cohen; [13]) If X and Y are k-spaces one of which is locally compact, then $X \times Y = X \times_W Y$.

2.3. In order to obtain the theorems one would like, it turns out to be necessary to change the set which is to constitute the function space of maps $Y \to X$, and also to change the topology. That the set which occurs naturally in this context is the set of weakly continuous maps $Y \to X$ is a remark of Kelley [29;p225].

Definition 2.31 Let X, Y be spaces. Then $Y \pitchfork X$ is the set of all weakly continuous functions $Y \to X$. Thus $Y \pitchfork X = X^{\langle Y \rangle}$ (as a set).

The following theorem is 7.5 of [29]. The Hausdorff assumption we are making is essential here.

Theorem 2.32 (Kelley). Let $\varepsilon : (Y \pitchfork X) \times Y \longrightarrow X$ be the evaluation map. The compact-open topology is the smallest topology on $Y \pitchfork X$ such that $\varepsilon \,|\, (Y \pitchfork X) \times A$ is continuous for all compact subsets A of Y.

Definition 2.33 The weak topology on $Y \pitchfork X$ is the weakened compact-open topology; that is, a set $A \subset Y \pitchfork X$ is closed in the weak topology if and only if $A \cap C$ is closed in C for every subset $C \subset Y \pitchfork X$ which is compact in the compact-open topology. That the compact-open topology on $Y \pitchfork X$ is Hausdorff is Theorem 7.4 of [29].

Obviously $Y \pitchfork X = \langle X^{\langle Y \rangle} \rangle$. In general the weak and the compact-open topologies do not coincide; for example if Y is discrete, so that $Y = \langle Y \rangle$, then X^Y is homeomorphic to the product of disjoint copies of X, one for each $y \in Y$. So if Y contains two elements and is discrete, X^Y is homeomorphic to $X \times X$; yet $X \times X$ may not have the weak topology even if X does.

The following proposition is an obvious corollary of 2.32.

Proposition 2.34 The evaluation map $\varepsilon : (Y \pitchfork X) \times Y \longrightarrow X$ is weakly continuous, and so continuous on $(Y \pitchfork X) \times_W Y$.

Definition 2.35 Let $f \in (Z \times_W Y) \pitchfork X$. We define a function $\mu f : Z \to Y \pitchfork X$ by

$$(\mu f) \ (z) \ (y) = f(z, y) \qquad z \in Z, \ y \in Y.$$

Since $f: Z \times Y \to X$ is continuous on compact subsets of $Z \times Y$, so also is $(\mu f)(z)$, $z \in Z$. Therefore μf is well-defined.

The notation $(Z \times_W Y) \pitchfork X$ is slightly redundant (since $(Z \times_W Y) \pitchfork X = (Z \times Y) \pitchfork X$) but seems more revealing as it stands.

Theorem 2.36 If $f \in (Z \times_W Y) \pitchfork X$, then $\mu f: Z \to Y \pitchfork X$ is weakly-continuous.

Proof. We first note that it is no loss to assume Z is a k-space and prove μf is continuous. Second, by 2.25, it is sufficient to prove that μf is continuous when $Y \pitchfork X$ has the compact-open topology.

Let then $W = M(K, U) \subset Y \pitchfork X$ be a sub-basic open set for the compact-open topology of $Y \pitchfork X$; thus K is a compact subset of Y, U is an open subset of X. Let $\mu f = g$, and $z \in g^{-1}(W)$.

Now $f^{-1}(U)$ is open in $Z \times_W Y$ and so $f^{-1}(U) \cap Z \times_W K$ is open in $Z \times_W K$. Since Z is a k-space and K is compact, it follows from 2.27 that $Z \times_W K = Z \times K$. Therefore $f^{-1}(U) \cap Z \times K$ is open in $Z \times K$. But $z \times K \subset f^{-1}(U)$, and so there exists a set V open in Z such that $z \times K \subset V \times K \subset f^{-1}(U)$. Therefore $z \in V \subset g^{-1}(W)$. Therefore $g^{-1}(W)$ is open in Z. Therefore g is continuous.

Theorem 2.37 μ defines a homeomorphism $\mu: (Z \times_W Y) \pitchfork X \to Z \pitchfork (Y \pitchfork X)$.

Proof. We prove μ is continuous, and then define a continuous inverse.

By 2.36, the evaluation map

$$\varepsilon: ((Z \times_W Y) \pitchfork X) \times_W Z \times_W Y \to X$$

corresponds to a continuous map

$$\mu': ((Z \times_W Y) \pitchfork X) \times_W Z \to Y \pitchfork X,$$

and μ' itself corresponds to a map

$$\mu'': (Z \times_W Y) \pitchfork X \to Z \pitchfork (Y \pitchfork X).$$

which also is continuous since $(Z \times_W Y) \pitchfork X$ has the weak topology. But obviously $\mu'' = \mu$. Therefore μ is continuous.

By 2.36 the composition

$$(Z \pitchfork (Y \pitchfork X)) \times_W Z \times_W Y \xrightarrow{\varepsilon \times 1} (Y \pitchfork X) \times_W Y \xrightarrow{\varepsilon} X$$

corresponds to a continuous map

$$\nu : Z \pitchfork (Y \pitchfork X) \longrightarrow (Z \times_W Y) \pitchfork X .$$

In formulae, if $f \in Z \pitchfork (Y \pitchfork X)$, $z \in Z$, $y \in Y$, then $(\nu f)(z,y) = \varepsilon(\varepsilon \times 1)(f,z,y) = f(z)(y)$. Obviously $\mu\nu = 1$, $\nu\mu = 1$.

<u>Remark 2.38</u> If we give $Y \pitchfork X$ the compact-open topology, all that may be proved is that μ is weakly continuous with a weakly continuous inverse. But it is not unreasonable to regard this as sufficient.

2.4. Let $\mathcal{W}$ be the class of the class of spaces of the homotopy type of a CW-complex. Milnor [35] has proved the useful theorem that if $X \in \mathcal{W}$ and Y is compact then $X^Y \in \mathcal{W}$. We prove this is true for $Y \pitchfork X$.

<u>Lemma 2.41</u> If X and Y are of the same homotopy type, then so are $\langle X \rangle$ and $\langle Y \rangle$.

<u>Proof.</u> Let $f: X \to Y$, $g: Y \to X$ be continuous functions and $F: fg \simeq 1$, $G: gf \simeq 1$ homotopies. The functions $\langle f \rangle : \langle X \rangle \longrightarrow \langle Y \rangle$, $\langle g \rangle : \langle Y \rangle \longrightarrow \langle X \rangle$, defined in 2.23, are continuous, as are $\langle G \rangle : X \times_W I \to \langle X \rangle$, $\langle F \rangle : Y \times_W I \to \langle Y \rangle$. Since I is compact, 2.27 implies that $\langle F \rangle$, $\langle G \rangle$ define continuous functions

$F' : \langle Y\rangle \times I \longrightarrow \langle Y\rangle$, $G' : \langle X\rangle \times I \longrightarrow \langle X\rangle$. Clearly F', G' are homotopies $F' : \langle f\rangle \langle g\rangle \simeq 1$, $G' : \langle g\rangle \langle f\rangle \simeq 1$.

<u>Lemma 2.42</u> Let Y be a k-space. If $X \simeq Y$, then $\langle X\rangle \simeq Y$ and the inclusion $i_X : \langle X\rangle \longrightarrow X$ is a homotopy equivalence.

<u>Proof.</u> Let $f: X \to Y$, $g : Y \to X$ be continuous functions such that $fg \simeq 1$, $gf \simeq 1$. Let $f' = fi_X : \langle X\rangle \to Y$, $g' = \langle g\rangle : Y \to \langle X\rangle$. By 2.41, $f'g' \simeq 1$, $g'f' \simeq 1$. Let $h = g'f : X \longrightarrow \langle X\rangle$. Then

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle i_X}\uparrow & \nearrow{\scriptstyle f'} & \\ \langle X\rangle & & \end{array}$$

$hi_X = g'fi_X = g'f' \simeq 1$, $i_Xh = i_Xg'f = gf \simeq 1$.

Therefore $i_X : \langle X\rangle \simeq X$.

<u>Theorem 2.43</u> Let $X \in \mathcal{W}$ and let Y be compact. Then $Y \pitchfork X \in \mathcal{W}$, and the inclusion $Y \pitchfork X \longrightarrow X^Y$ is a homotopy equivalence.

<u>Proof.</u> Since a CW-complex is a k-space, the first assertion follows from 2.41 and Milnor's theorem that $X^Y \in \mathcal{W}$; the second assertion follows from 2.42 and Milnor's theorem.

<u>Remark 2.44</u> The assertions about m-ads corresponding to 2.41, 2.42, 2.43, are obviously true.

§3. WEAK IDENTIFICATION MAPS.

3.1. An unpleasant feature of the theory of identification maps is that the product of identification maps is not in general an identification map. Because of this certain natural products, such as the join and the smash product, turn out to be non-associative, even for CW-complexes.

If we restrict attention to k-spaces, then we prove below that the weak product of identification maps is an identification map. It seems unlikely that this is true for all spaces. However, by modifying the notion of identification map to that of "weak identification map" we obtain a theorem valid for all spaces.

3.2. Definition 3.21 Let X be a (Hausdorff) space and $f: X \to Y$ a function onto Y. The weak identification topology on Y with respect to f is the finest topology on Y making f weakly continuous. (That such a topology exists is the content of Lemma 3.22). If Y has this topology, then f is called a weak identification map, or, simply, a weak identification.

Lemma 3.22 For any $f : X \to Y$ mapping onto Y, the weak identification topology on Y with respect to f exists and is the set T of all subsets C of Y such that $f^{-1}(C) \cap A$ is closed in X for all compact sets $A \subset X$.

Proof. The verification that T is a topology is trivial and is omitted. But given T is a topology, it is obviously the finest topology on Y making f weakly continuous.

The following Lemma is obvious.

Lemma 3.23 Let $f : X \to Y$ be weakly continuous and onto. Then f is a weak identification map if and only if $fi_X : \langle X \rangle \to Y$ is an identification map.

3.3. Our main purpose is to prove

Theorem 3.31 Let $f: P \to X$, $g: Q \to Y$ be weak identification maps. Then $f \times_W g : P \times_W Q \to X \times_W Y$ is an identification map.

By 3.23, this theorem is equivalent to

Theorem 3.32 Let $f : P \to X$, $g: Q \to Y$ be identification map, and let P, Q be k-spaces. Then $f \times_W g: P \times_W Q \to X \times_W Y$ is an identification map

Proof of 3.32 The spaces X, Y are k-spaces, since they are identifications of k-spaces [14; 1.81].

It is sufficient to prove that $f \times_W 1: P \times_W Q \to X \times_W Q$ is an identification. For then equally $1 \times_W g: X \times_W Q \to X \times_W Y$ is an identification, and therefore so also is $(1 \times_W g)(f \times_W 1) = f \times_W g$.

Let $h = f \times_W 1 : P \times_W Q \to X \times_W Q$, and let B be a compact subset of Q. By 2.27, $P \times_W B = P \times B$. By the classical result on the product of identification maps [55 ; Lemma 4], $h | P \times B : P \times B \to X \times B$ is an identification.

Let $Z \subset P \times_W Q$ be open and saturated with respect to h. We must prove $h(Z)$ is open in $X \times_W Q$. This will be true if for any compact subset $B \subset Q$, $h(Z) \cap X \times B$ is open in $X \times B$. But this is clearly true, since $h(Z) \cap X \times B = h(Z \cap P \times B)$, and $Z \cap P \times B$ is open in $P \times B$ and saturated with respect to the identification map $h | P \times B$.

3.4. We give an application of Theorem 3.32.

<u>Definition 3.41</u> If X, Y are spaces with base points x_o, y_o respectively, the <u>weak smash product</u> $X \divideontimes_W Y$ is the identification space obtained from $X \times_W Y$ by collapsing to a point the subspace $X \times_W y_o \cup x_o \times_W Y$.

<u>Proposition 3.42</u> The weak smash product is associative; i.e. for all spaces with base point X, Y, Z, the spaces $(X \divideontimes_W Y) \divideontimes_W Z$ and $X \divideontimes_W (Y \divideontimes_W Z)$ are canonically homeomorphic.

<u>Proof.</u> By 3.12 the natural maps

$$f : X \times_W Y \times_W Z \longrightarrow (X \divideontimes_W Y) \divideontimes_W Z$$

$$g : X \times_W Y \times_W Z \longrightarrow X \divideontimes_W (Y \divideontimes_W Z)$$

are identification maps. Since fg^{-1}, gf^{-1} are single-valued, they are continuous. Obviously $(fg^{-1})(gf^{-1}) = 1$, $(gf^{-1})(fg^{-1}) = 1$.

3.5. The weak smash product also occurs in connection with function spaces. If X, Y are spaces with base point, let $Y \pitchfork X$ be the subspace of the weak function space consisting of weakly continuous functions $Y \longrightarrow X$ which respect base point; the base point of $Y \pitchfork X$ is the constant map.

<u>Theorem 3.5</u> Let X, Y, Z be spaces with base point. There is a (continuous) evaluation map $\varepsilon : (Y \pitchfork X) \divideontimes_W Y \longrightarrow X$ and a homeomorphism

$$\mu : (X \divideontimes_W Y) \pitchfork X \longrightarrow Z \pitchfork (Y \pitchfork X)$$

such that when $Z = Y \pitchfork X$, then $\mu(\varepsilon) = 1 : Y \pitchfork X \longrightarrow Y \pitchfork X$.

We do not give the proof as the situation is parallelled later in css-theory.

II. HOMOLOGY OF FUNCTION SPACES.

In this chapter we discuss our present knowledge of the homology of function spaces. In fact our knowledge is confined to the homology of $Y \pitchfork X$ when Y and X are spaces with base point (so that the functions of $Y \pitchfork X$ preserve base points) and when Y or X is a sphere S^n.

When $Y = S^n$, the methods used are spectral sequences or homology operations (§ 1). When $X = S^n$, the fundamental result is Moore's theorem (2.1), which gives the low-dimensional homology of $Y \pitchfork S^n$. In § 3 we obtain a result on duality which has application to function spaces.

In this chapter and the next we use the function space $Y \pitchfork X$ instead of the classical X^Y. This does not affect results on the usual homotopy invariants, for these depend only on singular homotopy type. However, we do assume without further comment that certain classical results on X^Y carry over to $Y \pitchfork X$; in each case the proof of the result for $Y \pitchfork X$ is a simple modification of the proof for X^Y. An example of the type of result we mean is the fibring theorem for function spaces. [27; III 13.1] which in fact is true more generally for $Y \pitchfork X$ than for X^Y.

Throughout this chapter, spaces are (Hausdorff) spaces with base point, and the functions of $Y \pitchfork X$ respect base points.

§1. Loop Spaces.

The loop space of X is $\Omega X = S^1 \divideontimes X$, where $S^m (m = 0, 1, \ldots)$ is the m-sphere. The iterated loop space $\Omega^n X$ is defined inductively by $\Omega^n X = \Omega(\Omega^{n-1} X)$, $\Omega^1 X = \Omega X$. Now $S^m \divideontimes_W S^n$ is canonically homeomorphic to S^{m+n}. Hence

$$\Omega^n X = S^1 \wedge (S^1 \wedge \Omega^{n-2} X) \approx (S^1 \ast_W S^1) \wedge \Omega^{n-2} X \approx S^2 \wedge \Omega^{n-2} X,$$

and so by induction $\Omega^n X$ is canonically homeomorphic to $S^n \wedge X$ (this is, in the classical case, due to Hurewicz [28]).

Let $PX = I \wedge X$, where $I = [0,1]$ has base point 0. The projection $p : PX \longrightarrow X$ defined by $p(f) = f(1)$ is a continuous fibre map [43] with fibre ΩX. Information on the homology of ΩX is mainly obtained from the spectral sequence of this fibring. For example Serre in [43] calculates the cohomology ring of ΩS^n.

These methods apply only inconveniently to the study of $\Omega^r X$ $(r>1)$. Dyer and Lashof [15] following Kudo and Araki [30], use the fact that $\Omega^r X$ has a great deal of multiplicative structure to define homology operations on $H_*(\Omega^r X, Z_p)$ (p prime); the case $p = 2$ was covered in [30]. Using these operations they determine completely $H_*(\Omega^r S^n, Z_p)$ $(r < n)$ and obtain some information on $H_*(\Omega^n S^n, Z_p)$ and $H_*(\Omega^r X, Z_p)$.

§2. A theorem of J.C.Moore.

In this section and the next, homology and cohomology are singular, modulo base point and with coefficients the integers Z.

Let $s_n \in H^n(S^n)$ be a generator. For any X,

$\varepsilon^*(s_n) \in H^n((X \wedge S^n) \ast_W X)$, where $\varepsilon : (X \wedge S^n) \ast_W X \longrightarrow S^n$ is the evaluation map. For any $z \in H_q(X \wedge S^n)$ the slant product $\varepsilon^*(s_n)/z \in H^{n-q}(X)$ [36,46] is defined. Let

$$\phi : H_q(X \wedge S^n) \longrightarrow H^{n-q}(X)$$

be given by $\phi(z) = \varepsilon^*(s_n)/z$. Moore has shown [36; Theorem 3]

<u>Theorem 2.1</u> (Moore) If X is a compact space of dimension $\leq m$, and $m<n$, then

(a) ϕ is an isomorphism for $0\leq q< \min\{2(n-m), n\}$,

(b) $H_q(X \pitchfork S^n) = 0$ for $n\leq q<2(n-m)$ if $2(n-m)>n$.

No other result, of similar power, on the homology of function spaces seems to be known.

§3. SPANIER-WHITEHEAD DUALITY.

3.1. Spanier-Whitehead duality, originally given in [48], has been shown by Spanier [47] to have its roots in function spaces and Moore's theorem 2.1. Spanier's emphasis in [47] is on using the function spaces $X \pitchfork S^n$ to determine a dual of X (c.f. the last paragraph on p.364 of [47].) We show how, when $X = S^r \cup e^{r+q}$, Spanier's results may be used to obtain an explicit n-dual of X and to determine the low dimensional homotopy type of $X \pitchfork S^n$.

3.2. Let $\mathcal{W}$ denote the category of finite connected CW-complexes with base point. (In this category we may replace $\ast_W$ by $\ast$). According to [47] a duality map is a map (in $\mathcal{W}$)

$$u : X' \ast X \longrightarrow S^n$$

for some n, such that the slant product $u^*(s_n)/z \in H^{n-q}(X)$ (s^n a generator of $H^n(S^n)$, $z \in H_q(X')$) induces an isomorphism

$$\phi_u : H_q(X') \longrightarrow H^{n-q}(X)$$

Given such a u, X' is called an n-dual of X by means of u.

3.3. Let $u : X' * X \to S^n$ be any map (in $\mathcal{W}$). By I.3.5 u corresponds uniquely to a map $g_u : X' \to X \pitchfork S^n$; further, the following diagram is commutative [47; 2.10]

$$\begin{array}{ccc} H_q(X') & \xrightarrow{g_u} & H_q(X \pitchfork S^n) \\ & \searrow \phi_u \quad \swarrow \phi & \\ & H^{n-q}(X), & \end{array}$$

where ϕ is Moore's map (2.1). Let $\dim X = m < n$. By 2.1 ϕ is iso for $0 < q < \min\{2(n-m), n\}$. So if u is a duality map, then g_u is iso for $0 < q < \min\{2(n-m), n\}$. If further $\pi_1(X') = \pi_1(X \pitchfork S^n) = 0$, then $g_u : \pi_q(X') \to \pi_q(X \pitchfork S^n)$ is iso for $0 < q < \min\{2(n-m), n\}$, by a theorem of J.H.C.Whitehead. Thus information on the low-dimensional homotopy type of $X \pitchfork S^n$ may be deduced from knowledge of an n-dual X' of X.

3.5. Let $E : \pi_r(S^n) \rightarrow \pi_{r+1}(S^{n+1})$ denote the classical suspension homomorphism (c.f. for example [27]). We prove

<u>Theorem 3.51</u>* Let $X = S^r \cup_\alpha e^{r+q}$. Then X has an n-dual if $E^{n-r-1}(\alpha) \in \operatorname{Im} E^{r+q-1}$. In particular, X has an n-dual if $n \geqslant 2r+q$, or if $n \leqslant 2r+q$ and $\alpha \in \operatorname{Im} E^{2r+q-n}$.

<u>Proof.</u> The identifications $u : S^{n-r-1} * S^r \longrightarrow S^{n-1}$, $v : S^{n-r-q} * S^{r+q-1} \longrightarrow S^{n-1}$ are duality maps $(r+q \leqslant n)$. Let $f : S^{r+q-1} \longrightarrow S^r$ represent the homotopy class α, and let $f' : S^{n-r-1} \longrightarrow S^{n-r-q}$ be a map such that the following diagram is homotopy commutative

$$\begin{array}{ccc} S^{n-r-1} * S^{r+q-1} & \xrightarrow{\;1 * f\;} & S^{n-r-1} * S^r \\ {\scriptstyle f' * 1}\downarrow & & \downarrow{\scriptstyle u} \\ S^{n-r-q} * S^{r+q-1} & \xrightarrow[\;v\;]{} & S^{n-1} \end{array} \qquad (*)$$

By [47; Theorem 6.10], $X' = S^{n-r-q} \cup_{f'} e^{n-r}$ is an n-dual of X.

Now (*) is equivalent to $E^{n-r-1}(\alpha) = \pm E^{r+q-1}(\beta)$, where β is the homotopy class of f' (the sign, which in fact may be recovered from [6], is unimportant here). Thus if $E^{n-r-1}(\alpha) \in \operatorname{Im} E^{r+q-1}$, then there exists an f' such that (*) is homotopy commutative.

*I have learnt that this result has been given by P.J.Hilton and E.H.Spanier in their paper "On the embeddability of certain complexes in Euclidean spaces" Proc. Amer. Math. Soc. (11) 1960, 523-526.

<u>Corollary 3.53</u>. Let $X = S^r \cup e^{r+q}$, and let $n \geqslant 2r+q$. There is a map

$$g : E^{n-2r-q}X = S^{n-r-q} \cup e^{n-r} \longrightarrow X \wedge S^n \qquad (E^nX = S^n \divideontimes_W X)$$

such that $g : \pi_t(E^{n-2r-q}X) \longrightarrow \pi_t(X \wedge S^n)$ is an isomorphism for

$$0 \leqslant t < \begin{cases} 2\ (n-r-q) & \text{if } 2r+q \leqslant n \leqslant 2r+2q, \\ n & \text{if } 2r+2q \leqslant n\ . \end{cases}$$

<u>Proof</u>. If $n \geqslant 2r+q$ we may clearly in 3.51 take $f' = E^{n-2r-q}f$.
Thus an n-dual of X is $X' = E^{n-2r-q}X$, that is, X is self-dual up to suspension (this is proved in [48] when X is the suspension of the real or complex projective plane). So the corollary follows from the remarks of 3.3.

3.6. By an A^r_q-polyhedron is meant an (r-1)-connected polyhedron of dimension $\leqslant r+q$. On p.65 of [48] it is stated that n-duals of A^r_2-polyhedron may be effectively constructed for $n \geqslant 2r+4$ (n-dual here is in the sense of [47]). It is clear then from 3.3. that the n-type of $X \wedge S^n$ may be effectively computed if X is an A^r_2-polyhedron and $n \geqslant 2r+4$.

III. HOMOTOPY GROUPS OF FUNCTION SPACES.

This chapter contains only classical material (except for the type of reformulation mentioned at the beginning of Chapter II).

§ 1. Loop Spaces.

For any space X with base point, the homeomorphism $S^r \wedge \Omega X \approx S^{r+1} \wedge X$ implies that $\pi_r(\Omega X) \approx \pi_{r+1}(X)$. This isomorphism is due to Hurewicz [28]. The homotopy groups of ΩX are thus completely known.

§ 2. Track groups.

2.1. If Y, X are spaces with base point, let $EY = S^1 \divideontimes_W Y$, so that

$E Y \wedge X = (S^1 \divideontimes_W Y) \wedge X \approx S^1 \wedge (Y \wedge X) = \Omega (Y \wedge X)$.

The multiplication of loops in $\Omega (Y \wedge X)$ induces a multiplication in $\pi_0(EY \wedge X)$ so that this set becomes a group, the track group, written $\pi_1^Y(X)$. Clearly $\pi_1^Y(X) \approx \pi_1(Y \wedge X)$. More generally $\pi_r^Y(X)$ is defined to be either of $\pi_r(Y \wedge X)$ or $\pi_0(E^r Y \wedge X)$.

These track groups were first discussed by S. Wylie (unpublished) and S.T.Hu [26]. It was proved in [26] that $\pi_1^Y(X)$ is a solvable group.

2.2. Important results on track groups were given by M.G.Barratt in [3].

Theorem 2.21 (Barratt; [3]). Let Y be an A_1^n-complex.† For $m + n > 1$, $\pi_m^Y(X)$ is a central extension* of $H^{n+1}(Y, \pi_{m+n+1}(X))$ by $H^n(Y, \pi_{m+n}(X))$.

† i.e. a CW-complex $(n-1)$-connected and of dimension $\leqslant n + 1$.

* A group E is a central extension of G by Q if G is contained in the centre of E and $Q = E/G$.

Barratt determines this extension for all $m + n > 1$. This extension is abelian for $m + n > 2$ and is determined if further Y is finite by the behaviour of the Steenrod operation Sq^1 in Y with respect to a certain pairing $\pi_{m+n}(X) \otimes \pi_{m+n}(X) \longrightarrow \pi_{m+n+1}(X)$. [3; p.290].

Theorem 2.22 (Barratt; [3]). Let Y be an A_2^n-complex. For $m + n > 2$, $\pi^Y_m(X)$ is a central extension of A by B where

(i) A is a quotient group of $H^{n+2}(Y, \pi_{m+n+2}(X))$,

(ii) B is a central extension of $H^{n+1}(Y, \pi_{m+n+1}(X))$ by a subgroup of $H^n(Y, \pi_{m+n}(X))$.

Further this quotient group and subgroup may be determined when Y is finite in terms of certain (primary) cohomology operations acting on Y with respect to certain pairings of the homotopy groups of X.

2.3. Barratt's methods in [3] may be described roughly as induction on the skeletons of Y. His method of obtaining the extensions is geometric.

We can obtain by the methods of Part B results equivalent to Barratt's. The method is the one "dual" to Barratt's, by induction on the Postnikov system of X. This results in an algebraic description of the extensions of 2.21, together with additional information on k-invariants.

§3. Function spaces with and without base points.

Let X be a space, Y a CW-complex and $e \in Y$ a vertex.

We define $\tau : Y \pitchfork X \to X$ by $\tau(f) = f(e)$, $f \in Y \pitchfork X$. Then τ is a fibre map (c.f. [27, III 13.1] for example). For any $x \in X$, let $F^Y(X, x) = \tau^{-1}(x)$; $F^Y(X, x)$ is the function space of base point preserving functions, providing X and Y are given base points x and e respectively.

Let $\vartheta \in F^Y(X,x)$ be taken as base point.

Now the components of $S^r \wedge F^Y(X,x)$

$= F^{E^r Y}(X,x)$ $(r \geqslant 1)$ all have the same homotopy type. Hence for any space with base point Z, $\pi_r^Z(F^Y(X,x), \vartheta) = \pi_r^Z(F^Y(X,x), *)$

$= \pi_r^{Z \# Y}(X,x)$, where $* : Y \to X$ is the constant map with value x.

So the (track) homotopy exact sequence of the fibring $F^Y(X,x) \to Y \wedge X \xrightarrow{\tau} X$ becomes

$$\cdots \xrightarrow{\Delta} \pi_r^{Z \# Y}(X) \to \pi_r^Z(Y \wedge X, \vartheta) \to \pi_r^Z(X) \xrightarrow{\Delta} \pi_{r-1}^{Z \# Y}(X) \to \cdots \qquad (*)$$

This sequence for Y, Z spheres is due to G.W.Whitehead [53], who expresses the transgression as a Whitehead product (the correct signs are given in [58]). It is well-known that this result may be generalized to the sequence (*), giving that the transgression Δ is (up to sign) the Whitehead product $[\ \ ,\vartheta]$, where this product is defined as for example in [4].

It follows immediately that not all components of $Y \wedge X$ have the same homotopy type.

§ 4. Federer's Spectral Sequence.

4.1. Let Y be a CW-complex and X a path-connected space which is n-simple for each $n \geqslant 1$. Federer [21] has shown that there is a spectral sequence whose E^2 term is

$$E^2_{p,q} = H^p(Y, \pi_q(X))$$

and whose E^{∞} term is the graded group associated with a certain filtration on $\pi_*(Y \wedge X, \nu)$ $(\nu \in Y \wedge X)$.

An important consequence is that if the homotopy groups of X and the integral cohomology groups of Y all belong to a class $\mathcal{C}$ of abelian groups satisfying axioms (I) and (IIA) of [45], then for all $p \geq 2$, $\pi_p(Y \wedge X, \nu) \in \mathcal{C}$.

4.2. Federer (ibid) also proves the following theorem

<u>Theorem 4.2</u> Let $K(\pi, m)$ be an Eilenberg-Maclane space where π is an abelian group and $m \geq 1$. If Y is a finite dimensional CW-complex, then

$$\pi_q(Y \wedge K(\pi, m)) \approx H^{m-q}(Y, \pi) \qquad q \geq 0.$$

Theorem 4.2. was proved independently by A. Heller (unpublished) and R. Thom [51]. Another proof is given by Spanier [46]; we shall prove a somewhat more general theorem later (IX 1.32).

IV. POSTNIKOV SYSTEMS OF FUNCTION SPACES.

We assume the theory of Postnikov systems [42,37,38]. These give a complete description of the singular homotopy type of a topological space.

The general problem we are concerned with is the determination of the Postnikov system of $Y \pitchfork X$ in terms of the Postnikov systems of Y and X.

§1. Loop spaces.

Let $K(\pi, m+1)$ be an Eilenberg-Maclane complex, and $k: X \to K(\pi, m+1)$ a map; k determines a cohomology class, also written k, in $H^{m+1}(X, \pi)$. The loop space $\Omega K(\pi, m+1)$ is a space of type (π, m), and so there is a map $f: \Omega K(\pi, m+1) \longrightarrow K(\pi, m)$ inducing an isomorphism of homotopy groups. The map $f(\Omega k) : \Omega X \longrightarrow K(\pi, m)$ determines a cohomology class $\ell \in H^m(\Omega X, \pi)$. Suzuki [50] gives an expression for ℓ in terms of k as follows. Let $\Omega X \to PX \longrightarrow X$ be the canonical fibring. Since PX is contractible, the coboundary

$$\delta : H^m(\Omega X) \longrightarrow H^{m+1}(PX, \Omega X)\ ^*$$

is an isomorphism. Let σ be the composition

$$H^{m+1}(X) \xrightarrow{\approx} H^{m+1}(X, *) \xrightarrow{p^*} H^{m+1}(PX, \Omega X) \underset{\approx}{\xrightarrow{\delta^{-1}}} H^m(\Omega X).$$

Suzuki proves that $\ell = \sigma(k)$. (This result is also a special case of the work of Part B here).

This result determines inductively the k-invariants of ΩX in terms of those of X.

* The (understood) coefficient group is π.

§2. Abelian groups.

2.1. In this, and the next section, we shall work in the css-category, as this is the most convenient for the questions considered. The css-function complex (c.f. Chapter V) is written X^Y.

2.2. Let X be a css-abelian group. The Postnikov system of X is completely determined by a theorem of J.C. Moore [38] which states that the k-invariants of X are zero. Equivalently $X \simeq \prod_{r=0}^{\infty} K(\pi_r(X), r)$.

If π is an abelian group, and $m \geqslant 1$, the complex $K(\pi, m)$ is a css-abelian group. Hence $K(\pi, m)^Y$ is a css-abelian group for all Y. So the homotopy type of $K(\pi, m)^Y$ is completely determined by its homotopy groups, which are given by III 4.2.

2.3. This simple form for $K(\pi, m)^Y$ suggests that a successful attack on determining the Postnikov system of X^Y may be obtained by induction on the Postnikov system of X. Precisely, the Postnikov system of X represents X by a sequence Γ of fibrations with fibres $K(\pi_m, m)$ $(\pi_m = \pi_m(X))$. So X^Y is represented by a sequence Γ^Y of fibrations with fibres $K(\pi_m, m)^Y$. This sequence Γ^Y is not a Postnikov system. However, if Y is finite dimensional, any finite part of the Postnikov system of X^Y is represented by a finite section of the sequence Γ^Y.

The general problem is then to "re-sort" such a finite section of the sequence Γ^Y into a Postnikov system. This problem involves three particular problems, which are discussed in the next section.

§3. Statement of Problems.

The following problems arise naturally out of the preceding discussion. Similar problems have been considered by Thom [51]

Problem 3.1 Let $k: X \longrightarrow K(\pi, m)$ be a map, and Y a complex. Determine the map k' which makes the following diagram commutative

$$\begin{array}{ccc} X^Y & \xrightarrow{k^Y} & K(\pi, m)^Y \\ & \searrow^{k'} & \downarrow f \\ & & \prod_{r=0}^{r=m} K(H^r(Y,\pi), m-r) \end{array}$$

where f is an equivalence.

Problem 3.2 Let $\ell : K(\pi, m) \longrightarrow K(G,n)$ be a map and Y a complex. Determine the map ℓ' which makes the following diagram commutative

$$\begin{array}{ccc} K(\pi,m)^Y & \xrightarrow{\ell^Y} & K(G,n)^Y \\ \downarrow f & & \downarrow f' \\ \prod_{r=0}^{m} K(H^r(Y,\pi), m-r) & \xrightarrow{\ell'} & \prod_{s=0}^{n} K(H^s(Y,G), n-s), \end{array}$$

where f, f' are equivalences.

Problem 3.3 Let $k: X \longrightarrow \prod_{r=1}^{m} K(\pi_r, r+1)$ induce a fibration $\prod_{r=1}^{m} K(\pi_r, r)$ $\longrightarrow E \longrightarrow X$. Supposing k and the Postnikov system of X known, determine the Postnikov system of E .

We shall be largely concerned in Part B with the solutions of Problems 3.1, 3.2. We state the solutions roughly here, give an example and make some comments on the solutions.

Solution 3.4 Let $\varepsilon: X^Y \times Y \longrightarrow X$ be the evaluation map. We may select a "Künneth isomorphism"

$$\kappa : H^m(X^Y \times Y, \pi) \longrightarrow \sum_{r=0}^{m} H^{m-r}(X^Y, H^r(Y,\pi))$$

and an equivalence f , such that $k' = \kappa\varepsilon^*(k)$, $k \in H^m(X, \pi)$.

<u>Solution 3.5</u> Let $A = \prod_{r=0}^{m} K(H^r(Y,\pi),m-r)$, $B = \prod_{s=0}^{n} K(H^s(Y,G),n-s)$.

In $\sum_{r=0}^{m} H^{m-r}(A,H^r(Y,\pi))$ a "fundamental class" $\omega(A)$ may be defined which is the ordinary fundamental class on each factor of A. Let ℓ be the cohomology operation of type $(\pi,m;G,n)$ corresponding to $\ell \in H^n(K(\pi,m),G)$. We may select "Künneth isomorphisms"

$$\kappa_1 : H^m(A \times Y,\pi) \xrightarrow{\approx} \sum_{r=0}^{m} H^{m-r}(A, H^r(Y,\pi))$$

$$\kappa_2 : H^n(A \times Y,G) \xrightarrow{\approx} \sum_{s=0}^{n} H^{n-s}(A, H^s(Y,G)),$$

and equivalences f, f' such that $\ell' = \kappa_2\, \ell\, \kappa_1^{-1}\, \omega(A)$

<u>Example 3.6</u> That these solutions (particularly 3.5) may be put in a form suitable for computation is illustrated by the following example. Let $\ell = Sq^n : K(Z,m) \longrightarrow K(Z_2,m+n)$, and let $Y = S^r \cup_2 e^{r+1}$. Then ℓ' is given by the following (self-explanatory) diagram

$$\begin{array}{ccccccc}
K(Z,m)^Y & \simeq & K(Z,m) & \times & & & K(Z_2,m-r-1) \\
\downarrow \ell^Y & & \downarrow Sq^n & & \swarrow Sq^n + Sq^{n-1}Sq^1 & & \downarrow Sq^n Sq^1 \\
K(Z_2,m+n)^Y & \simeq & K(Z_2,m+n) & \times & K(Z_2,m+n-r-1) & \times & K(Z_2,m+n-r)
\end{array}$$

3.7. A discussion of the relation between 3.4 and 3.5 may illuminate the following chapters. We shall prove all the statements made in this subsection, but not necessarily in the order given here.

Suppose we are in the situation of 3.2. According to 3.4, we must calculate $\varepsilon^*(\ell) \in H^n(K(\pi,m)^Y \times Y,G)$. The following diagram is commutative

$$\begin{array}{ccccc}
H^m(K(\pi,m),\pi) & \xrightarrow{\ \varepsilon^*\ } & H^m(K(\pi,m)^Y \times Y, \pi) & \xleftarrow[\approx]{(f\times 1)^*} & H^m(A\times Y, \pi) \\
\boldsymbol{\ell}\downarrow & & \boldsymbol{\ell}\downarrow & & \downarrow\boldsymbol{\ell} \\
H^n(K(\pi,m),G) & \xrightarrow[\varepsilon^*]{} & H^n(K(\pi,m)^Y \times Y, G) & \xleftarrow[(f\times 1)^*]{\approx} & H^n(A\times Y, G)
\end{array}$$

where $f: K(\pi,m)^Y \simeq A$ and $\boldsymbol{\ell}$ is the cohomology operation corresponding to ℓ, so that $\boldsymbol{\ell}(\omega') = \ell$, where $\omega' \in H^m(K(\pi,m),\pi)$ is the fundamental class. The class we require is that obtained by acting with a Künneth isomorphism on $((f \times 1)^*)^{-1}\varepsilon^*(\ell) = ((f \times 1)^*)^{-1} \boldsymbol{\ell}\varepsilon^*(\omega')$.

The evaluation map $\varepsilon : K(\pi,m)^Y \times Y \to K(\pi,m)$ corresponds to the identity $K(\pi,m)^Y \longrightarrow K(\pi,m)^Y$; the identity $A \to A$ determines the fundamental class $\omega(A) \in \sum_{r=0}^{m} H^{m-r}(A, H^r(Y,\pi))$. The crux of the argument is now that the Künneth isomorphism κ_1, which is defined entirely in terms of chain complexes, is related to the css-exponential map which is used to define ε. Since both ε and $\omega(A)$ correspond to identities, we obtain that $\kappa_1((f \times 1)^*)^{-1} \varepsilon^*(\omega') = \omega(A)$, and so Solution 3.5.

That ε and $\omega(A)$ both correspond to the identity, and so are related as we have described, is the intuition which suggests the following theory. This theory gives a method of passing from complexes to chain complexes, and so to homology, all the time preserving the exponential map μ. Actually the only fact about $K(\pi,m)$ which is used is that $K(\pi,m)$ is a css-abelian group, and we prove the theorems more generally for arbitrary css-abelian groups.

The actual steps made in the theory are the following. First, for any css-abelian group A, A^Y is isomorphic to $C(Y) \pitchfork A$, where $C(Y)$ is essentially the free css-abelian group on Y, and $\pitchfork$ is a map product defined for all css-abelian groups. This turns the problem into one involving only the category of css-abelian groups, or, as they are usually called, FD-complexes.

There is a functor N assigning to any FD-complex A its normalised chain complex NA ; the category of chain complexes has a natural hom product, written $\pitchfork$. To make the transition from FD-complexes to chain complexes, we first define a hom product $\pitchfork$ of FD-complexes such that $N(C(Y) \pitchfork A)$ is closely related to $C_N(Y) \pitchfork NA$, and then prove that $C(Y) \pitchfork A$, $C(Y) \underline{\pitchfork} A$ are of the same FD-homotopy type.

Each of the above products has an appropriate exponential law, and we prove that the isomorphism $A^Y \longrightarrow C(Y) \underline{\pitchfork} A$, the equivalence $C(Y) \underline{\pitchfork} A \rightarrow C(Y) \pitchfork A$, and the functor N, preserve these exponential laws. This is the fundamental part of the theory, as the final transition from chain complexes to homology is quite simple. These elements of structure are sufficient to obtain 3.4, 3.5 above.

3.8 As regards problem 3.3, the homotopy groups of the total space E can be determined up to isomorphism by a well-known method (c.f. Chapter XII). It is not known how to determine the k-invariants of E, even if $\pi_r = 0$, $r \neq m$, though solutions of some special cases are easy to obtain. A special case of this problem is solved in [1; Addendum] as a method of calculating the value of a secondary operation.

PART B

INTRODUCTION.

Part B of this thesis is largely concerned with the solution of Problems 3.1, 3.2 of Chapter IV. Most of the basic functors and relations necessary for this solution are given in Chapters V,VI,VII. Some of the material is classical, but it has usually been found necessary to extend and generalise the scope of the functors to apply them to present problems.

A further purpose of Chapter VII, and also Chapter VIII, is to show how various parts of the classical theory of Eilenberg-Maclane complexes, such as the classification of maps into a $K(\pi,m)$, and the theory of operations, may be derived more simply, and in more general form, using the theory of Dold-Kan.

The solution of the function complex problem is constructed in Chapters IX, X, XI. Some applications are given in Chapters XI and XII.

The theory we present is in terms of complexes with base point. The modifications necessary for the non-base point case are given in XI § 4.

CHAPTER V. SEMI-SIMPLICIAL FUNCTION COMPLEXES.

In § 1 of this chapter, we extend the classical definition of function complexes in the category of css-complexes to the css-category of M-ads (where M is an indexing set), and prove the exponential law in this category. From this we obtain in § 2 the exponential law in the category of css-complexes with base point. In § 3 we relate css-function complexes and topological function spaces.

§1. M-ads.

(1.1) The definitions of css-complex (complete semi-simplicial complex), css-map, product of css-complexes, Kan complex, homotopy, homotopy groups, as given for example in [31,37] are assumed here. The term css-complex is often abbreviated to complex.

(1.2) We recall that the complex of the standard q-simplex, denoted by Δ^q, may be defined as the free complex on one generator δ^q of dimension q. So if X is a complex and $x \in X_q$, there is a unique map $\hat{x} : \Delta^q \to X$ such that $\hat{x}(\delta^q) = x$.

A <u>categorical css-operator</u> is for each complex X a natural function $\phi : X_q \to X_r$. Any such operator may be written as a product of face and and degeneracy operations. If $\phi : X_q \to X_r$ is a categorical css-operator, then ϕ defines a unique map $\phi^* : \Delta^r \to \Delta^q$ such that $\phi^*(\delta^r) = \phi\,\delta^q$. This sets up a 1-1 correspondence between the set of categorical css-operators $X_q \to X_r$ and the set Map (Δ^q, Δ^r).

Let $\underset{\sim}{\Delta}$ denote the category whose objects are the complexes Δ^q $(q \geqslant 0)$ and whose maps $\Delta^q \longrightarrow \Delta^r$ are the css-maps. Let $\mathcal{C}$ be any category. We recall [31; I§4] that a css-$\mathcal{C}$-complex S may be defined as a contravariant functor $S : \underset{\sim}{\Delta} \rightarrow \mathcal{C}$, and that if T is a css-$\mathcal{C}$-complex, a map $S \longrightarrow T$ is a natural transformation of functors.

The complex Δ^1 will also be denoted by I.

(1.3) L,M,N will denote indexing sets, possibly empty. An M-ad will mean a particular sort of carrier [3,49] $\underline{X} = (X;M)$ consisting of a complex X and a family $(X_\lambda)_{\lambda \in M}$ of sub-complexes; $(X; \emptyset)$ will mean the same as X. These are the objects of the css-category of M-ads, where a map $f: \underline{X} \rightarrow \underline{Y}$ will mean a css-map $f: X \rightarrow Y$ such that $f(X_\lambda) \subset Y_\lambda$, $\lambda \in M$. Clearly the category of $\emptyset$-ads is the usual css-category. For all M, Δ^q denotes the M-ad $(\Delta^q; M)$ in which each indexed subcomplex of Δ^q is empty; the exact interpretation of Δ^q is to be understood in each case from the context.

Remark. It is convenient to allow the indexing set to be replaced by an isomorph at will.

(1.4) The intersection $\underline{X} \cap A$ of an M-ad (X; M) and complex A will mean the M-ad consisting of $X \cap A$ and the family $(X_\lambda \cap A)_{\lambda \in M}$.

If $L \subset M$, the restriction of an M-ad $\underline{X}$ to L will mean the L-ad consisting of X and the family $(X_\lambda)_{\lambda \in L}$. Thus the restriction of an M-ad to $\emptyset$ is the complex X.

The product $\underline{X} \times \underline{Y}$ of an M-ad $\underline{X}$ and N-ad $\underline{Y}$ is the L-ad $\underline{Z}$, where $Z = X \times Y$, L is the union of disjoint copies M' of M, N' of N and $Z_{\lambda'} = X_\lambda \times Y (\lambda \in M)$, $Z_{\nu'} = X \times Y_\nu$ $(\nu \in N)$.

The <u>smash product</u> $\underline{X} * \underline{Y}$ of two M-ads is the M-ad consisting of $X \times Y$ and the family $(X_\lambda \times Y \cup X \times Y_\lambda)_{\lambda \in M}$.

(1.5.) The function complex $\underline{X}^{\underline{Y}}$ of M-ads $\underline{X}$, $\underline{Y}$ is the complex K defined by

$$\begin{cases} K_q = \mathrm{Map}\,(\Delta^q * \underline{Y}\,,\,\underline{X}\,) & q = 0,1,\ldots \\ \emptyset = \mathrm{Map}\,(\emptyset^* * 1,1)\,:\,K_q \to K_r & \emptyset^* \in \mathrm{Map}(\Delta^r, \Delta^q) \end{cases}$$

(where Map denotes the set of maps of M-ads.)

Obviously the restriction of the M-ads $\underline{X}$, $\underline{Y}$ to L-ads $\underline{X}_o$, $\underline{Y}_o$ (for $L \subset M$) will induce an embedding of $\underline{X}^{\underline{Y}}$ in $\underline{X}_o^{\underline{Y}_o}$. In particular, on taking $L = \emptyset$, we have an embedding of $\underline{X}^{\underline{Y}}$ in X^Y. The following lemma is obvious.

<u>Lemma 1.51</u> $\underline{X}^{\underline{Y}}$ is the sub-complex of X^Y containing precisely those $f : \Delta^q \times Y \to X$ such that $f(\Delta^q \times Y_\lambda) \subset X_\lambda$ $(\lambda \in M)$.

(1.6) The following theorem is due to Cartan [10; Exposé 3, § 2]

<u>Theorem 1.61 (Cartan)</u> If X, Y, Z are complexes, there is a natural isomorphism

$$\mu :\; X^{Z \times Y} \to (X^Y)^Z\,,$$

given by the formula

$$\mu(\,f)\,(w) = f\,(\,\hat{w} \times 1) \quad , \; f \in (X^{Y \times Z})_q, \quad w \in (\Delta^q \times Z)_p.$$

<u>Definition 1.62</u> Let $\underline{X}$, $\underline{Y}$ be M-ads. We define the M-ad $\underline{Y} \pitchfork \underline{X}$ (called the M-ad of maps $\underline{Y} \to \underline{X}$) ; the total space of $\underline{Y} \pitchfork \underline{X}$ is $\underline{X}^{\underline{Y}}$ and the family of subspaces is $((\underline{X} \cap X_\lambda)^{\underline{Y}})_{\lambda \in M}$, embedded in the obvious way.

Let $\underline{X}$, $\underline{Y}$, $\underline{Z}$ be M-ads and let μ be as in 1.61. Then we prove

Theorem 1.62 μ induces an isomorphism

$$\mu : (\underline{Z} \circledast \underline{Y}) \pitchfork \underline{X} \longrightarrow \underline{Z} \pitchfork (\underline{Y} \pitchfork \underline{X}).$$

Proof. The complexes of the left-hand M-ad are in $X^{Z\times Y}$, and those in the other are in $(X^Y)^Z$. It is therefore necessary to show that the isomorphism μ maps the indexed subcomplexes on the correct images. That is we must prove

(i) $\mu(\underline{X}^{\underline{Z} \circledast \underline{Y}}) = (\underline{Y} \pitchfork \underline{X})^{\underline{Z}}$, (ii) $\mu\, (\underline{X} \cap X_\lambda)^{\underline{Z} \circledast \underline{Y}} = ((\underline{Y} \pitchfork \underline{X}) \cap (\underline{X} \cap X_\lambda)^{\underline{Y}})^{\underline{Z}}$, $\lambda \in M$

It is obvious that

$$(\underline{Y} \pitchfork \underline{X}) \cap (\underline{X} \cap X_\lambda)^{\underline{Y}} = \underline{Y} \pitchfork (\underline{X} \cap X_\lambda) , \quad \lambda \in M .$$

So (ii) follows from (i).

Now $f : \Delta^q \times Z \times Y \rightarrow X$ is in $\underline{X}^{\underline{Z} \circledast \underline{Y}} \iff f(\Delta^q \times (Z \times Y_\lambda \cup Z_\lambda \times Y)) \subset X_\lambda$ for all $\lambda \in M$. So, by the formula for μ, and lemma 1.51,

$$f \in \underline{X}^{\underline{Z} \circledast Y} \iff \begin{cases} (\mu f)\ (\sigma, z) \in \underline{X}^{\underline{Y}} \text{ for all } (\sigma, z) \in \Delta^q \times Z \\ \text{and} \\ (\mu f)\ (\sigma, z_\lambda) \in (\underline{X} \cap X_\lambda)^{\underline{Y}} \text{ for all } (\sigma, z_\lambda) \in \Delta^q \times Z_\lambda. \end{cases}$$

So $f \in \underline{X}^{\underline{Z} \circledast \underline{Y}} \iff \mu f \in (\underline{Y} \pitchfork \underline{X})^{\underline{Z}}$.

Let us define a homotopy between maps $f_0, f_1 : \underline{X} \rightarrow \underline{Y}$ of M-ads to be a map $F : I \circledast \underline{X} \rightarrow \underline{Y}$ of M-ads such that

$$F(s_0^q \partial_i \delta^1, \quad x) = f_i(x) \quad , \qquad x \in X_q, \quad i = 0,1.$$

Then 1.62 clearly implies

Corollary 1.63 μ induces a homotopy preserving isomorphism

$$\mu : \text{Map}\ (\underline{Z} \circledast \underline{Y}, \underline{X}) \longrightarrow \text{Map}(\underline{Z}, \underline{Y} \pitchfork \underline{X}) .$$

We can now make the important

Definition 1.64 The evaluation map $\varepsilon : \underline{X}^{\underline{Y}} \circledast \underline{Y} \longrightarrow \underline{X}$, for M-ads $\underline{X}, \underline{Y}$, is defined by the condition $\mu(\varepsilon) = 1: \underline{X}^{\underline{Y}} \to \underline{X}^{\underline{Y}}$.

Explicitly, $\varepsilon(f,y) = f(\delta^1,y)$, $f \in (\underline{X}^{\underline{Y}})_q$, $y \in Y_q$.

§2. Base Points.

(2.1) The category $\mathcal{K}$ of complexes with base point* is a subcategory of the css-category of 2-ads. So we have

Theorem 2.11 There is a natural isomorphism of complexes with base point

$$\mu : X^{Y ⋇ Z} \longrightarrow (X^Z)^Y$$

(where $X,Y,Z \in \mathcal{K}$) which yields a homotopy preserving isomorphism

$$\mu : \text{Map}\ (Y ⋇ Z, X) \longrightarrow \text{Map}(Y, X^Z).$$

Here the base point of X^Y is $*^Y$, where $*$ is the base point of X and X_o^Y is embedded in X^Y in the obvious way for all subcomplexes $X_o \subset X$.

Definition 2.12 Let X,Y denote complexes with base point. The evaluation map $\varepsilon : X^Y ⋇ Y \to X$ is that defined by $\mu(\varepsilon) = 1: X^Y \to X^Y$.

2.2. In 2.1, $Y ⋇ Z$ is a 2-ad but is not an object of $\mathcal{K}$. However, if $X \in \mathcal{K}$, then Map $(Y ⋇ Z, X)$ may obviously be identified with the set of maps from $Y \times Z / Y \times * \cup * \times Z$ to X , where the former complex has base point $\{Y \times * \cup * \times Z\}$. It is convenient, then, to abuse our present language and make the

Definition 2.21 The smash product $X ⋇ Y$ of complexes with base points is the complex $X \times Y / X \times * \cup * \times Y$ with base point the collapsed complex $\{X \times * \cup * \times Y\}$. Since Δ^1 has no base point, the smash product $\Delta^1 ⋇ X$ is the complex $\Delta^1 \times X / \Delta^1 \times *$ with base point $\{\Delta^1 \times *\}$.

* The base point $*$ of an $X \in \mathcal{K}$ is the sub-complex of X generated by a given vertex, also written $*$, of X .

If $X,Y \in \mathcal{K}$, then we identify $(X^Y)_q$ with $\mathrm{Map}(\Delta^q ⋇ Y,X)$, the set of maps of complexes with basepoint.

Clearly §2.1, with these interpretations, remains valid as it stands.

(2.3) For reference, we state explicitly the notions of homotopy in $\mathcal{K}$.

If $f_i: X \longrightarrow Y$ $(i = 0,1)$ are two maps in $\mathcal{K}$, then a homotopy $F: f_0 \simeq f_1$ is a map $F: \Delta^1 ⋇ X \longrightarrow Y$ such that

$F(s_0^q \partial_i \delta^1, x) = f_i(x) \qquad i = 0,1, \quad x \in X_q$. The set of homotopy classes of maps $X \to Y$ is defined for all X if and only if Y is Kan, and is written $[X,Y]$.

A map $f: X \to Y$ is an equivalence if it has a homotopy inverse, that is a map $f': Y \to X$ such that $ff' \simeq 1$, $f'f \simeq 1$. Then X and Y are said to be equivalent. These are written in symbols $f: X \simeq Y$, $X \simeq Y$.

§3. Relation with topological function spaces.

(3.1) The solution of problems 3.1, 3.2 of Chapter IV, which is our chief concern, will be given entirely in the css-category. The purpose of this section is to show that the topological and css-function spaces are sufficiently closely related for such a css-solution to be translatable into a solution of the corresponding topological problem.

The results of this section may be well-known for the classical function space, but even there do not seem to be in the literature.

(3.2) Let $\mathcal{T}$ be the category of topological spaces with base point. Let $S: \mathcal{T} \to \mathcal{K}$, $|\ |: \mathcal{K} \to \mathcal{T}$ be respectively the singular and realisation functor [33]. These functions are adjoint in the sense that there is a natural isomorphism

$$\Phi : \mathrm{Map}\,(|K|,X) \to \mathrm{Map}(K,\ S(X)),\quad K \in \mathcal{K},\ X \in \mathcal{T} \qquad (3.21),$$

where Map denotes the set of maps in the respective category.

We let $j: |S(X)| \longrightarrow X$, $i: K \longrightarrow S(|K|)$ be the unique maps such that $\Phi(j) = 1$, $i = \Phi(1)$, where the first identity map is on $S(X)$, and the second on $|K|$.

(3.3) A <u>singular homotopy equivalence</u>* $f: X \longrightarrow Y$ $(X, Y \in \mathcal{T})$, which we write $f: X \simeq_s Y$, is a map of spaces with base point such that $S(f): S(X) \longrightarrow S(Y)$ is a homotopy equivalence in $\mathcal{K}$. For all $X \in \mathcal{T}$, $j: |S(X)| \longrightarrow X$ is a singular homotopy equivalence [33], and is an ordinary homotopy equivalence if and only if X is of the homotopy type of a CW-complex (with base point).

(3.4) If $X \in \mathcal{T}$, the map $i_X: \langle X \rangle \longrightarrow X$ (c.f. Chapter I) induces an isomorphism $S(\langle X \rangle) \approx S(X)$, by which we identify these complexes [46,2.2]. If $K, L \in \mathcal{K}$, there is a natural homeomorphism $|K \times L| \approx |K| \times_W |L|$, by which we identify these spaces [46; 3.5] This homeomorphism induces a homeomorphism $|K \# L| \approx |K| \#_W |L|$.

(3.5) For $X, Y \in \mathcal{T}$, let $Y \pitchfork X$ denote the subspace of the weak function space of maps $Y \longrightarrow X$ consisting of maps respecting base point.

<u>Theorem 3.51</u> If $K \in \mathcal{K}$, $X \in \mathcal{T}$, there is a natural isomorphism

$$\lambda : S(|K| \pitchfork X) \approx S(X)^K.$$

<u>Proof</u>

$$\begin{aligned} S(|K| \pitchfork X)_q &= \mathrm{Map}(|\Delta^q|,\ |K| \pitchfork X) \\ &\approx \mathrm{Map}(|\Delta^q| \#_W |K|, X) \\ &\approx \mathrm{Map}(|\Delta^q \# K|,\ X) \\ &\approx \mathrm{Map}(\Delta^q \# K,\ S(X)) \\ &= (S(X)^K)_q. \end{aligned}$$

* Because of the definitions in Chapter I, we avoid the term "weak homotopy equivalence".

<u>Corollary 3.52</u> If $Y \in \mathcal{T}$ is of the homotopy type of a CW-complex, then there is a homotopy equivalence $S(Y \pitchfork X) \simeq S(X)^{S(Y)}$.

<u>Proof</u> For then $j : |S(Y)| \simeq Y$ induces $j' : Y \pitchfork X \simeq |S(Y)| \pitchfork X$. Therefore $\lambda S(j') : S(Y \pitchfork X) \simeq S(X)^{S(Y)}$.

<u>Corollary 3.53</u> Let $K, L \in \mathcal{K}$ and let K be Kan. There is a singular homotopy equivalence

$$|K^L| \simeq_S |L| \pitchfork |K|$$

which is a homotopy equivalence if L is finite.

<u>Proof</u> By 3.51, $|\lambda| : |S(|K|)^L| \approx |S(|L| \pitchfork |K|)|$. Also $j : |S(|L| \pitchfork |K|)| \simeq_S |L| \pitchfork |K|$.

Since K is Kan, $i : K \to S(|K|)$ is a homotopy equivalence; so i induces a homotopy equivalence $i^L : K^L \to S(|K|)^L$. The composition $j\,|\lambda|\,|i^L|$ is a singular equivalence $|K^L| \simeq_S |L| \pitchfork |K|$.

If L is finite, $|L| \pitchfork |K|$ is of the homotopy type of a CW-complex (I. 2.43). So the singular equivalence $|K|^L \simeq_S |L| \pitchfork |K|$ is a homotopy equivalence.

(3.6) The following theorem shows that S preserves μ. We leave the proof to an Appendix.

<u>Theorem 3.6</u> If $K, L \in \mathcal{K}$, $X \in \mathcal{T}$, there is a commutative diagram

$$\begin{array}{ccc} S((|K| *_W |L|) \pitchfork X) & \xrightarrow[\approx]{\lambda} & S(X)^{K * L} \\ S(\mu) \Big\downarrow \approx & & \approx \Big\downarrow \mu \\ S(|K| \pitchfork (|L| \pitchfork X)) & \xrightarrow[\lambda^2]{\approx} & (S(X)^L)^K \end{array},$$

where λ^2 is the composition

$$S(|K| \pitchfork (|L| \pitchfork X)) \xrightarrow{\lambda} S(|L| \pitchfork X)^K \xrightarrow{\lambda^K} (S(X)^L)^K.$$

VI. CHAIN COMPLEXES.

§1. Graded Groups.

(1.1) It is convenient to follow several authors and vary the usual definition of a graded group. By a graded group A is meant a family of disjoint (abelian) groups $(A_r)_{r \in Z}$ indexed by the integers Z ; and element of A is a member of any one of the groups A_r . If a is an element of A , we write $a \in A$. If $a \in A_r$, we write $r = \deg(a)$, the degree of a . The (distinct) zeros of the groups A_r are all written 0 .

Associated with A is the (weak) direct sum $\sum_{r \in Z} A_r$, which is the more usual notion of a graded group: the groups A_r are embedded in the direct sum, and their elements called the homogeneous elements of the sum. Our definition readily permits us to use the associated direct product where convenient, and to refer without qualification to the degree of an element.

(1.2) A map $f: A \rightarrow B$ of degree p of graded groups A,B is a family of homomorphisms $(f_r: A_r \rightarrow B_{r+p})_{r \in Z}$; it is trivial if each f_r is the trivial homomorphism. The maps $A \rightarrow B$ of degree p form a group which will be written $(A \pitchfork B)_p$, where the group operation is defined by addition of values; the maps $A \rightarrow B$ of all degrees therefore form a graded group $A \pitchfork B$, sometimes called the map or hom product of A and B . Composition of suitable maps f, g of degrees p,q forms a map fg of degree $p + q$. The trivial map of any degree will be written 0 .

(1.3) Let $f: A' \rightarrow A$, $g: B \rightarrow B'$ be maps (of graded groups) of degrees r and s respectively. A map $(f \pitchfork g) : A \pitchfork B \rightarrow A' \pitchfork B'$ of degree $r + s$ is defined by

$$(f \pitchfork g)(h) = (-1)^{r(p+s)} ghf \qquad h \in (A \pitchfork B)_p .$$

The sign given here is determined by the convention of Milnor [34] that when two terms of degrees α, β are interchanged, a sign $(-1)^{\alpha\beta}$ is introduced. This convention seems easiest to apply and to lead readily to a consistent system of signs.*

If f, g are as above, and $f': A'' \to A', g' : B' \to B''$ are of degrees r', s' respectively, then

$$(f' \pitchfork g')(f \pitchfork g) = (-1)^{r(s+r')} (ff' \pitchfork g'g) .$$

(1.4) The tensor product of graded groups A, B is the graded group $A \otimes B$ such that $(A \otimes B)_p = \sum_{r+s=p} A_r \otimes B_s$.

A pairing $\alpha : (A \pitchfork B) \otimes (A' \pitchfork B') \to (A \otimes A') \pitchfork (B \otimes B')$ is defined by setting $\alpha(f \otimes g)(a \otimes a') = (-1)^{ps} fa \otimes ga'$, $f \in A \pitchfork B$, $g \in (A' \pitchfork B')_s$, $a \in A_r$, $a' \in A'$; $\alpha(f \otimes g)$ is written $f \times g$ and is called the cartesian product of f and g. If $h : B \otimes B' \to C$ is any map, then the cartesian product of f and g with respect to h is $h(f \times g)$; this is usually written $f \times g$, the pairing being understood.

(1.5) There is a natural isomorphism of degree 0, $T : A \otimes B \longrightarrow B \otimes A$ defined by

$$T(a \otimes b) = (-1)^{pq} \, b \otimes a , \quad a \in A_p, \quad b \in B_q .$$

(1.6) There is a natural isomorphism of degree 0 $\mu : (A \otimes B) \pitchfork C \to A \pitchfork (B \pitchfork C)$ defined by

$$\mu(f)(a)(b) = f(a \otimes b) \qquad , \quad f \in (A \otimes B) \pitchfork C, \quad a \in A, \quad b \in B .$$

The following lemma will be useful in the proofs of two theorems of Chapter X.

* This system differs from that of [11; IV §5].

<u>Lemma 1.61</u> Let A, B, C be graded groups. In the composition

$$(A \pitchfork B) \otimes (C \pitchfork Z) \xrightarrow{\alpha} (A \otimes C) \pitchfork (B \otimes Z) \xrightarrow{T \pitchfork 1} (C \otimes A) \pitchfork B \xrightarrow{\mu} C \pitchfork (A \pitchfork B)$$

we have identified $B \otimes Z$ and B. Let $f \in A \pitchfork B$, $g \in C \pitchfork Z$, and let $g' \in C$ be the unique element such that $g(g') = 1$. Then

$$\left\{ \mu (T \pitchfork 1)\, \alpha \, (f \otimes g) \right\} (g') = f \; .$$

The proof is easy, and is omitted.

(1.7) Let A be a graded group. The groups $\eta A, \eta^- A$ and the maps $\eta : A \to \eta A$, $\eta^- : A \longrightarrow \eta^- A$ of degrees $+1$, -1 respectively are defined by

$$(\eta A)_{r+1} = A_r, \qquad \eta_r(a) = a \in (\eta A)_{r+1},$$

$$(\eta^- A)_{r-1} = A_r, \qquad \eta^-_r(a) = a \in (\eta^- A)_{r-1} .$$

If $f : A \to B$ is a map of degree p, then $\eta f : \eta A \longrightarrow \eta B$ is a map of degree p defined by $(\eta f)_{r+1} = (-1)^{rp} \ \eta_r f_r \ \eta^-_{r+1}$. Thus η becomes a functor and so similarly does η^-.

The maps and functors η, η^- are inverse to each other: $\eta \eta^- = 1$ and $\eta^- \eta = 1$. Also the functors η, η^- are adjoint; that is, there are natural identifications $(\eta^- A) \pitchfork B = A \pitchfork \eta B$, $(\eta A) \pitchfork B = A \pitchfork \eta^- B$.

(1.8) Let G be an abelian group. The graded group G is defined by

$$G_r = \begin{cases} G & r = 0 \\ 0 & r \neq 0 . \end{cases}$$

Therefore the graded group $\eta^p G$ satisfies

$$(\eta^p G)_r = \begin{cases} G & r = p \\ 0 & r \neq p . \end{cases}$$

If A is a graded group, the graded group which is A_p in degree p and 0 otherwise is $\eta^p A_p$.

For all graded groups A there are natural identifications $Z \otimes A = A$, $Z \pitchfork A = A$, $A \otimes Z = A$. We also identify $(\eta^p Z) \otimes A$ and $\eta^p A$ by the map sending $1 \otimes a \to a \in (\eta^p A)_r$, $a \in A_r$, $1 \in (\eta^p Z)_p$.

(1.9) If $f : A \to B$ is a map of degree p, then Ker f, Im f, Coker f, Coim f will denote the graded groups given by

$$(\mathrm{Ker}\ f)_r = \mathrm{Ker}(f_r : A_r \longrightarrow B_{r+p}) \ ,$$

$$(\mathrm{Im}\ f)_r = \mathrm{Im}(f_{r-p} : A_{r-p} \longrightarrow B_r) \ ,$$

$$(\mathrm{Coker}\ f)_r = \mathrm{Coker}(f_{r-p} : A_{r-p} \longrightarrow B_r) \ ,$$

$$(\mathrm{Coim}\ f)_r = \mathrm{Coim}(f_r : A_r \longrightarrow B_{r+p}) \ .$$

Thus f admits a factorisation

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & B \\ {\scriptstyle f^1}\downarrow & & \downarrow{\scriptstyle f^3} \\ \mathrm{coim}\ f & \xrightarrow[\ f^2\ ]{} & \mathrm{Im}\ f \end{array}$$

in which f^1, f^3 are of degree 0 and f^2 is of degree p. Let us say that f is an α-morphisn (where α = epi, mono or iso). iff each f_r is an α-morphism. Then in the above diagram, f^1 is an epimorphism, f^3 is a mono-morphism and f^2 is an iso-morphism. The expression " f is an α-morphism" is usually abbreviated to " f is α ".

If $i : A \to B$ is an inclusion, i.e. if each $i_r : A_r \longrightarrow B_r$ is an inclusion, then we write B/A for Coker i.

§ 2. Chain Complexes.

(2.1) A <u>chain complex</u> (A, ∂), or, simply, A, is a graded group A together with a "differential" ∂, i.e. a map $\partial : A \to A$ of degree -1 such that $\partial^2 = 0$. The <u>group of cycles</u> of A is the graded group $Z(A) = \mathrm{Ker}\,\partial$, the <u>group of boundaries</u> is $B(A) = \mathrm{Im}\,\partial$. Clearly $B(A) \subset Z(A)$, and we define the <u>homology of A</u> to be the graded group $H(A) = Z(A)/B(A)$.

Usually $Z(A)_p$, $B(A)_p$, $H(A)_p$ are written $Z_p(A), B_p(A), H_p(A)$.

(2.2) If A, B are chain complexes, a <u>chain map</u> $f : A \to B$ <u>of degree p</u> is a map of graded groups of degree p such that $\partial f = (-1)^p f \partial$. The graded group of chain maps $A \to B$ is written $\mathfrak{Z}_*(A,B)$ (for reasons given later), the group of chain maps of degree p being written $\mathfrak{Z}_p(A,B)$. As usual, if $f \in \mathfrak{Z}_p(A,B)$, then f induces a map $f_* : H(A) \to H(B)$ of degree p.

(2.3) The group ηA forms with the differential $\eta \partial$, a chain complex; thus

$$\partial_{r+1} = (-1)^r \eta \partial_r \eta^- : \quad (\eta A)_{r+1} \longrightarrow (\eta A)_r .$$

Then $\eta : A \to \eta A$ is a chain map of degree $+1$. Similarly, $\eta^- A$ has differential $\eta^- \partial$, and η^- is a chain map inverse to η.

(2.4) The <u>tensor product</u> $A \otimes B$ of chain complexes A, B is made into a chain complex by giving it * the differential $\partial \otimes 1 + 1 \otimes \partial$; i.e. in accordance with the convention of 1.4,

$$(\partial \otimes 1 + 1 \otimes \partial)(a \otimes b) = \partial a \otimes b + (-1)^p a \otimes \partial b, \quad a \in A_p, \quad b \in B .$$

It follows that if f, g are chain maps of degrees p, q respectively, $f \otimes g$ is a chain map of degree $p + q$. Thus the tensor product is a covariant functor of two chain complexes.

(2.5) The <u>hom product</u> $A \pitchfork B$ of chain complexes A, B is made into a chain complex by giving it * the differential $\delta = 1 \pitchfork \partial - \partial \pitchfork 1$.

Thus $(\delta f)(a) = \partial f a + (-1)^{p+1} f \partial a$, $\quad a \in A$, $f \in (A \pitchfork B)_p$.

* That is, the tensor or hom product of the underlying graded groups.

Remark $A \pitchfork B$ is more usually given the grading opposite in sign to the present conventions, so that it becomes a co-chain complex, hence the use of the traditional δ . This usual grading has been changed to avoid awkwardness in considering the natural isomorphism (§ 1.6) $\mu : (A \otimes B) \pitchfork C \approx A \pitchfork (B \pitchfork C)$ which is fundamental to our treatment of function spaces. The effect of the change, of course, is to make the cohomology of a (css) complex into homology which is non-zero only in nonpositive dimensions. This convention has been used previously by S. Lefschetz [30a] and V.K.A. Guggenhein [24a] ; we are using the differential for $A \pitchfork B$ given in [24a] .

It follows at once that if f, g are chain maps of degree p, q respectively, then $f \pitchfork g$ is a chain map of degree $p + q$. Thus the hom product is a functor of two chain complexes, contravariant in the first, covariant in the second.

If A, B are chain complexes, the cohomology of A with coefficicnts in B is defined to be $H^*(A,B) = H(A \pitchfork B)$;that is $H^p(A,B) = H_p(A \pitchfork B)$.

(2.6) Lemma 2.6 $\mu : (A \otimes B) \pitchfork C \approx A \pitchfork (B \pitchfork C)$ is a chain isomorphism if A, B, C are chain complexes.

μ, defined in 1.6, is an isomorphism of degree 0 of the underlying graded groups. The proof of the lemma is trivial and is omitted.

(2.7.) Lemma 2.7 Let A, A', B, B' be chain complexes. The map

$$\alpha : (A \pitchfork B) \otimes (A' \pitchfork B') \longrightarrow (A \otimes A') \pitchfork (B \otimes B')$$

of 1.4, is a chain map of degree 0 .

The proof is easy and is also omitted.

Corollary 2.71. If $f \in (A \pitchfork B)_p$, $g \in A' \pitchfork B'$, then

$$\delta (f \times g) = (\delta f) \times g + (-1)^p f \times \delta g \ .$$

The corollary follows immediately from 2.7 since $f \times g = \alpha (f \otimes g)$.

§ 3. Chain homotopy.

(3.1) Chain maps $f: A \to B$ are divided into equivalence classes by the relation of chain homotopy. A chain homotopy $D: f \simeq g$, where $f, g \in \mathfrak{F}_p(A,B)$, is an element of $(A \pitchfork B)_{p+1}$ such that

$$f - g = D\partial + (-1)^p \partial D \ .$$

This is obviously an equivalence relation in $\mathfrak{F}_p(A,B)$ such that if also $D' : f' \simeq g$, then $D + D' : f + f' \simeq g + g'$, $-D : -f \simeq -g$. Thus the chain homotopy classes of chain maps of degree p inherit a group structure from $\mathfrak{F}_p(A,B)$. When $p = 0$, this group is written $\langle A,B \rangle$. It is clear that if $f \simeq g$, then $f_* = g_* : H(A) \longrightarrow H(B)$.

(3.2) A chain map $f: A \rightarrow B$ of degree 0 is called a <u>chain equivalence</u> if there exists a <u>chain homotopy inverse f'</u> to f, that is, a chain map $f' : B \rightarrow A$ of degree 0 such that $ff' \simeq 1$, $f'f \simeq 1$, the appropriate identity maps. Then A, B are said to be equivalent. These are written in symbols $f: A \simeq B$, $A \simeq B$.

(3.3) The following proposition is essentially Exercise 5 of Chapter IV of [11].

<u>Lemma 3.31</u> For chain complexes A, B

$$\mathfrak{F}_p(A,B) = Z_p(A \pitchfork B)$$

and $B_p(A \pitchfork B)$ is the subgroup of chain maps of degree p homotopic to the trivial map. Therefore there is an isomorphism

$$\gamma' : \langle A,B \rangle \approx H_0(A \pitchfork B)$$

<u>Proof</u>. The proof is obvious: for $\mathfrak{F}_p(A,B) \subset (A \pitchfork B)_p$ and for any $f \in (A \pitchfork B)_p$, $\delta f = 0$ if and only if $\partial f = (-1)^p f \partial$, in which case $f \in \mathfrak{F}_p(A,B)$. Similarly, $f = \delta c$ if and only if $(-1)^p c : f \simeq 0$.

<u>Proposition 3.32</u> If A, B, C are chain complexes, then $\mu : (A \otimes B) \pitchfork C \to A \pitchfork (B \pitchfork C)$ induces a homotopy preserving isomorphism

$$\mu : \mathfrak{F}_p(A \otimes B, C) \longrightarrow \mathfrak{F}_p(A, B \pitchfork C),$$

and so an isomorphism

$$\mu : \langle A \otimes B, C\rangle \longrightarrow \langle A, B \pitchfork C\rangle$$

The proposition follows immediately from 3.31 since μ is an isomorphism of chain complexes.

VII. FD-COMPLEXES.

In §1 we give some of the basic notions of FD-complexes. In §2 we define for FD-complexes A, B an FD-complex A⋔B which is a "map product" of A and B ; the "exponential law" for ⋔ is proved. In § 3 we discuss the normalisation functor, and we introduce the important notion of homology of a css-complex with coefficients in a chain complex. In § 4 we recall the properties of the Dold-Kan functor, and use these properties to prove the fundamental classification theorem (4.7). In §5 we give the theory of operations, extending the classical theory to the more general coefficients we are using.

§1. FD-complexes.

(1.1) An FD-complex is a css-abelian group [31; 4]. An FD-map is a css-homomorphism. The category of FD-complexes is written $\mathfrak{FD}$, the set of maps A → B in $\mathfrak{FD}$ being written $\mathfrak{F}$(A,B). We regard $\mathfrak{FD}$ as a subcategory of $\mathfrak{K}$, the category of css-complexes with base point, the base point of any $A \in \mathfrak{FD}$ being the subcomplex 0 of zeros of A .

(1.2) There is a functor A from the category of sets to the category of free abelian groups, assigning to a set S the free abelian group on the elements of S , and to a set transformation f the homomorphism with the same values on the generators. This induces a functor B from the css-category to the category $\mathfrak{FD}$ such that $(B(X))_q = A(X_q)$, $\phi = A(\phi)$ for any categorical css-operator ϕ , and $B(f)_q = A(f_q)$ for any css-map $f: X \rightarrow Y$.

We define $K(q) = B(\Delta^q)$. It is clear that $K(q)$ is the free css-abelian group (free with respect to the css-operators and the group structure) on one generator δ^q in dimension q. So if A is an FD-complex, and $a \in A_q$, there is a unique map $\hat{a} : K(q) \to A$ such that $\hat{a}(\delta^q) = a$. This defines an isomorphism $A_q \approx \mathfrak{F}(K(q), A)$.

(1.3) For any css-complex with base point X there is an embedding $B(*) \subset B(X)$. So there is a functor $C : \mathcal{X} \longrightarrow \mathfrak{F}\mathcal{D}$ such that $C(X) = B(X)/B(*)$. This functor C is one of the standard functors used throughout.

(1.4) [18] The <u>cartesian product of FD-complexes</u> A, B is the FD-complex $A \underline{\times} B$ such that

$$\begin{cases} (A \underline{\times} B)_q = A_q \otimes B_q & q = 0, 1, \ldots \\ \phi = \phi \otimes \phi & \phi \text{ a categorical css-operator.} \end{cases}$$

Clearly is an additive covariant functor of two variables.

<u>Lemma 1.41</u> If $X, Y \in \mathcal{X}$, there are isomorphisms

$$C(X \mathbin{⋇} Y) \approx C(X) \underline{\times} C(Y), \quad C(\Delta^q \mathbin{⋇} X) \approx K(q) \underline{\times} C(X)^*$$

by which we identify these complexes.

The proof is obvious.

(1.5) [16] Two FD-maps $f_0, f_1 : A \longrightarrow B$ are FD-homotopic, written $f_0 \cong f_1$, if there is an FD-map $F : K(1) \underline{\times} A \to B$ such that

$$F(s_0^q \partial_0 \delta^1 \otimes a) = f_0(a), \quad F(s_0^q \partial_1 \delta^1 \otimes a) = f_1(a), \quad a \in A_q$$

* We recall that $\Delta^q \mathbin{⋇} X = \Delta^q \times X / \Delta^q \times *$.

The notions of FD-homotopy inverse, FD-homotopy equivalence are defined in the usual way, and we use, as for chain complexes, the notations $f : A \cong B$, $A \cong B$.

The relation of FD-homotopy equivalence is an additive equivalence relation on $\mathfrak{F}(A,B)$. So the FD-homotopy classes of FD-maps inherit, from $\mathfrak{F}(A,B)$, a group structure, and this group is written $\langle A,B \rangle$.

The following Lemma is obvious.

Lemma 1.51. Let $X, Y \in \mathcal{X}$. If $f_0 \simeq f_1 : X \to Y$, then $C(f_0) \cong C(f_1) : C(X) \to C(Y)$. Thus C induces a function $C : [X,Y] \to \langle C(X), C(Y) \rangle$.

§2. The functor $\pitchfork$

(2.1) The map product $A \pitchfork B$ of FD-complexes A, B, is the FD-complex defined by

$$\begin{cases} (A \pitchfork B)_q = \mathfrak{F}(K(q) \times A,\ B) & q = 0,1,\ldots \\ \phi = \mathfrak{F}(\phi^* \times 1,\ 1) & \phi \text{ a categorical css-operator.} \end{cases}$$

Clearly $\pitchfork$ is an additive functor of two FD-complexes, contravariant in the first, covariant in the second. We identify $(A \pitchfork B)_0$ and $\mathfrak{F}(A,B)$ in the obvious way.

(2.2) Let $X \in \mathcal{X}$, $A \in \mathfrak{FD}$. Let Map (X,A), the set of css-maps $X \to A$ of complexes with base point, be given the structure of an abelian group by addition of values. This addition includes an addition in $[X,A]$.

Lemma 2.21 There is an isomorphism of groups

$$D : \text{Map}(X, A) \longrightarrow \mathcal{Z}(C(X), A) .$$

Further D preserves homotopy and induces an isomorphism

$$D : [X, A] \longrightarrow \langle C(X), A\rangle .$$

Proof. If $f : X \to A$, then $D(f) : C(X) \longrightarrow A$ is the unique map whose value on each generator of $C(X)$ is exactly f. The first part of the lemma is obvious.

The second part follows from the first by the definitions of homotopy and 1.41.

Proposition 2.22 If $X \in \mathcal{K}$, $A \in \mathcal{F}\mathcal{D}$, there is an isomorphism

$$D : A^X \longrightarrow C(X) \pitchfork A$$

by means of which we identify these complexes.

Here A is also regarded as an object of $\mathcal{K}$. The proposition follows immediately from 2.21, 1.41 and the definitions.

(2.3) We now prove the "exponential law" for the functor $\pitchfork$, closely following Cartan's proof in the css-category [10; Expose 3, § 2].

First we prove

Proposition 2.31 Let $A, B, C \in \mathcal{F}\mathcal{D}$. There is a natural isomorphism

$$\mu : \mathcal{F}(A \underline{\times} B, C) \to \mathcal{F}(A, B \pitchfork C) .$$

Proof. Let $f \in \mathcal{F}(A \underline{\times} B, C)$, $a \in A_q$. We define an FD-map $(\mu f)(a) : K(q) \underline{\times} B \to C$ by $(\mu f)(a) = f(\hat{a} \times 1)$.

An inverse ϑ to μ is defined by setting

$$(\vartheta g)(a \otimes b) = g(a)(\delta^q \otimes b), \quad g \in \mathcal{F}(A, B \pitchfork C), \quad a \in A_q, \quad b \in B_q .$$

It is trivial to show $\nu\mu = 1, \mu\nu = 1$.

The naturality of μ is obvious from its definition.

<u>Theorem 2.32</u> Let $A, B, C \in \mathfrak{F}\mathcal{D}$. There is a natural isomorphism

$$\mu : (A \underline{\times} B) \pitchfork C \to A \pitchfork (B \pitchfork C) .$$

<u>Proof.</u> In dimension q, μ is the isomorphism

$$\mathfrak{F}(K(q) \underline{\times} A \underline{\times} B,\ C) \to \mathfrak{F}(K(q) \underline{\times} A,\ B \pitchfork C)$$

of 2.31. That this isomorphism is natural implies that μ is an FD-map.

<u>Corollary 2.33</u> The isomorphism μ of 2.31 is homotopy preserving and induces an isomorphism

$$\mu : \langle A \underline{\times} B,\ C \rangle \to \langle A,\ B \pitchfork C \rangle .$$

We must also show that the isomorphism D preserves μ.

<u>Proposition 2.34</u> Let $X, Y \in \mathcal{X}$, $A \in \mathfrak{F}\mathcal{D}$. The following is a commutative diagram of isomorphisms

$$\begin{array}{ccc} \text{Map}\ (X \# Y, A) & \xrightarrow{D} & \mathfrak{F}(C(X) \underline{\times} C(Y),\ A) \\ \mu \downarrow & & \downarrow \mu \\ \text{Map}\ (X,\ A^Y) & \xrightarrow[D]{} & \mathfrak{F}(C(X),\ C(Y) \pitchfork A) \end{array} .$$

The proof is trivial and is omitted.

<u>§3. The Normalisation Functor N.</u>

(3.1) The category of chain complexes in which the maps are the chain maps of degree 0 is written $\mathcal{C}$. The group of maps $A \to B$ in $\mathcal{C}$ is written $\mathfrak{F}(A,B)$. The full subcategory of $\mathcal{C}$ consisting of chain complexes A such that $A_i = 0,\ i < 0$, is written $\mathcal{C}_0$.

(3.2) <u>Definition</u> (J.C.Moore; [37]). The <u>normalisation functor</u> is the additive functor $N : \mathfrak{F}\mathcal{D} \to \mathcal{C}_0$ defined by

$$\begin{cases} N(A)_q = \bigcap_{i>0} \operatorname{Ker}(\partial_i : A_q \to A_{q-1}), & q = 0,1,\ldots \quad A \in \mathfrak{F}\mathcal{D} \\ \partial = \partial_0 | N(A)_q : N(A)_q \to N(A)_{q-1} , & \\ N(f) = f|N(A), & f : A \to B . \end{cases}$$

Classically [18] the normalised chain complex of $A \in \mathfrak{F}\mathcal{D}$ is formed by taking the boundary $\partial = \sum_{i \geq 0} (-1)^i \partial_i$ on the complex $A/D(A)$, where $D(A)$ is the subcomplex of degenerate elements. In [16] Dold proves that there is an isomorphism of chain complexes $N(A) \approx A/D(A)$.

The chain complex $N(q)$ is $N\,K(q)$, $q = 0,1,\ldots$

(3.3) We now make the important

<u>Definition 3.3</u> If $X \in \mathcal{X}$, the <u>normalised chain complex of X</u> is $C_N(X) = NC(X)$. If $A \in \mathcal{C}$, <u>the cohomology of X with coefficients in A</u> is $H^*(X,A) = H_*(C_N(X) \pitchfork A)$. If $B \in \mathfrak{F}\mathcal{D}$, the <u>cohomology of X with coefficients in B</u> is $H^*(X,B) = H^*(X,N\,B)$.

The introduction of this cohomology is essential for the theory. It also leads to a gain in conciseness. For example, if A is a product of Eilenberg-Maclane complexes, the classical theorem on the classification of maps of a complex X into A reads, in our notation, $[X,A] = H^0(X,A)$ (c.f. 4.7). This conciseness is very useful when dealing with complexes such as A^Y.

(3.4) For later purposes we will need a classical theorem of Eilenberg-Zilber [20].

<u>Theorem 3.41</u> (Eilenberg-Zilber). Let $A,B \in \mathfrak{F}\mathcal{D}$. There are natural chain maps

$$N(A \times B) \underset{\Delta}{\overset{\xi}{\rightleftarrows}} N(A) \otimes N(B)$$

such that $\xi\Delta = 1$, and such that there is a natural chain homotopy $\Delta\xi \simeq 1$.

The theorem that $N(A \times B) \cong N(A) \otimes N(B)$ in a natural manner is given in [20]. Actual formulae for maps Δ, ξ ~~such that $\xi\Delta = 1$~~ as above are given by Eilenberg-Maclane, in [18].

(3.42) The naturality of the maps in 3.41, and of the homotopies, is important. The naturality of Δ, ξ means that if $f : A \longrightarrow A'$, $g : B \longrightarrow B'$ are FD-maps, then there is a commutative diagram

$$\begin{array}{ccccc}
N(A \times B) & \xrightarrow{\xi} & N(A) \otimes N(B) & \xrightarrow{\Delta} & N(A \times B) \\
\downarrow{\scriptstyle N(f \times g)} & & \downarrow{\scriptstyle N(f) \otimes N(g)} & & \downarrow{\scriptstyle N(f \times g)} \\
N(A' \times B') & \xrightarrow[\xi]{} & N(A') \otimes N(B') & \xrightarrow[\Delta]{} & N(A' \times B') .
\end{array}$$

An implication of naturality is that the maps Δ, ξ, and the homotopy $\Delta\xi \simeq 1$ may be entirely expressed in terms of categorical css-operators; that is, these maps and homotopies are given in each dimension by formulae of the type

$$\sum_i \phi^i \otimes \psi^i$$

where ϕ^i, ψ^i are categorical css-operators.

(This remark is due to Eilenberg-Maclane [18]).

§4. The Dold-Kan Functor.

In this section we give the results of Dold-Kan [16] which furnish an equivalence between the categories $\mathcal{C}_0$ and $\mathcal{FD}$. Using these results we prove the Classification Theorem 4.7.

The original Dold-Kan functor in fact maps $\mathcal{C}_0 \to \mathfrak{F}\mathcal{D}$. For our purposes, we require this to be extended to a functor $\mathcal{C} \to \mathfrak{F}\mathcal{D}$. Information on this extended Dold-Kan functor is given in Chapter VIII.

<u>Definition 4.1</u> (Dold-Kan; [16]) The additive functor $R : \mathcal{C} \to \mathfrak{F}\mathcal{D}$ is defined by

$$\begin{cases} R(A)_q = \mathfrak{F}(N(q), A) & q = 0,1,\ldots, \quad A \in \mathcal{C} \\ \phi = \mathfrak{F}(N(\phi^*), 1) & \phi^* \in \mathrm{Map}(\Delta^q, \Delta^r) \\ R(f) = \mathfrak{F}(f,1) & f \text{ a map in } \mathcal{C} \end{cases}$$

The restriction $R|\mathcal{C}_0$ is written R_0.

<u>Theorem 4.2</u> (Dold-Kan) There are natural equivalences

$\Phi : 1 \to R_0 N, \quad \Psi : NR_0 \to 1$.

<u>Corollary 4.3</u> (Dold-Kan) (a) Let $A, A' \in \mathfrak{F}\mathcal{D}$. Then

$$N : \mathfrak{F}(A, A') \approx \mathfrak{F}(NA, NA') .$$

(b) Let $B, B' \in \mathcal{C}_0$. Then

$$R_0 : \mathfrak{F}(B, B') \approx \mathfrak{F}(R_0 B, R_0 B') .$$

<u>Theorem 4.4</u>. (Dold-Kan). The functors N, R_0 preserve homotopy, i.e.

(a) if f_1, f_2 are two maps in $\mathfrak{F}\mathcal{D}$, then $f_1 \simeq f_2 \Leftrightarrow Nf_1 \simeq Nf_2$,

(b) if g_1, g_2 are two maps in $\mathcal{C}_0$, then $g_1 \simeq g_2 \Leftrightarrow R_0 g_1 \simeq R_0 g_2$.

<u>Corollary 4.5</u> (Dold-Kan). The functors N, R_0 induce isomorphisms

$$N : \langle A, A' \rangle \approx \langle NA, NA' \rangle, \quad R_0 : \langle B, B' \rangle \approx \langle R_0 B, R_0 B' \rangle$$

for all $A, A' \in \mathfrak{F}\mathcal{D}$, $B, B' \in \mathcal{C}_0$. Also $A \simeq A' \Leftrightarrow NA \simeq NA'$, $B \simeq B' \Leftrightarrow R_0 B \simeq R_0 B'$.

<u>Remark 4.6</u> The results 4.2 - 4.3 give the reason for the parallel notation that has been adopted for FD- and chain complexes.

It might be thought convenient to go further and identify the categories $\mathcal{C}_0$ and $\mathcal{FD}$. This course is not adopted here, one reason being that such a course would mean regarding $\mathcal{C}_0$ as a subcategory of $\mathcal{X}$; and this seems unnatural.

(4.7) The following theorem generalises (as we shall see; c.f. VIII §1) a classical theorem on maps into an Eilenberg-Maclane complex. [9; Expose 14; § 2]

Theorem 4.71. (Classification Theorem) Let $X \in \mathcal{X}$, $A \in \mathcal{FD}$.
Then there is a natural isomorphism

$$\gamma : [X, A] \approx H^0(X, NA) = H^0(X,A).$$

Proof. This isomorphism is the composition of the natural isomorphisms

$$[X,A] \xrightarrow{D} \langle C(X),A\rangle \xrightarrow{N} \langle C_N(X),NA\rangle \xrightarrow{\gamma'} H^0(X\ NA),$$

where D is given by 2.21, and γ' is the isomorphism of VI 3.31.

Definition 4.72 Let $A \in \mathcal{FD}$. The fundamental class $\omega(A) \in H^0(A,A)$ is the class corresponding to the homotopy class of the identity $A \to A$ under the isomorphism γ of 4.71.

Proposition 4.73 The isomorphism γ of 4.71 satisfies

$$\gamma\{f\} = f^*\omega(A) \quad , \qquad f:X \to A .$$

The proof is obvious.

§5. Operations.

(5.1) Definition 5.1 Let $A,B \in \mathcal{C}$ or $\mathcal{FD}$. An operation θ of type (A,B) is a function assigning to each $X \in \mathcal{X}$ a function $\theta_X : H^0(X,A) \to H^0(X,B)$ such that for any map $f : X \to Y$ in $\mathcal{X}$, the following diagram commutes

$$\begin{array}{ccc} H^0(X,A) & \xrightarrow{\theta_X} & H^0(X,B) \\ f^* \uparrow & & \uparrow f^* \\ H^0(Y,A) & \xrightarrow[\theta_Y]{} & H^0(Y,B); \end{array}$$

that is, θ is a natural transformation $H^0(\ ,A) \longrightarrow H^0(\ ,B)$ when these are regarded as functors from $\mathcal{X}$ to the category of abelian groups and set maps.

θ is additive if θ_X is a homomorphism for all X.

Usually θ_X is written simply θ.

The set of operations of type (A,B) is written $\mathcal{O}(A,B)$, and this set is given the structure of an abelian group by addition of values. Strictly, that $\mathcal{O}(A,B)$ <u>is</u> a set, is a corollary of 5.2 below.

(5.2) <u>Theorem 5.2</u> Let $A,B \in \mathcal{J}\mathcal{D}$. There are natural isomorphisms

$$\mathcal{O}(A,B) \approx [A,B] \approx H^0(A, NB).$$

<u>Proof</u>. The proof is exactly the same (using the fundamental class of 4.72) as the classical case. (Serre [43]).

<u>Corollary 5.21</u> Let $A,B \in \mathcal{C}$. There is a natural isomorphism

$$\mathcal{O}(A,B) \approx H^0(RA, B)$$

<u>Proof</u>. If $X \in \mathcal{X}$, we prove in VIII 1.34 that $H^0(X,A) = H(X,NRA)$, $H^0(X,B) = H^0(X,NRB)$. Therefore

$$\begin{aligned} \mathcal{O}(A,B) &= \mathcal{O}(NRA, NRB) \\ &= \mathcal{O}(RA, RB) && \text{by definition,} \\ &= H^0(RA,NRB) && \text{by 5.2} \\ &= H^0(RA, B). \end{aligned}$$

<u>Corollary 5.22.</u> Operations preserve zero. I.e. if $A, B \in \mathcal{D}$, $k \in \mathcal{O}(A, B)$ $X \in \mathcal{X}$ and 0 is the zero of $H^0(X, A)$, then $k(0) = 0 \in H^0(A, B)$.

<u>Proof.</u> This follows easily from the isomorphism $\mathcal{O}(A, B) \approx [A, B]$ and the fact that the maps of $[A, B]$ preserve base point.

<u>Remark 5.23.</u> It is clear that many theorems on $[A, B]$ may be stated equally in terms of $H^0(A, B)$ or $\mathcal{O}(A, B)$, and conversely and we shall in any given case use whichever of these is convenient.

(2.3) <u>Definition 2.31.</u> Let $S^1 = \Delta^1/\dot{\Delta}^1$ be the 1-sphere, so that $C_N(S^1) \approx \eta Z$. Let $A \in \mathcal{C}$, $X \in \mathcal{X}$. The suspension isomorphism $\sigma' : H^0(S^1 * X, A) \approx H^0(X, \eta^- A)$ is defined to be the unique map making the following diagram commutative

$$\begin{array}{ccccc} H^0(S^1 * X, A) & \overset{\Delta^*}{\approx} & H_0((C_N(S^1) \otimes C_N(X)) \pitchfork A) & \approx & H_0((\eta Z \otimes C_N(X)) \pitchfork A) \\ \sigma \downarrow & & & & \downarrow \approx \\ H^0(X, \eta^- A) & \xleftarrow{\approx} & H_0(C_N(X) \pitchfork \eta^- A) & \xleftarrow{\approx} & H_0(\eta\, C_N(X) \pitchfork A) \end{array}$$

Let $A, B \in \mathcal{C}$; the <u>suspension</u> $\sigma : \mathcal{O}(A, B) \to \mathcal{O}(\eta^- A, \eta^- B)$ is defined by letting σk be for each $k \in \mathcal{O}(A, B)$ the operation which for each $X \in \mathcal{X}$ makes the following diagram commutative.

$$\begin{array}{ccc} H^0(S^1 \divideontimes X, A) & \xrightarrow[\approx]{\sigma'} & H^0(X, \eta^- A) \\ \theta \downarrow & & \downarrow \sigma^- \theta \\ H^0(S^1 \divideontimes X, B) & \xrightarrow[\sigma']{\approx} & H^0(X, \eta^- B) \end{array}$$

That $\sigma^- \theta$ is always an "additive" operation may be proved as in the classical case.

(5.4) We wish now to say something about the relation between these operations and the classical cohomology operations. This is most easily expressed if we first give some additivity lemmas.

Lemma 5.41 Let $A, B^i \in \mathcal{C}$ for $i \in I$. There is a natural isomorphism

$$\mathcal{O}(A, \sum_{i \in I} B^i) \approx \sum_{i \in I} \mathcal{O}(A, B^i)$$

Proof. The lemma is immediate using the definition of addition in $\mathcal{O}$, and the fact that $H^0(X, \sum_{i \in I} B^i) \approx \sum_{i \in I} H^0(X, B^i)$.

Lemma 5.42 Let $A^j, B \in \mathcal{C}$ for $j \in J$. There is an injection

$$i : \sum_{j \in J} \mathcal{O}(A^j, B) \longrightarrow \mathcal{O}(\sum_{j \in J} A^j, B)$$

and a projection

$$p : \mathcal{O}(\sum_{j \in J} A^j, B) \longrightarrow \prod_{j \in J} (A^j, B)$$

such that pi is the injection of the direct sum into the direct product, and p is onto.

Proof. Let us identify, for each $X \in \mathfrak{X}$, the groups $H^0(X, \sum_{j\in J} A^j)$ and $\sum_{j\in J} H^0(X,A^j)$; we write $K^j = H^0(X,A^j)$, $K = \sum_{j\in J} K^j$. Let $\theta \in \sum_{j\in J} \mathcal{O}(A^j,B)$ have components θ^j. Then we define $i(\theta)$ to be θ^j on the component K^j of K.

Let $\phi \in \mathcal{O}(\sum_{j\in J} A^j,B)$. Then we define the j^{th} component of $p\phi$ to be on K^j, the composition

$$K^j \longrightarrow K \xrightarrow{\phi} H^0(X,B)$$

It is clear that pi is the injection of the direct sum into the direct product. Let h be the composition

$$\prod_{j\in J} \mathcal{O}(A^j,B) \xrightarrow{i'} \mathcal{O}(\prod_{j\in J} A^j,B) \xrightarrow{i''} \mathcal{O}(\sum_{j\in J} A^j,B)$$

in which i' is defined analagously to i above and i'' is induced by the injection of the direct sum into the direct product. Then clearly $ph = 1$. Therefore p is onto, and in fact $\prod_{j\in J}\mathcal{O}(A^j,B)$ is a direct summand of $\mathcal{O}(\sum_{j\in J} A^j,B)$.

In general p has non-trivial kernel. For example, when J has two elements, this follows from the isomorphism

$\mathcal{O}(A_1 + A_2, B) \approx H^0(A_1 + A_2, B)$.

5.5. Theorem 5.51 If $X \in \mathcal{X}$, $A \in \mathcal{C}$, there is an isomorphism

$$\lambda : H^0(X, A) \to H^0(X, H(A))$$

which is natural with respect to maps of X .

Proof. Let F be a free complex and $f : F \to A$ a map such that $f^* : H(F) \approx H(A)$; let $g : F \to H(F)$ be a (chain) map such that $g^* : H(F) \approx H(F)$; that maps such as f and g exist is well known. By X 1.11, f and g induce isomorphisms

$$H^0(X, A) \underset{\approx}{\xleftarrow{f^*}} H^0(X, F) \underset{\approx}{\xrightarrow{g^*}} H^0(X, H(F)) \underset{\approx}{\xrightarrow{f^{**}}} H^0(X, H(A)).$$

Clearly $\lambda = f^{**} g^* (f^*)^{-1}$ is natural with respect to maps of X .

Corollary 5.52. If $A, B \in \mathcal{C}$, there is a (non-natural) isomorphism

$$\lambda : \mathcal{O}(A,B) \approx \mathcal{O}(H(A), H(B)) .$$

It follows from 5.52 that, at the expense of naturality, we may consider $\mathcal{O}(A,B)$ only in the case where A and B are chain complexes with trivial differential. In this case we may write (c.f. VI 1.8)

$$A = \sum_{r \in Z} \eta^r A_r \ , \qquad B = \sum_{s \in Z} \eta^s B_s \ .$$

Now classically an operation of type $(A_r, r; B_s, s)$ is a natural transformation in the category of abelian groups and set maps

$$H^{-r}(X, A_r) \longrightarrow H^{-s}(X, B_s)$$

(using our present conventions as to grading). But $H^{-r}(X, A_r) = H^0(X, \eta^r A_r)$, $H^{-s}(X, B_s) = H^0(X, \eta^s B_s)$. So an operation of type $(A_r, r : B_s, s)$ in the classical sense is exactly an operation of type $(\eta^r A_r, \eta^s B_s)$ in our sense.

From 5.41 it follows that

$$\mathcal{O}(A, B) \approx \sum_{s \in Z} \mathcal{O}(A, \eta^s B_s)$$

Clearly $\mathcal{O}(A, \eta^s B_s) \neq \prod_{r \in Z} \mathcal{O}(\eta^r A_r, \eta^s B_s)$. However there is a projection

$$\rho_{r,s} : \mathcal{O}(A, B) \longrightarrow \mathcal{O}(\eta^r A_r, \eta^s B_s)$$

which will be of use later.

VIII. THE FUNCTOR $\hat{K}(\ ,m)$

In this chapter we show the relationship between the Dold-Kan functor and the classical construction of the Eilenberg-Maclane Complex $K(\pi,m)$; in fact $K(\pi,m) = R(\eta^m \pi)$. More generally, by replacing π by a chain complex, we obtain a functor $\hat{K}(\ ,m) : \mathcal{C} \to \mathfrak{F}\mathcal{D}$ such that $\hat{K}(A,m) = R(\eta^m A)$.

In § 1 we define $\hat{K}(\ ,m)$ and prove a number of simple propositions about $\hat{K}(\ ,m)$, and so about R , which we need elsewhere. We also relate these constructs and the $\overline{W}$ construction, and give a simple proof of the well-known fact that any FD-complex is of the homotopy type of a product of Eilenberg-Maclane complexes.

In § 2 we discuss the exactness properties of $\hat{K}(\ ,m)$; the situation here is not as simple as for the classical $K(\pi,m)$ since $\hat{K}(\ ,m)$ is only left-exact.

§ 1. Definitions and Basic Properties.

(1.1) Definition 1.1 Let $A \in \mathcal{C}$. The complex $\hat{K}(A,m) \in \mathfrak{F}\mathcal{D}$ is defined by

$$\hat{K}(A,m)_q = Z^{-m}(\overset{q}{\Delta},A) = Z_{-m}(N(q) \pitchfork A) \qquad q = 0,1,\ldots$$

$$\phi = \phi^* \pitchfork 1 \qquad \phi \text{ a categorical css-operator.}$$

Clearly $\hat{K}(\ ,m)$, which we also write $\hat{K}^m$, is an additive functor $\mathcal{C} \to \mathfrak{F}\mathcal{D}$.

(1.2) Proposition 1.2 $K^0 = R : \mathcal{C} \to \mathfrak{F}\mathcal{D}$

Proof. By VI 3.31, if $A \in \mathcal{C}$, $Z_0(N(q) \pitchfork A) = \mathfrak{F}(N(q), A) = R(A)_q$.

(1.3.) Definition 1.3. The additive functor $S : \mathcal{C} \to \mathcal{C}_0$ is defined by

$$S(A)_r = \begin{cases} A_r & r > 0 \\ Z_0(A) & r = 0 \\ 0 & r < 0 \end{cases} \qquad , \ A \in \mathcal{C} \ ,$$

with differential induced by that of A .

We write s for the natural inclusion of chain complexes $s : S(A) \subset A$.

<u>Proposition 1.32</u> If $A \in \mathcal{C}$, then $\hat{K}^0(s) : \hat{K}^0(SA) \approx \hat{K}^0(A)$.

<u>Proof.</u> If $C \in \mathcal{C}_0$, then any chain map $C \to A$ factors uniquely through $s : SA \subset A$. Since $N(q) \in \mathcal{C}_0$, the proposition follows.

<u>Proposition 1.33</u> $NR = N\hat{K}^0$ is naturally equivalent to S .

<u>Proof.</u> Let $A \in \mathcal{C}$. Then

$$\begin{aligned} N\hat{K}^0(A) &\approx N\hat{K}^0(SA) && \text{by } 1.32 \\ &= NR(SA) && \text{by } 1.2 \\ &\approx SA && \text{by VII } 4.2\text{, since } SA \in \mathcal{C}_0. \end{aligned}$$

<u>Corollary 1.34</u> Let $X \in \mathcal{X}$, $A \in \mathcal{C}$. There is a natural isomorphism $H^0(X,A) \approx H^0(X,NRA)$.

<u>Proof.</u> Clearly (as in 1.32) $H^0(X,A) \approx H^0(X,SA)$. By 1.33, $H^0(X,SA) \approx H^0(X,NRA)$.

<u>Proposition 1.35.</u> There are natural equivalences $\hat{K}^m \approx \hat{K}^0\eta^m = R\eta^m$, $N\hat{K}^m \approx S\eta^m$.

<u>Proof.</u> The first equivalence follows from the isomorphism $Z_{-m}(N(q) \wedge A) \approx Z_0(N(q) \wedge \eta^m A)$ (c.f. VI 2.3 for our conventions on η). The second equivalence follows from the first and 1.33.

<u>Proposition 1.36.</u> Let $h_1, h_2 : A \to B$ be maps in $\mathcal{C}$. Then $h_1 \simeq h_2 \Rightarrow \hat{K}^m(h_1) \simeq \hat{K}^m(h_2)$. If further $A_i = 0,\ i < -m$, $\hat{K}^m(h_1) \simeq \hat{K}^m(h_2) \Rightarrow h_1 \simeq h_2$.

__Proof.__ $h_1 \cong h_2 \Leftrightarrow \eta^m h_1 \cong \eta^m h_2 \Rightarrow S\eta^m h_1 \cong S\eta^m h_2 \Leftrightarrow RS\eta^m h_1 \cong RS\eta^m h_2$.

If $A_i = 0$, $i < -m$, then $\eta^m A = S\eta^m A$; so $S\eta^m h_1 \cong S\eta^m h_2 \Rightarrow \eta^m h_1 \cong \eta^m h_2$.

__Corollary 1.37__ Let $A, B \in \mathcal{C}$. Then $A \simeq B \Longrightarrow \hat{K}^m(A) \cong \hat{K}^m(B)$.

(1.4) There is a well-known $\bar{W}$ construction assigning a classifying space to any css-group [18,37]. The notation of [37; 2.17] is used here.

__Theorem 1.41__ Let $A \in \mathfrak{Z}\mathcal{D}$. There is a natural isomorphism $\bar{W}A \approx \hat{K}(NA, 1)$.

__Proof.__ Let $\underline{a} = [a_{q-1}, \ldots, a_0] \in (N\bar{W}A)_q$. Then

$\partial_1 \underline{a} = 0 \Rightarrow [\partial_0 a_{q-1} + a_{q-2}, a_{q-3}, \ldots, a_0] = 0$

$\Rightarrow a_{q-3} = \ldots = a_0 = 0$ and $\partial_0 a_{q-1} + a_{q-2} = 0$. So $\partial_i \underline{a} = 0$, $i > 0$,

implies $\underline{a} = [a, -\partial_0 a, 0, \ldots, 0]$ where $a \in (NA)_{q-1}$. Further $\partial_0 [a, -\partial_0 a, 0, \ldots, 0] = [-\partial_0 a, 0, \ldots, 0]$. Hence the map $f: \eta NA \longrightarrow N\bar{W}A$ given by

$$f(a) = (-1)^q [a, -\partial_0 a, 0, \ldots, 0] \quad , \qquad a \in (\eta NA)_{q+1}$$

is an isomorphism of chain complexes.

So

$$\bar{W}A \approx \hat{K}(N\bar{W}A, 0) \approx \hat{K}(\eta NA, 0) \approx \hat{K}(NA, 1).$$

__Corollary 1.42.__ Let $A \in \mathcal{C}$ satisfy $A_i = 0$, $i < -m$. Then $\bar{W}\hat{K}(A,m) \approx \hat{K}(A, m+1)$

__Proof.__ We have $N\bar{W}\hat{K}(A,m) \approx \eta N\hat{K}(A,m) \approx \eta S \eta^m A$. By assumption $S\eta^m A = \eta^m A$, and so $N\bar{W}\hat{K}(A,m) = \eta^{m+1} A = S\eta^{m+1} A = N\hat{K}(A, m+1)$.

(1.5) Theorem 1.51 Let $A \in \mathcal{C}$, $X \in \mathcal{X}$. There is a natural isomorphism

$$[X, \hat{K}(A,m)] \approx H^{-m}(X,A).$$

Proof

$$\begin{aligned}[X, \hat{K}(A,m)] &\approx H^0(X, N\hat{K}(A,m)) && \text{by VII. 4.71}\\ &\approx H^0(X, S\eta^m A) && \text{by 1.35}\\ &\approx H^0(X, \eta^m A) && \text{since } C_N(X) \in \mathcal{C}.\\ &\approx H^{-m}(X,A).\end{aligned}$$

Corollary 1.52 Let $A \in \mathcal{C}$. There is a natural isomorphism

$$\alpha : \pi_q \hat{K}(A,m) \approx H_{q-m}(A).$$

Corollary 1.53. If $f : A \to A'$ is a map in $\mathcal{C}$ such that $f_*: H_r(A) \approx H_r(A')$ for $r \geqslant m$, then $\hat{K}(f,m)$ is a homotopy equivalence $\hat{K}(A,m) \simeq \hat{K}(A',m)$.

Proof. By 1.52, $K(f,m)$ induces an isomorphism of homotopy groups.

It should be noted that we cannot assert $\hat{K}(f,m)$ is an FD-homotopy equivalence.

(1.6) We conclude this section by giving a simple proof, without the theory of Postnikov systems, of the well-known fact that any FD-complex is of the homotopy type of a product of Eilenberg-Maclane complexes.

Theorem 1.61 Let $A \in \mathfrak{FD}$. There is an equivalence

$$h : A \simeq \sum_{r=0}^{\infty} K(\pi_r(A), r)$$

Proof. Let F be a free chain complex and $f : F \to NA$ a chain map such that $f_*: H(F) \approx H(NA)$ is an isomorphism. Let $g : F \to H(NA) = \pi_*(A)$ be a chain map such that $g_*: H(F) \approx H(NA)$ is an isomorphism.

By 1.53, $Rf : RF \to A$ is a homotopy equivalence, and so has a homotopy inverse $(Rf)' : A \to RF$. Clearly $h = (Rg)(Rf)' : A \to R(\pi_*(A))$ is a homotopy equivalence.

Since $\pi_*(A)$ has trivial differential, $\pi_*(A) \approx \sum_{r=0}^{\infty} \eta^r \pi_r(A)$,

and $R(\pi_*(A)) \approx \sum_{r=0}^{\infty} R(\eta^r \pi_r(A)) \approx \sum_{r=0}^{\infty} K(\pi_r(A),r)$, by 1.35

§ 2. Exactness properties of $\hat{K}(\,,m)$

The results of this section are not used elsewhere in this thesis.

(2.1) Proposition 2.11 (Cartan) Let $E : 0 \to \Gamma' \xrightarrow{i} \Gamma \xrightarrow{j} \Gamma'' \to 0$ be an exact sequence of css-groups. Then j is a fibre map.

Proof. Clearly Γ is a principal fibre bundle with structural group in the sense of [10; Exposé 1], the group Γ' acting on Γ by acting on the cosets of Γ' in Γ. So j is a fibre map [10; Expose 1, Proposition 2]

Proposition 2.12 * Let $E : 0 \to \Gamma' \xrightarrow{i} \Gamma \xrightarrow{j} \Gamma'' \to 0$ be an exact sequence of css-groups. The homotopy sequence of E coincides with the homology sequence of $NE : 0 \to N\Gamma' \xrightarrow{i} N\Gamma \xrightarrow{j} N\Gamma'' \to 0$.

Proof. Since j is a fibre map with fibre Γ', and $\Gamma', \Gamma, \Gamma''$ are group complexes, and so Kan, the homotopy sequence of E is well-defined [31; Theorem 14].

The functor N is Moore's normalisation functor [37; 2.6]. The complexes $N\Gamma', N\Gamma, N\Gamma''$ are in general non-abelian chain complexes. However, they are normal in the sense of Frohlich [23], so their homology groups are defined. These homology groups are the same as the homotopy groups of the respective css-group [37; 2.7].

* This proposition is probably well-known, but does not seem to be in the literature.

We prove that the sequence NE is exact. Clearly i' is mono and $j'i' = 0$. Let $\gamma \in N\Gamma$ and suppose $j\gamma = 0$. Then $\gamma = i\gamma'$ for some $\gamma' \in \Gamma'$. But i is mono and $(N\Gamma)_q = \bigcap_{i>0} \operatorname{Ker} \partial_i$; so $\gamma' \in N\Gamma'$, and we have proved exactness at N. Let $\gamma'' \in N\Gamma''$. Since j is a fibre map, there is a $\gamma \in \Gamma$ such that $j\gamma = \gamma''$ and $\partial_i \gamma = 0$, $i > 0$; so $\gamma \in N\Gamma$, and j is epi.

It is proved in §4 of [23] that any exact sequence of normal chain complexes has an exact homology sequence.

Let z'' be a q-cycle of $N\Gamma''$. The homology boundary $\partial_* [z'']$ is found as follows: an element $\gamma \in N\Gamma$ is chosen so that $j'\gamma = z''$, and $\partial_* [z'']$ is defined to be the homology class $[\gamma']$, where γ' is a (q-1)-cycle of $N\Gamma'$ such that $i'\gamma' = \partial\gamma$. But this process is exactly the same as finding the homotopy transgression of z'' with respect to E, when z'' is regarded as a representative of an element of $\pi_q(\Gamma'')$. So the boundary operators of E and NE coincide; the other maps of the exact sequences obviously coincide.

(2.2) <u>Proposition 2.21</u> $\hat{K}^m : \mathcal{C} \longrightarrow \mathfrak{F}\mathcal{D}$ is left-exact.

<u>Proof.</u> If $A \in \mathcal{C}$, then $K(A,m)_q \approx \mathfrak{F}(C_N(\Delta^q), \eta^m A)$. Since η is exact and $\mathfrak{F}$ is left-exact, the proposition follows.

<u>Theorem 2.22</u> Let $E : 0 \to A' \xrightarrow{i} A \xrightarrow{j} A'' \to 0$ be an exact sequence in $\mathcal{C}$. Let $\hat{j} = \hat{K}(j,m)$. Then (i) Im $\hat{j}$ contains the identity component of $\hat{K}(A'',m)$ (ii) for each $q > 0$, there is a commutative diagram

$$\begin{array}{ccc} \pi_q(\mathrm{Im}\hat{j}) & \xrightarrow{\partial_*} & \pi_{q-1}(\hat{K}(A',m)) \\ \alpha \downarrow \approx & & \approx \downarrow \alpha \\ H_{q-m}(A'') & \xrightarrow[\partial_*]{} & H_{q-m-1}(A') \end{array}$$

Proof. That $\alpha : \pi_q(\mathrm{Im}\hat{j}) \longrightarrow H_{q-m}(A'')$ is defined and is an isomorphism follows from part (i) of the proposition and 1.52; the commutativity of the diagram follows from 2.12. So it is sufficient to prove (i).

To prove (i), it is sufficient to prove that if $\gamma \in \hat{K}(A'',m)_q = Z^{-m}(\Delta^q, A'')$ is in the identity component of $\hat{K}(A'',m)$, then γ is a boundary : for if $\gamma = \delta c''$, where $c'' \in (C_N(\Delta^q) \pitchfork A'')_{-m+1}$, then $c'' = j'c$ for some $c \in (C_N(\Delta^q) \pitchfork A)_{-m+1}$; so $\gamma = j'\delta c \in \mathrm{Im}\hat{j}$.

Now $Z^{-m}(\Delta^q, A'') = Z^0(\Delta^q, \eta^m A'') = \mathfrak{Z}(C_N(\Delta^q), \eta^m A'')$,

and under these equalities a boundary corresponds to a chain map homotopic to 0. So the theorem follows from the following general lemma on FD-complexes.

Lemma 2.23 Let $B \in \mathfrak{FD}$, and let $\Phi : B_q \approx \mathfrak{Z}(C_N(\Delta^q), NB)$ be the isomorphism of VII 4.2. An element $\gamma \in B_q$ is in the identity component of B if and only if $\Phi\gamma \simeq 0 : C_N(\Delta^q) \longrightarrow NB$.

Proof An element $\gamma \in B_q$ is in the identity component of B $\Longleftrightarrow$ $\hat{\gamma} \simeq 0 : \Delta^q \longrightarrow B \Longleftrightarrow D\hat{\gamma} \simeq 0 : C(\Delta^q) \longrightarrow B$ (VII 2.21) $\Longleftrightarrow$ $ND\hat{\gamma} \simeq 0 : C_N(\Delta^q) \longrightarrow NB$ (VII 4.41). Since $ND\hat{\gamma} = \Phi(\gamma)$ [16;p.59], the lemma is proved.

Remark 2.24 Proposition 2.21 suggests determining the right-derived functors $R^{\#}\hat{K}^m$. These derived functors exist since $\mathcal{C}$ has enough injectives [24], and are given by the next proposition.

<u>Proposition 2.25.</u> There are natural equivalences for $n > 0$

$$\pi_q R^n \hat{K}^m \approx \begin{cases} H_{-m-n} & q = 0 \\ 0 & q > 0 \end{cases}$$

<u>Proof.</u> Let $A \in \mathcal{C}$, and $0 \longrightarrow A \xrightarrow{i} I \xrightarrow{j} M \longrightarrow 0$ be an exact sequence where $I \in \mathcal{C}$ is an injective object. Let $M' = \operatorname{Im} \hat{K}^m(j)$; then $R^1\hat{K}^m(A) \approx \operatorname{Coker} \hat{K}^m(j)$, so there are exact sequences

$$0 \longrightarrow \hat{K}^m(A) \xrightarrow{i'} \hat{K}^m(I) \xrightarrow{j'} M' \longrightarrow 0 \qquad (*)$$

$$0 \longrightarrow M' \xrightarrow{i''} \hat{K}^m(M) \longrightarrow R^1\hat{K}^m(A) \longrightarrow 0 \qquad (**)$$

Since I is injective, $H(I) = 0$. So from the exact homotopy sequence of (*) and 2.22

$$\pi_q(M') \approx \begin{cases} H_{q-m-1}(A) & q > 0 \\ 0 & q = 0 . \end{cases}$$

By 2.22, $i''_* : \pi_q(M') \approx \pi_q(\hat{K}^m(M))$, $q > 0$. Therefore from the exact sequence of (**)

$$\pi_q\, R^1\hat{K}^m(A) \approx \begin{cases} \pi_0\, \hat{K}^m(M) & q = 0 \\ 0 & q > 0 \end{cases} \approx \begin{cases} H_{-m-1}(A) & q = 0 \\ 0 & q > 0 \end{cases}$$

The proposition follows by induction.

(2.3) In [9, Exposé 14] Cartan introduces a complex $L(\pi, m)$ which is a contractible fibre space over $K(\pi, m+1)$. This construction also has a place in the present theory.

<u>Definition 2.31</u> The additive functor $\hat{L}(\ ,m) : \mathcal{C} \to \mathcal{FD}$ is defined by

$$\begin{cases} \hat{L}(A,m)_q = (C_N(\Delta^q) \pitchfork A)_{-m} & A \in \mathcal{C} \\ \phi = \phi^* \pitchfork 1 & \phi \text{ a categorical css-operator} \\ \hat{L}(f,m) = f \pitchfork 1 & f \text{ a map in } \mathcal{C} \end{cases}$$

The natural transformation $\hat{\delta} : \hat{L}(\ ,m) \longrightarrow \hat{K}(\ ,m+1)$ is defined by

$$\hat{\delta}(A) = \delta : (C_N(\Delta^q) \pitchfork A)_{-m} \longrightarrow Z^{-m-1}(\Delta^q, A) , \qquad A \in \mathcal{C}.$$

<u>Proposition 2.32</u> There is a natural FD-homotopy equivalence $\hat{L}(\ ,m) \cong 0$.

<u>Proof.</u> Let $A \in \mathcal{C}$. It is sufficient to find a natural contraction $N\,\hat{L}(A,m) \cong 0$.

Let $D(q) \subset N(q)$ denote the subcomplex generated by the images of $N(\partial_i^*) : N(q-1) \to N(q)$ for $i > 0$. From the argument of [16;p.59] it is clear that

$$\begin{aligned} N\,\hat{L}(A,m)_q &\approx (N(q)/D(q) \pitchfork A)_{-m} \\ &\approx Z \pitchfork A_{q-m} + Z \pitchfork A_{q-m-1} \\ &\approx A_{q-m} + A_{q-m-1} , \end{aligned}$$

and that these isomorphisms give a natural representation of $N\,\hat{L}(A,m)$ as the direct sum of elementary complexes of the form $\ldots 0 \to A_r \to A_r \to 0 \ldots$ So $N\,\hat{L}(A,m)$ has a natural contraction.

<u>Proposition 2.33</u> Let $A \in \mathcal{C}$. The complex $\mathrm{Im}\,\hat{\delta}$, where $\hat{\delta} : L(A,m) \to K(A,m+1)$, is the identity component of $K(A,m+1)$.

<u>Proof.</u> In the course of proving 2.22 it was shown that the identity component of $\hat{K}(A,m+1)$ contains $\mathrm{Im}\,\hat{\delta}$. Since $\hat{L}(A,m)$ is connected, $\mathrm{Im}\,\hat{\delta}$ is contained in the identity component of $\hat{K}(A,m+1)$.

IX. k-INVARIANTS OF FUNCTION COMPLEXES.[1]

In § 2 of this chapter we give theorems which are half-way towards solving Problems 3.1, 3.2 of Chapter IV. These theorems make the transition from FD-complexes to chain complexes. The transition from css-complexes to FD-complexes was made in Chapter VII, and the transition from chain-complexes to homology will be covered in Chapter X,XI. This latter step is quite simple theoretically, and involves essentially only "coefficient homomorphisms"; the main problem is to put the results in a form suitable for computations.

The course we adopt here is closely related to, and has the same motivation as, the Eilenberg-Zilber Theorem (VII. 3.41). This theorem, we recall, replaces the chain-complex $N(A \underline{\times} B)$, where $A,B \in \mathfrak{F}\mathfrak{D}$ by the chain-complex $NA \otimes NB$. Now the Dold-Kan theorem (VII, 4.2) shows that $N(A \underline{\times} B)$ may be written as a functor of NA and NB; nonetheless, the homological algebra of this functor is much less readily understood than that of $NA \otimes NB$, so that the replacement of $N(A \underline{\times} B)$ by $NA \otimes NB$ is indeed convenient.

At this stage there is a choice of working in the category either of FD-complexes or of chain complexes; the expositions in the two cases are "dual", in the sense that propositions have to be proved about the functor which is not natural to the particular category chosen. Thus $\underline{\times}$ is natural to the FD-category, but not to the chain complex category; $\otimes$, conversely, is a natural construct for chain complexes, but not for FD-complexes.

We shall work in the FD-category, and accordingly we define $A \otimes B$, for FD-complexes A,B, by $A \otimes B = R(NA \otimes NB)$. The Eilenberg-Zilber Theorem then furnishes an FD-homotopy equivalence $\Delta : A \otimes B \simeq A \underline{\times} B$.

In VII § 2 we defined, for FD-complexes A, B a map product $A \underline{\pitchfork} B$ such that if $Y \in \mathcal{X}$ then $B^{Y} \approx C(Y) \underline{\pitchfork} B$. This functor $\underline{\pitchfork}$ has the same deficiency as $\underline{X}$, namely that the normalised chain complex $N(A \underline{\pitchfork} B)$ is an inconvenient object. Accordingly, we define in § 1 a new complex $A \pitchfork B$ such that $A \pitchfork B \approx R(NA \pitchfork NB)$, and construct an FD-homotopy equivalence $\hat{\Delta}: A \underline{\pitchfork} B \longrightarrow A \pitchfork B$.

There is for the functor $\pitchfork$ an exponential law which gives an isomorphism $\mu : (A \otimes B) \pitchfork C \longrightarrow A \pitchfork (B \pitchfork C)$. To complete the picture of the transition from FD-complexes to chain complexes we prove that $N\mu$ is essentially the exponential map for chain complexes, and that the equivalences $\Delta, \hat{\Delta}$ preserve the exponential law.

These constructs are applied in § 2 to determine the compositions

$$X^{Y} \xrightarrow{k^{Y}} A^{Y} \xrightarrow{\hat{\Delta}} C(Y) \pitchfork A .$$

$$C(Y) \pitchfork A \xrightarrow{\hat{\Delta}'} A^{Y} \xrightarrow{\ell^{Y}} B^{Y} \xrightarrow{\hat{\Delta}} C(Y) \pitchfork B ,$$

where $\hat{\Delta}'$ denotes a homotopy inverse of $\hat{\Delta}$, $X, Y \in \mathcal{X}$, $A, B \in \mathcal{FD}$, $k : X \longrightarrow A$, $\ell : A \longrightarrow B$. In Chapter XI we define equivalences $C(Y) \pitchfork A \rightarrow R\, H(Y, A)$ and obtain the cohomological solution.

It should be noted that the procedure we have adopted is essential for the solution. If we choose an arbitrary equivalence $A^{Y} \xrightarrow{h} R\, H(Y, A)$, then we cannot say much about the composition $X^{Y} \xrightarrow{k^{Y}} A^{Y} \xrightarrow{h} R\, H(Y, A)$.

§ 1. The functors $\otimes$, $\pitchfork$ of FD-complexes.

(1.1) Definition 1.11. Let $A, B \in \mathcal{FD}$. The tensor product $A \otimes B \in \mathcal{FD}$ is the complex

$$A \otimes B = R\,(NA \otimes NB) .$$

The tensor product is an additive functor of two FD-complexes; thus if f, g

are two maps in $\mathcal{FD}$, then

$$f \otimes g = R(Nf \otimes Ng) .$$

Let Ψ: $NR \longrightarrow 1$ be the natural equivalence of the Dold-Kan Theorem (VII, 4.2). For $A,B,C \in \mathcal{FD}$ the composition

$$R(NA \otimes N(B \otimes C)) \xrightarrow{R(1 \otimes \Psi)} R(NA \otimes NB \otimes NC) \xrightarrow{R(1 \otimes \Psi)^{-1}} R(N(A \otimes B) \otimes NC)$$

is a natural isomorphism $A \otimes (B \otimes C) \approx (A \otimes B) \otimes C$ by means of which we identify these complexes.

<u>Definition 1.12.</u> Let $A,B \in \mathcal{FD}$. The <u>hom product</u> $A \pitchfork B \in \mathcal{FD}$ is defined by

$$\begin{cases} (A \pitchfork B)_q = \mathcal{F}(K(q) \otimes A, B) & q = 0,1,\ldots \\ \phi = \mathcal{F}(\phi^* \otimes 1, 1) & \phi \text{ a categorical css-operator.} \end{cases}$$

Clearly the hom product is an additive functor of two FD-complexes, contravariant in the first, covariant in the second.

<u>Definition 1.13.</u> Let $A,B,C \in \mathcal{FD}$. The map

$$\mu : \mathcal{F}(A \otimes B, C) \longrightarrow (A, B \pitchfork C) \qquad (*)$$

is defined by

$$\mu(f)(a) = f(\hat{a} \otimes 1) \qquad , f \in \mathcal{F}(A \otimes B, C), \quad a \in A .$$

Clearly μ is a natural map, and so defines an FD-map

$$\mu : (A \otimes B) \pitchfork C \longrightarrow A \pitchfork (B \pitchfork C)$$

which in dimension q is obtained from $(*)$ by writing $K(q) \otimes A$ for A.

(1.2) To prove that the above exponential map is an isomorphism we shall relate it with the exponential map for chain complexes. This necessitates introducing temporarily two other hom products, which we shall later identify with $\pitchfork$.

<u>Definition 1.21</u> Let $A,B \in \mathfrak{F}\mathfrak{D}$. We define hom products $A \pitchfork B$, $A \pitchfork'' B \in \mathfrak{F}\mathfrak{D}$ and isomorphisms $\lambda^1 : A \pitchfork B \to A \pitchfork' B$, $\lambda^2 : A \pitchfork' B \to A \pitchfork'' B$ by

$$\begin{cases} (A \pitchfork' B)_q = \mathfrak{F}(N(q) \otimes NA,\ NB), & q = 0,1,\ldots \\ \phi = \mathfrak{F}(\phi^* \otimes 1,\ 1) & \phi \text{ a categorical css-operator} \end{cases}$$

$$\begin{cases} (A \pitchfork'' B)_q = \mathfrak{F}(N(q),\ NA \pitchfork NB) = R(NA \pitchfork NB)_q, & q = 0,1,\ldots \\ \phi = \mathfrak{F}(\phi^*,\ 1) & \phi \text{ a categorical css-operator} \end{cases}$$

$\lambda^1(f) = N(f)\Psi^{-1}$, $f \in \mathfrak{F}(K(q) \otimes A, B)$; $\lambda^2(g) = \mu(g)$, $g \in \mathfrak{F}(N(q) \otimes NA,\ NB)$, where Ψ is the Dold-Kan map (VII, 4.2) and μ is the exponential map for chain complexes. Clearly λ^1, λ^2, are natural FD-isomorphisms.

<u>Theorem 1.22</u> Let $A,B,C \in \mathfrak{F}\mathfrak{D}$. The following diagram is commutative*

$$\begin{array}{ccc}
\mathfrak{F}(A \otimes B, C) & \xrightarrow{\ \mu\ } & \mathfrak{F}(A,\ B \pitchfork C) \\
 & & \approx\downarrow \lambda^1 \\
 & & \mathfrak{F}(A, B \pitchfork' C) \\
N \downarrow \approx & & \approx\downarrow \lambda^2 \\
 & & \mathfrak{F}(A, B \pitchfork'' C) \\
 & & \approx\downarrow N \\
\mathfrak{F}(N(A \otimes B), NC) & & \mathfrak{F}(NA,\ N(B \pitchfork'' C)) \\
\Psi^{\cdot}\uparrow\approx & & \approx\downarrow \Psi. \\
\mathfrak{F}(NA \otimes NB,\ NC) & \xrightarrow[\mu]{\ \approx\ } & \mathfrak{F}(NA,\ NB \pitchfork NC),
\end{array}$$

where the maps μ are exponential maps, and we have identified $\mathfrak{F}(NA,\ S(NB \pitchfork NC))$ and $\mathfrak{F}(NA,\ NB \pitchfork NC)$.

We defer the proof to an Appendix.

* Here, and later, we find the following notation convenient, if f is any map, then a map induced contravariantly by f is written f^*, and a map induced covariantly by f is written f_* or, simply, f.

<u>Corollary 1.23</u> Let $A, B, C \in \mathfrak{F}\mathfrak{D}$. Then

$$\mu : \mathfrak{F}(A \otimes B,\ C) \longrightarrow \mathfrak{F}(A,\ B \pitchfork C)$$

is a homotopy preserving isomorphism, and

$$\mu : \ (A \otimes B) \pitchfork C \longrightarrow A \pitchfork (B \pitchfork C)$$

is an isomorphism.

<u>Proof.</u> The first statement follows from the fact that all the other maps of the diagram of 1.22 are homotopy preserving isomorphisms. The second statement clearly follows from the first statement.

(1.24) It is now convenient to make the identifications

$$A \pitchfork B = A \pitchfork' B = A \pitchfork'' B \quad \text{for} \quad A, B \in \mathfrak{F}\mathfrak{D} \ .$$

We shall also identify A and RNA $(A \in \mathfrak{F}\mathfrak{D})$, and NRK and SK $(K \in \mathcal{C})$.

The diagram of 1.22 can now be written as a commutative diagram of homotopy preserving isomorphisms

$$\begin{array}{ccc} \mathfrak{F}(A \otimes B,\ C) & \xrightarrow[\approx]{\mu} & \mathfrak{F}(A,\ B \pitchfork C) \\ N \downarrow \approx & & \approx \downarrow N \\ \mathfrak{F}(NA \otimes NB,\ NC) & \xrightarrow[\mu]{\approx} & \mathfrak{F}(NA,\ NB \pitchfork NC) \ . \end{array}$$

(1.3) The natural maps of the Eilenberg-Zilber Theorem (VII, 3.41) transform under R to maps

$$A \underline{\times} B \ \underset{\Delta}{\overset{\xi}{\rightleftarrows}} \ A \otimes B$$

such that $\xi \Delta = 1$ and such that there is a natural FD-homotopy $\Delta \xi \simeq 1$.

We now prove

<u>Theorem 1.31</u> Let $A, B \in \mathfrak{F}\mathfrak{D}$. There are natural maps

$$A \underline{\pitchfork} B \ \underset{\xi}{\overset{\hat{\Delta}}{\rightleftarrows}} \ A \hat{\pitchfork} B$$

such that $\hat{\Delta}\hat{\xi} = 1$ and such that there is a natural FD-homotopy $\hat{\xi}\hat{\Delta} \simeq 1$.

Proof. The complexes $A \pitchfork B$, $A \pitchfork B$ are given by

$$(A \underline{\pitchfork} B)_q = \mathcal{F}(K(q) \underline{\times} A, B), \quad (A \pitchfork B)_q = \mathcal{F}(K(q) \otimes A, B).$$

Let $\hat{\Delta}_q = \mathcal{F}(\Delta, 1)$, $\hat{\xi}_q = \mathcal{F}(\xi, 1)$. The fact that Δ, ξ are natural implies that $\hat{\Delta}$, $\hat{\xi}$ are FD-maps. Also $\xi\Delta = 1$ implies $\hat{\Delta}\hat{\xi} = 1$.

Let $H : K(1) \underline{\times} K(q) \underline{\times} A \longrightarrow K(q) \underline{\times} A$ be the natural homotopy of the Eilenberg-Zilber Theorem. Let $H' : K(1) \underline{\times} (A \underline{\pitchfork} B) \longrightarrow A \underline{\pitchfork} B$ be defined in dimension q by

$$H'_q(\phi\delta^1 \otimes f)(\psi\delta^q \otimes a) = f\, H(\psi\phi\delta^1 \otimes \psi\delta^q \otimes a) \qquad \begin{cases} f \in \mathcal{F}(K(q) \times A, B) \\ a \in A_p, \phi^* \in Map(\Delta^q, \Delta^1) \\ \psi^* \in Map(\Delta^p, \Delta^q) \end{cases}$$

Since each H'_q is natural, the map H' is an FD-map. Further

$$H'(s_0^q \partial_0 \delta^1 \otimes f)(\psi\delta^q \otimes a) = f\, H(s_0^p \partial_0 \delta^1 \otimes \psi\delta^q \otimes a) = f(\psi\delta^q \otimes a),$$

$$H'(s_0^q \partial_1 \delta^1 \otimes f)(\psi\delta^q \otimes a) = f\, H(s_0^p \partial_1 \delta^1 \otimes \psi\delta^q \otimes a) = f\Delta\xi(\psi\delta^q \otimes a)$$

$$= (\hat{\xi}\hat{\Delta} f)(\psi\delta^q \otimes a).$$

Therefore $H' : 1 \simeq \hat{\xi}\hat{\Delta}$.

Actually H' is the map corresponding (under μ) to the map

$$K(1) \underline{\times} (A \underline{\pitchfork} B) \underline{\times} A \xrightarrow{H} (A \underline{\pitchfork} B) \underline{\times} A \xrightarrow{\varepsilon} B$$

where ε is the evaluation map (defined by $\mu(\varepsilon) = 1$). However, the explicit formula for H' is relevant to Remark 1.33 below.

Corollary 1.32 Let $Y \in \mathcal{K}$, and let $A \in \mathcal{C}$ be such that $A_i = 0$, $i < m$. The homotopy groups of $\hat{K}(A,m)^Y$ are given by

$$\pi_q(K(A,m)^Y) \approx H^{q-m}(Y, A).$$

Proof.

$$\begin{aligned}
\pi_q(K(A,m)^Y) &\approx \pi_q(C(Y) \pitchfork K(A,m)) && \text{by VII. 2.22} \\
&\approx \pi_q(C(Y) \pitchfork K(A,m)) && \text{by 1.31} \\
&\approx H_q(C_N(Y) \pitchfork S\eta^m A) && \text{by VIII. 1.35} \\
&= H^q(Y, S\eta^m A) \\
&= H^q(Y, \eta^m A). && \text{since } S\eta^m A = \eta^m A \\
&= H^{q-m}(Y, A).
\end{aligned}$$

Remark 1.33 Let T be an additive, covariant functor of two variables from abelian groups to abelian groups. Then T may be extended to a functor $\underline{T} : \exists\mathcal{D} \times \exists\mathcal{D} \longrightarrow \exists\mathcal{D}$ by setting

$$\begin{cases} \underline{T}(A, B)_q = T(A_q, B_q) & A,B \in \exists\mathcal{D}, \quad q = 0,1,\ldots \\ \phi = T(\phi, \phi) & \phi \text{ a categorical css-operator.} \end{cases}$$

There is also a well-known way of extending T to a functor $T : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ by setting, for $C, C' \in \mathcal{C}$,

$$T(C, C')_n = \sum_{p+q=n} T(C_p, C'_q)$$

with differential $T(\partial, 1) + T(1, \partial)$ (using the sign convention of VI. 1). So we may define a functor $T : \exists\mathcal{D} \times \exists\mathcal{D} \longrightarrow \exists\mathcal{D}$ by

$$T(A, B) = R(T(NA, NB)) \qquad A,B \in \exists\mathcal{D}.$$

Further, two "function complexes of type T,"

$\pitchfork_T$ and $\pitchfork_T$ may be defined by setting, for $A,B \in \exists\mathcal{D}$

$$(A \underline{\pitchfork}_T B)_q = \exists(\underline{T}(K(q), A), B),$$
$$(A \pitchfork_T B)_q = \exists(T(K(q), A), B) .$$

It may be proved, however, that $\underline{T}(A,B) \cong T(A,B)$, $A \underline{\pitchfork}_T B \cong A \pitchfork_T B$. The first of these equivalences is found by recalling (VI. 3.42) that the maps and homotopies of the Eilenberg-Zilber theorem may be expressed as linear combinations of pairs of categorical css-operators. So the functor T (of abelian groups) may be applied to these pairs to give maps and homotopies relating the functors (of FD-complexes) $\underline{T}$ and T.

The second of these equivalences follows from the first in an analagous manner to the proof of 1.31.

<u>Remark 1.34.</u> The naturality of $\hat{\Delta}$, $\hat{\xi}$ implies commutativity in various diagrams of the type

$$\begin{array}{ccccc}
(A \otimes B)\underline{\pitchfork} C & \xrightarrow{\xi \underline{\pitchfork} 1} & (A \underline{\times} B)\underline{\pitchfork} C & \xrightarrow{\Delta \underline{\pitchfork} 1} & (A \otimes B)\underline{\pitchfork} C \\
\downarrow{\scriptstyle\hat{\Delta}} & & \downarrow{\scriptstyle\hat{\Delta}} & & \downarrow{\scriptstyle\hat{\Delta}} \\
(A \otimes B)\pitchfork C & \xrightarrow[\xi \pitchfork 1]{} & (A \underline{\times} B)\pitchfork C & \xrightarrow[\Delta \pitchfork 1]{} & (A \otimes B)\pitchfork C .
\end{array}$$

<u>(1.4)</u> The fundamental theorem (1.41) is now simple of proof. The corollary 1.44 is a form of the theorem which is useful in applications later, and which shows clearly that we have made the transition from css-complexes to chain complexes in such a way as to preserve μ.

<u>Theorem 1.41</u> Let $A,B,C \in \exists\mathcal{D}$. The following diagram is commutative

$$\begin{array}{ccc}
\exists(A \underline{\times} B, C) & \xrightarrow{\Delta^{\cdot}} & \exists(A \otimes B, C) \\
{\scriptstyle\mu}\downarrow{\scriptstyle\approx} & & {\scriptstyle\approx}\downarrow{\scriptstyle\mu} \\
\exists(A, B \underline{\pitchfork} C) & \xrightarrow[\hat{\Delta}_{\cdot}]{} & \exists(A, B \pitchfork C)
\end{array}$$

<u>Proof.</u> Let $f \in \exists(A \underline{\times} B, C)$, $a \in A$. Then

$$
\begin{aligned}
(\mu\Delta' f)(a) &= (\Delta' f)(\hat{a}\otimes 1) && \text{by definition of } \mu \\
&= f\Delta(\hat{a}\otimes 1) && \\
&= f\,(\hat{a}\,\underline{\times}\,1)\Delta && \text{by naturality of } \Delta \\
&= (\mu f)(a)\Delta && \text{by definition of } \mu\text{, VII, 2.31} \\
&= \{\hat{\Delta}.\mu f\}(a) && \text{by definition of } \hat{\Delta}\text{ .}
\end{aligned}
$$

<u>Corollary 1.42</u> Let $A, B, C \in \mathcal{JD}$. There is a commutative diagram of isomorphisms

$$
\begin{array}{ccc}
\langle A\,\underline{\times}\,B,\ C\rangle & \xrightarrow[\approx]{\Delta'} & \langle A\otimes B,\ C\rangle \\
\mu\downarrow\approx & & \approx\downarrow\mu \\
\langle A,\ B\,\underline{\pitchfork}\,C\rangle & \xrightarrow[\hat{\Delta}.]{\approx} & \langle A,\ B\pitchfork C\rangle
\end{array}
$$

<u>Definition 1.43</u> Let $X, Y \in \mathcal{X}$, $A \in \mathcal{JD}$. We define

$$\theta : H^0(X \ast Y, A) \approx H^0(X, C_N(Y)\pitchfork NA) = H^0(X,\ C(Y)\pitchfork A)$$

to be the composition

$$
\begin{aligned}
H^0(X \ast Y, A) = {} & H^0(N(C(X)\,\underline{\times}\,C(Y)),\ NA) \\
& \xrightarrow[\approx]{\Delta^*} H^0(C_N(X)\otimes C_N(Y),\ NA) \\
& \xrightarrow[\approx]{\mu} H\ (X,\ C_N(Y)\pitchfork NA)\ .
\end{aligned}
$$

<u>Corollary 1.44</u> Let $X, Y \in \mathcal{X}$, $A \in \mathcal{JD}$. There is a commutative diagram of isomorphisms

$$
\begin{array}{ccc}
H^0(X \ast Y,\ A) & \xrightarrow[\approx]{\theta} & H^0(X,\ C_N(Y)\pitchfork NA) \\
\mu\downarrow\approx & & \| \\
H^0(X,\ C(Y)\,\underline{\pitchfork}\,A) & \xrightarrow[\hat{\Delta}_*]{} & H^0(X,\ C(Y)\pitchfork A)\ .
\end{array}
$$

<u>Proof.</u> In writing $\mu : H^0(X \ast Y, A) \xrightarrow[\approx]{} H^0(X,\ C(Y)\,\underline{\pitchfork}\,A)$, we have used the identifications

$$H^0(X \ast Y, A) = [X \ast Y, A] \xrightarrow{\mu} [X, A^Y] = [X, C(Y)\,\underline{\pitchfork}\,A] = H^0(X,\ C(Y)\,\underline{\pitchfork}\,A)\ .$$

The commutativity of the diagram follows immediately from 1.42, the definition of θ , and 1.24.

(1.5) Theorem 1.41 is a theorem about the dimension 0 of certain function complexes. We show that this theorem generalises to the whole function complexes. This result is not needed later, so we defer the proof to an Appendix.

<u>Theorem 1.51</u> Let $A, B, C \in \mathcal{FD}$. There are natural FD-homotopy equivalences

$$\Delta^1 : (A \underline{\times} B) \underline{\pitchfork} C \simeq (A \otimes B) \pitchfork C, \qquad \Delta^2 : A \underline{\pitchfork} (B \underline{\pitchfork} C) \simeq A \pitchfork (B \pitchfork C)$$

such that the following diagram is commutative

$$\begin{array}{ccc} (A \underline{\times} B) \underline{\pitchfork} C & \xrightarrow{\Delta^1} & (A \otimes B) \pitchfork C \\ \mu \downarrow \approx & & \approx \downarrow \mu \\ A \underline{\pitchfork} (B \underline{\pitchfork} C) & \xrightarrow[\Delta^2]{} & A \pitchfork (B \pitchfork C) \end{array}$$

§2. THE BASIC THEOREMS.

(2.1) Definition 2.11 Let $X, Y, Z \in \mathcal{X}$. A map

$$F^Y : Z^X \longrightarrow (Z^Y)^{X^Y}$$

is defined as follows: in dimension q , F^Y is the composition

$$\mathrm{Map}(\Delta^q \divideontimes X, Z) \xrightarrow{(1 \divideontimes \varepsilon)^*} \mathrm{Map}(\Delta^q \divideontimes X^Y \divideontimes Y, Z) \xrightarrow{\mu} \mathrm{Map}(\Delta^q \divideontimes X^Y, Z^Y),$$

where $\varepsilon : X^Y \divideontimes Y \longrightarrow X$ is the evaluation map

Let $k \in (Z^X)_q$, $(\phi\delta^q, g) \in (\Delta^q \divideontimes X^Y)_p$, $(\psi\delta^p, y) \in (\Delta^p \divideontimes Y)_r$

Then $F^Y(k)(\phi\delta^q, g)(\psi\delta^p, y) = k(\psi\phi\,\delta^q, g(\psi\delta^p, y))$.
In particular, when $q = 0$, identifying $\Delta^0 \divideontimes X$ and X ,

$$F^Y(k)\ (g) = kg .$$

In dimension 0, we write k^Y for $F^Y(k)$; it is clear that

$$F^Y = (F^Y)_0 : \mathrm{Map}(X, Z) \longrightarrow \mathrm{Map}(X^Y, Z^Y)$$

is homotopy preserving.

When $Z = A \in \mathcal{FD}$, then A^X and $(A^Y)^{X^Y}$ are FD-complexes, and F^Y is a homomorphism. We also write F^Y for the corresponding maps

$$[X, A] \longrightarrow [X^Y, A^Y], \qquad H^0(X, A) \longrightarrow H^0(X^Y, A^Y) .$$

The following theorem is now almost obvious.

Theorem 2.12 Let $X, Y \in \mathcal{X}$, $A \in \mathcal{FD}$. The following diagram is commutative

$$\begin{array}{ccc}
H^0(X, A) & \xrightarrow{\varepsilon^*} & H^0(X^Y \divideontimes Y, A) \\
F^Y \downarrow & & \| \\
H^0(X^Y, A^Y) & & H^0(X^Y \divideontimes Y, NA) \\
\hat{\Delta}_* \downarrow \approx & & \approx \downarrow \theta \\
H^0(X^Y, C(Y) \pitchfork A) & = & H(X^Y, C_N(Y) \pitchfork NA) ,
\end{array}$$

where θ is defined in 1.43.

Proof. We consider the following diagram

$$\begin{array}{ccc}
\mathrm{Map}(X,A) & \xrightarrow{\varepsilon^{\cdot}} & \mathrm{Map}(X^Y \ast Y, A) \\
F^Y \downarrow & \swarrow \mu\ \approx & \Vert \\
\mathrm{Map}(X^Y, A^Y) & & \Vert \\
\Vert & & \Vert \\
\mathfrak{F}(C(X^Y), A^Y) & \xleftarrow[\approx]{\mu} & \mathfrak{F}(C(X^Y) \times C(Y), A) \\
\hat{\Delta}. \downarrow & & \downarrow \Delta^{\cdot} \\
\mathfrak{F}(C(X^Y), C(Y) \pitchfork A) & \xleftarrow[\approx]{\mu} & \mathfrak{F}(C(X^Y) \otimes C(Y), A) .
\end{array}$$

The top triangle is commutative by definition of F^Y. The bottom square is commutative by 1.41. Passing to homotopy classes, and so to homology, we obtain the theorem.

(2.2) The next theorem has two purposes. First it gives an alternative description of the (homotopy class of the) evaluation map $\varepsilon : A^Y \ast Y \longrightarrow A$. Second, it shows the equivalence (up to homotopy) of two possible definitions of an "evaluation map" $(C(Y) \pitchfork A) \ast Y \longrightarrow A$.

Theorem 2.21. Let $A \in \mathfrak{F}\mathfrak{D}$, $Y \in \mathfrak{X}$. The maps of the diagram

$$\begin{array}{ccc}
H^0((C(Y) \pitchfork A) \ast Y, A) & \xrightarrow{(\hat{\Delta} \ast 1)^*} & H^0(A^Y \ast Y, A) \\
\theta \downarrow \approx & & \approx \downarrow \mu \\
H^0(C(Y) \pitchfork A, C(Y) \pitchfork A) & & H^0(A^Y, A^Y)
\end{array}$$

satisfy

$$\mu(\hat{\Delta} \ast 1)^* \theta^{-1} \quad \omega(C(Y) \pitchfork A) = \omega(A^Y)$$

where ω denotes, as usual, the fundamental class of an FD-complex.

Proof. The following is a commutative diagram of isomorphisms

$$\begin{array}{ccccc} H^0((C(Y)\pitchfork A) \divideontimes Y, A) & \xrightarrow{(\hat{\Delta}\divideontimes 1)^*} & H^0(A^Y \divideontimes Y, A) & \xrightarrow{\mu} & H^0(A^Y, A^Y) \\ \downarrow \theta & & \downarrow \theta & & \downarrow \hat{\Delta}_* \\ H^0(C(Y)\pitchfork A, C(Y)\pitchfork A) & \xrightarrow[\hat{\Delta}^*]{} & H^0(A^Y, C(Y)\pitchfork A) & = & H^0(A^Y, C(Y)\pitchfork A) \end{array} ;$$

for the left-hand square is commutative by naturality of θ , and the right-hand square is commutative by 1.44. So

$$\mu(\hat{\Delta}\divideontimes 1)^* \theta^{-1} \omega(C(Y)\pitchfork A) = (\hat{\Delta}_*)^{-1} \hat{\Delta}^* \omega(C(Y)\pitchfork A) .$$

But clearly $(\hat{\Delta}_*)^{-1} \hat{\Delta}^* \omega(C(Y)\pitchfork A) = \omega(A^Y)$, for the fundamental class corresponds to the identity map.

It should be noted that $\mu^{-1}\omega(A^Y)$ is the class of the evaluation map $\mathcal{E} : A^Y \divideontimes Y \to A$.

<u>Corollary 2.22</u> $\theta\,\mathcal{E}^*\omega(A) = \hat{\Delta}^*\omega(C(Y)\pitchfork A)$.

<u>Proof.</u>
$$\begin{aligned} \theta\,\mathcal{E}^*\omega(A) &= \theta\,\mu^{-1}\omega(A^Y) && \text{by definition of } \mathcal{E} \\ &= \theta(\hat{\Delta}\divideontimes 1)^*\theta^{-1}\omega(C(Y)\pitchfork A) && \text{by 2.21} \\ &= \hat{\Delta}^*\omega(C(Y)\pitchfork A) && \text{by naturality of } \theta . \end{aligned}$$

(2.3) Let $A,B \in \mathfrak{J}\mathfrak{D}$, $Y \in \mathfrak{X}$. We define a homomorphism

$$G^Y : \mathcal{O}(A,B) \longrightarrow \mathcal{O}(C(Y)\pitchfork A,\ C(Y)\pitchfork B)$$

(c.f. VII. § 5 for definition of $\mathcal{O}$). Let $k \in \mathcal{O}(A,B)$; then $G^Y(k)$ is the operation which, on $H^0(X, C(Y)\pitchfork A)$, is the composition

$$H^0(X, C(Y)\pitchfork A) \underset{\approx}{\xrightarrow{\theta^{-1}}} H^0(X\divideontimes Y, A) \xrightarrow{k} H^0(X\divideontimes Y, B) \underset{\approx}{\xrightarrow{\theta}} H^0(X, C(Y)\pitchfork B) .$$

We also write G^Y for the corresponding maps

$$[A,B] \longrightarrow [C(Y)\pitchfork A,\ C(Y)\pitchfork B] ,$$

$$H^0(A,B) \longrightarrow H^0(C(Y)\pitchfork A,\ C(Y)\pitchfork B) .$$

<u>Theorem 2.31</u> Let $A,B \in \mathfrak{J}\mathfrak{D}$, $Y \in \mathfrak{X}$. The following diagram is commutative

$$\begin{array}{ccc} H^0(A,B) & \xrightarrow{F^Y} & H^0(A^Y, B^Y) \\ G^Y \downarrow & & \approx \downarrow \hat{\Delta}_* \\ H^0(C(Y)\pitchfork A, C(Y)\pitchfork B) & \underset{\hat{\Delta}^*}{\xrightarrow{\approx}} & H^0(A^Y, C(Y)\pitchfork B) \end{array}$$

where $\hat{\Delta}_*$, $\hat{\Delta}^*$, are induced respectively by $\hat{\Delta} : B^Y \cong C(Y) \pitchfork B$, $\hat{\Delta} : A^Y \cong C(Y) \pitchfork A$.

Proof. For clarity, we write $\underline{k}$ for the operation in $\mathcal{O}(A,B)$ corresponding to $k \in H^0(A,B)$. The following diagram is commutative

$$\begin{array}{ccccc}
H^0(A,A) & \xrightarrow{\underline{k}} & H^0(A,B) & & \\
\downarrow \varepsilon^* & & \downarrow \varepsilon^* & & \\
H^0(A^Y * Y, A) & \xrightarrow{\underline{k}} & H^0(A^Y * Y, B) & \xrightarrow[\approx]{\mu} & H^0(A^Y, B^Y) \\
\theta \downarrow \approx & & \theta \downarrow \approx & & \approx \downarrow \hat{\Delta}_* \\
H^0(A^Y, C(Y) \pitchfork A) & \xrightarrow{G^Y(\underline{k})} & H^0(A^Y, C(Y) \pitchfork B) & = & H^0(A^Y, C(Y) \pitchfork B) \\
\hat{\Delta}^* \uparrow \approx & & \approx \uparrow \hat{\Delta}^* & & \\
H^0(C(Y) \pitchfork A, C(Y) \pitchfork A) & \xrightarrow[G^Y(\underline{k})]{} & H^0(C(Y) \pitchfork A, C(Y) \pitchfork B) & &
\end{array}$$

;

for the left-hand squares are commutative, the middle one by definition of $G^Y(\underline{k})$, and the others by naturality of operations with respect to maps. The right-hand square is commutative by 1.44.

Now $\mu \varepsilon^* \underline{k}\, \omega(A) = \mu \varepsilon^*(k) = F^Y(k)$, by definition of F^Y.

Also $G^Y(\underline{k})\, \omega(C(Y) \pitchfork A) = G^Y(k)$.

By 2.22,

$$\theta \varepsilon^* \omega(A) = \hat{\Delta}^* \omega(C(Y) \pitchfork A).$$

Therefore $\hat{\Delta}^* G^Y(k) = \hat{\Delta}_* F^Y(k)$.

X. KÜNNETH ISOMORPHISMS.

It is well-known that if C, C' are free finitely generated chain complexes, bounded below, then for any graded group G, $H^*(C \otimes C', G) \approx H^*(C, H^*(C', G))$. This isomorphism is sometimes called a Künneth isomorphism.

The usual construction of this isomorphism is via the Universal Coefficient Theorem and the Künneth Theorem (c.f. [25; Chapter 5, Exercise 5]). However the direct sum forms of both these theorems are non-natural, so that the naturality properties of a Künneth isomorphism constructed in this way are unclear. In fact, it is not difficult to give an example to show that a Künneth isomorphism $H^*(C \otimes C', G) \approx H^*(C, H^*(C',G))$ cannot be natural with respect to maps of C'.

In §1, a simple construction of a Künneth isomorphism κ is given. This construction is carried out at the chain level, and satisfies three conditions (i) naturality of κ with respect to maps of C is assured (ii) it is possible to discuss the behaviour under κ of elements of $H^*(C \otimes C', G)$, given for example as co-cycles (iii) there is, when $C' = C_N(Y)$, $G = NA$, a related css-equivalence $C(Y) \pitchfork A \longrightarrow R\,H\,(Y,A)$.
These three conditions on κ are in fact essential for a complete discussion of the function complex problem.

The construction of κ given here is related to a construction of Bott and Samelson in [8], who there construct a Künneth isomorphism in homology when the coefficients are Z.

In § 2 we give formulae for k when $G = Z_n (n \geqslant 0)$ and C' is elementary.

Applications to function complexes are given in Chapter XI.

§ 1. Construction of Künneth Isomorphisms.

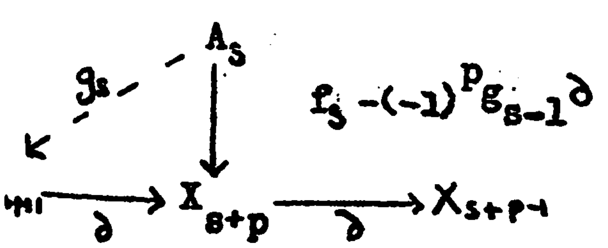
(1.1) In this section, Proposition 1.11 is of standard type, while Proposition 1.12 is well-known.

Proposition 1.11. Let $A \in \mathcal{C}_0$ be free. If $f : B \to C$ is any map in $\mathcal{C}_\#$ such that $f_* : H(B) \approx H(C)$, then

$$(1 \pitchfork f)_* : H(A \pitchfork B) \approx H(A \pitchfork C) .$$

Proof. It is sufficient (and in fact necessary) to prove that for all $X \in \mathcal{C}$, $H(X) = 0 \Longrightarrow H(A \pitchfork X) = 0$. For suppose this is true. Let M_f be the mapping cylinder [19] of a map $f : B \to C$ in $\mathcal{C}$. There is an exact sequence $0 \to C \to M_f \to B \to 0$ whose homology boundary coincides with f_*; since f_* is iso, $H(M_f) = 0$. Also, since A is free, $0 \to A \pitchfork C \to A \pitchfork M_f \to A \pitchfork B \to 0$ is exact and the homology boundary of this sequence is $(1 \pitchfork f)_*$. Since, by assumption, $H(A \pitchfork M_f) = 0$, it follows that $(1 \pitchfork f)_*$ is iso.

Let therefore $X \in \mathcal{C}$ satisfy $H(X) = 0$. Let $f \in Z_p(A \pitchfork X)$; we prove f is a boundary by constructing inductively an element $g \in (A \pitchfork X)_{p+1}$ such that $\delta g = f$. Let $g_r = 0$, $r < 0$. Suppose g_r has been defined for $r < s$ to satisfy $f_r = (-1)^p g_{r-1} \partial + \partial g_r$.



Then

$$\partial(f_s - (-1)^p g_{s-1}\partial) = (-1)^p f_{s-1}\partial - (-1)^p \partial g_{s-1}\partial$$
$$= (-1)^p((-1)^p g_{s-2}\partial + \partial g_{s-1})\partial - (-1)^p \partial g_{s-1}\partial$$
$$= 0.$$

Since $H(X) = 0$ and A_s is free, there exists $g_s : A_s \to X_{s+p+1}$ such that $g_s = f_s - (-1)^p g_{s-1}\partial$.

So we have proved f is a boundary. Hence $H(A \pitchfork X) = 0$.

<u>Proposition 1.12</u> Let $A, F \in \mathcal{C}$. Let F be free and let $\phi : H(F) \longrightarrow H(A)$ be any map. There is chain map $g : F \to A$ such that $g_* = \phi$.

If G is any graded group, there is a free complex $F \in \mathcal{C}$ such that $H(F) \approx G$,

(1.2) <u>Theorem 1.21.</u> Let $L, A, B \in \mathcal{C}$ and let $K \in \mathcal{C}_0$ be free. Let $\vartheta : H(L \pitchfork A) \approx H(B)$ be an isomorphism. There is an isomorphism

$$\kappa : H(K \otimes L, A) \approx H(K, B)$$

defined for all free $K \in \mathcal{C}_0$, natural with respect to maps of K and coinciding with ϑ (under the canonical identifications) if $K = Z$.

<u>Proof.</u> Let $F \in \mathcal{C}$ be a free complex and $f : F \longrightarrow L \pitchfork A$ a map such that $f_* : H(F) \approx H(L \pitchfork A)$. Since F is free, there is a map $g : F \longrightarrow B$ such that $g_* = \vartheta f_* : H(F) \to H(B)$. So if $K \in \mathcal{C}_0$ is free, there are isomorphisms

$$H(K \otimes L, A) \xrightarrow[\approx]{\mu} H(K, L \pitchfork A) \xleftarrow[\approx]{(1 \pitchfork f)_*} H(K, F) \xrightarrow[\approx]{(1 \pitchfork g)_*} H(K, B).$$

The composition of these isomorphisms $\kappa : H(K \otimes L, A) \approx H(B)$ is clearly natural with respect to maps of K. Further, if $K = Z$, then μ reduces to the identity and $\kappa = (1 \pitchfork g)_* (1 \pitchfork f)_*^{-1} = g_* f_*^{-1} = \vartheta$.

<u>Definition 1.22.</u> Let $L, A, B \in \mathcal{C}$. A <u>Künneth isomorphism of type $(L, A; B)$</u> is an isomorphism

$$\kappa : H(K \otimes L, A) \approx H(K, B)$$

which is defined for all free $K \in \mathcal{C}_0$, and which can be constructed as in the proof of 1.21.

Clearly if such an isomorphism exists then (putting $K = Z$) $\kappa : H(L,A) \approx H(B)$. If $B = L \pitchfork A$ or $H(L \pitchfork A)$ it will always be assumed that $\kappa : H(L,A) \approx H(L,A)$ is the identity.

The isomorphism κ is said to be associated with the triple $(F ; f,g)$, where F,f,g are as in 1.21, and with the isomorphism ∂ . A simpler construction of κ is obviously possible whenever a map $h : L \pitchfork A \longrightarrow B$ exists which induces an isomorphism in homology; such a map exists for example if $L \pitchfork A$ is free.

If κ is constructed by means of such a map h , then κ is said to be associated with h .

If $Y \in \mathcal{X}$, then a Künneth isomorphism of type $(Y,A;B)$ where $A,B \in \mathcal{C}$, is for each $X \in \mathcal{X}$ the composition

$$H(X \# Y,A) \xrightarrow{\Delta^*} H(C_N(X) \otimes C_N(Y),A) \xrightarrow{\kappa} H(X;B)$$

where κ is a Künneth isomorphism of type $(C_N(Y), A;B)$, and Δ is the Eilenberg-Zilber map.

If $Y \in \mathcal{X}$, $A,B \in \mathcal{FD}$, then a Künneth isomorphism of type $(Y,A; B)$ is simply a Künneth isomorphism of type $(Y,NA;NB)$.

<u>Remark 1.23</u> Each part of the above definition is essential for our purposes.

(1.3) <u>Definition 1.31</u> Let $Y \in \mathcal{X}$, and let κ_1, κ_2 be Künneth isomorphisms of types $(Y,A_1; B_1)$, $(Y, A_2; B_2)$ respectively, where $A_i,B_i \in \mathcal{C}$ or $\mathcal{FD}$ $i = 1,2$.

A homomorphism

$$\kappa_{12} : \mathcal{O}(A_1,A_2) \longrightarrow \mathcal{O}(B_1,B_2)$$

is defined as follows : if $k \in \mathcal{O}(A_1,A_2)$, then $\kappa_{12}(k)$ is, as a function on $H^0(X,B_1)$ ($X \in \mathcal{X}$) the composition

$$H^0(X,B_1) \xrightarrow{\kappa_1^{-1}} H^0(X \ast Y,A_1) \xrightarrow{k} H^0(X \ast Y,A_2) \xrightarrow{\kappa_2} H^0(X, B_2)$$

If $A_i,B_i \in \mathcal{FD}$, then we regard κ_{12} also as mapping

$$[A_1,A_2] \longrightarrow [B_1,B_2] , H^0(A_1,A_2) \longrightarrow H^0(B_1,B_2) .$$

<u>Remark 1.32</u> The determination of κ_{12} in general is difficult. In particular cases, in order to evaluate $\kappa_{12}(k)$ for a given k we shall simply use Definition 1.31.

Theorem 1.34 does determine one part of $\kappa_{12}(k)$ in general.

<u>Proposition 1.33</u> Let κ be a Künneth isomorphism of type $(Y,A \,;\, H(Y,A))$. The composition

$$H(S^1 \ast Y,A) \xrightarrow{\kappa} H(S^1,H(Y,A)) \approx \eta\, H(Y,A) \approx H(Y,\eta^- A)$$

coincides with the suspension σ' (VII.5.3).

The proposition is immediate from the definitions.

(1.4) Let now $A,B \in \mathcal{C}$, $Y \in \mathcal{X}$. Let κ_1 , κ_2 be Künneth isomorphisms of types $(Y,A;\, H(Y,A))$, $(Y,B;\, H(Y,B))$ respectively, so that

$\kappa_{12}: \mathcal{O}(A,B) \longrightarrow \mathcal{O}(H(Y,A),H(Y,B))$. According to the discussion of VII 5.4,5.5 there is a projection

$$p_m : \mathcal{O}(H(Y,A),\, H(Y,B)) \longrightarrow \mathcal{O}(\eta^m H^m(Y,A),\eta^m H^m(Y,B))$$

and $\mathcal{O}(\eta^m H^m(Y,A),\, \eta^m H^m(Y,B)) = \mathcal{O}(\eta^m H^0(Y,\, \eta^{-m}A),\eta^m H^0(Y,\eta^{-m} B))$.

But by a classical argument, if $m > 0$

$$\mathcal{O}(\eta^m H^0(Y,\eta^{-m} A),\eta^m H^0(Y,\eta^{-m} B) \approx \mathrm{Hom}\, (H^0(Y,\eta^{-m} A),\, H^0(Y,\eta^{-m}B)) .$$

Let us write $\rho_m : \mathcal{O}(H(Y,A),\ H(Y,B)) \longrightarrow \mathrm{Hom}(H^0(Y,\eta^{-m}A),\ H^0(Y,\eta^{-m}B))$ for the composition of these maps.

For $m=0$, we have

$$\mathcal{O}(H^0(Y,A)\ ,\ H^0(Y,B)) \approx \mathrm{Map}(H^0(Y,A),\ \ H^0(Y,B))\ ,$$

the latter set being the set of functions $H^0(Y,A) \longrightarrow H^0(Y,B)$ which preserve 0. So for $m = 0$, we take ρ_0 as a map

$$\rho_0 : \mathcal{O}(H(Y,A),\ H(Y,B)) \longrightarrow \mathrm{Map}(H^0(Y,A),\ \ H^0(Y,B)).$$

<u>Theorem 1.44.</u> Let $k \in \mathcal{O}(A,B)$, and $m \geqslant 0$. Then

$$\rho_m \kappa_{12}(k) = \sigma^m(k)\ ,$$

the m-fold suspension of k (VII.5.3).

<u>Proof.</u> To evaluate $\rho_m \kappa_{12}(k)$ it suffices to evaluate $\kappa_{12}(k)$ on $H^0(S^m,\ H(Y,A))$. The following diagram is commutative

$$\begin{array}{ccccccc}
H^0(S^m,\ H(Y,A)) & \xrightarrow{\kappa_1^{-1}} & H^0(S^m \# Y,A) & \xrightarrow{k} & H^0(S^m \# Y,B) & \xrightarrow{\kappa_2} & H^0(S^m, H(Y,B)) \\
\downarrow \approx & & \approx \downarrow \sigma^m & & \approx \downarrow \sigma^m & & \downarrow \approx \\
H^m(Y,A) & = & H^0(Y,\eta^{-m}A) & \xrightarrow[\sigma^m k]{} & H^0(Y,\eta^{-m}B) & = & H^m(Y,B).
\end{array}$$

The top row is $\kappa_{12}(k)$ acting on S^m, the bottom row is $\rho_m \kappa_{12}(k)$. So $\rho_m \kappa_{12}(k) = \sigma^m k$.

<u>§ 2. Determination of Some Künneth Isomorphisms.</u>

The presentation and results of this section owe a great deal to a paper of N. Palermo [39].

First we give an "additivity lemma".

(2.1) <u>Lemma 2.1.</u> (i) Let $A, L_i, B_i \in \mathcal{C}$ and let κ_i be a Künneth isomorphism of type $(L_i,\ A\ ;\ B_i)$ $(i = 1,2)$. Then

$$\kappa_1 + \kappa_2\ :\ H(K \otimes (L_1 + L_2),A) \approx H(K,\ B_1 + B_2)$$

is a Künneth isomorphism of type $(L_1 + L_2, A\ ;\ B_1 + B_2)$.

(ii) Let A_i, L, $B_i \in \mathcal{C}$ and let κ_i be a Kunneth isomorphism of type $(L, A_i; B_i)$ $(i = 1,2)$. Then

$$\kappa_1 + \kappa_2 \quad : H(K \otimes L, A_1 + A_2) \approx H(K, B_1 + B_2)$$

is a Kunneth isomorphism of type $(L, A_1 + A_2; B_1 + B_2)$.

The proof of the lemma is obvious.

(2.2) In discussing Kunneth isomorphisms of type $(L, A; B)$ explicitly, an obvious simplification is to suppose A, B have trivial differential, so that $B \approx H(L, A)$. If further A, L are finitely generated in each dimension, and L is free, then the "additivity lemma" 2.1 implies that it is sufficient to consider the case $A = Z_n (n \geqslant 0)$, $H(L) = \eta^q Z_t$ $(t \geqslant 0)$.

It will clearly be convenient to have a canonical system of generators and relations for $H(K \otimes L, Z_n)$. Such a system is given in [39]. To describe this, we need some notations.

<u>Notation 2.21.</u> [56, 39]

Let $X \in \mathcal{C}$ be a free complex.

If $x \in X \pitchfork Z$ is a cycle mod n, i.e. $\delta x = ny$ for some $y \in X \pitchfork Z$, then $(x)_n$ or x_n denotes the homology class of $h_n x \in K \pitchfork Z_n$, where $h_n x$ is the image of x under the map induced by the projection $Z \longrightarrow Z_n$. In particular, $(x)_o$ or x_o denote the homology class of a cycle x.

The Bockstein boundary δ_n is that associated with the exact sequence $0 \to Z \xrightarrow{n} Z \to Z_n \to 0$; i.e. $\delta_n x_n = (\frac{1}{n}\delta x)_o$, if x is a cycle mod n .

Let $m \geqslant 0$, $n > 0$ be integers. The coefficient homomorphism $h_{n,m}: H(X,Z_m) \longrightarrow H(X, Z_n)$ is defined by

$$h_{n,m}(x) = (\frac{n}{(n,m)} x)_o$$

for x a cycle mod m , where (n,m) is the HCF of n and m (if $m = 0$, then $(n,m) = n$) . The composition $h_{n,o}\delta_n$ is written $\delta_{n,n}$.

The maps $h_{n,m}$, δ_n satisfy the following relations [56]

$$h_{k,m}\, h_{m,n} = \frac{m(n,k)}{(m,k)(m,n)}\, h_{k,n} ,$$

$$\delta_m\, h_{m,n} = \frac{n}{(m,n)}\, \delta_n .$$

If $x \in X \pitchfork Z_n$, $y \in Y \pitchfork Z_n$ $(n \geqslant 0)$, then $x \times y \in (X \otimes Y) \pitchfork Z_n$ denotes the cartesian product (VI. 1.4) of x and y with respect to the ring pairing $Z_n \otimes Z_n \longrightarrow Z_n$. The cartesian product induces a pairing $\alpha_n : H(X, Z_n) \otimes H(Y,Z_n) \longrightarrow H(X \otimes Y, Z_n)$, and $\alpha_n(x_n \otimes y_n)$ is written $x_n \times y_n$.

The following two theorems are essentially theorems 3.1, 5.1 of [39].

<u>Theorem 2.22</u> (Palermo) If X,Y are finitely generated, free chain complexes, then $H(X \otimes Y, Z)$ is generated by elements of the form

$a_o \times x_o, \delta_d(a_d \times x_d)$ for d ranging over the integers (s,t), where s,t are torsion coefficients of X,Y respectively. Further, on $H(X \otimes Y, Z)$ the following relations hold

(i) $\delta_i(h_{i,j}\, a_j \times x_i) = \delta_j(a_j \times h_{j,i}\, x_i)$ $\quad i|j$ or $j|i$

(ii) $\delta_i\, a_i \times x_o = \delta_i\,(a_i \times h_{i,o}\, x_o)$ $\quad i > 0$

(iii) $(\omega a_o) \times \delta_i x_i = \delta_i\,(h_{i,o}\, a_o \times x_i)$ $\quad i > 0$

where ω is $(-1)^p$ in dimension p.

Theorem 2.23 (Palermo) Let X,Y be finitely generated free complexes, and let $n > 0$. Let c range over the integers $n, (n,s), (n,t), (n,s,t)$, and let d range over the integers (n,s,t), where s,t are torsion coefficients of X,Y respectively. Then $H(X \otimes Y, Z_n)$ is generated by elements of the form $h_{n,c}\,(a_c \times x_c)$, $\delta_{n,n}\, h_{n,d}\,(a_d \times x_d)$. Further in $H(X \otimes Y, Z_n)$ the following relations hold

(i) $h_{n,i}(h_{i,j}\, a_j \times x_i) = h_{n,j}\,(a_j \times h_{j,i} x_i)$ $\quad i,j|n\,;\ i,j > 0$

(ii) $\delta_{n,n}(a_n \times x_n) = (\delta_{n,n} a_n) \times x_n + (\omega a_n) \times \delta_{n,n}\, x_n$.

The relations given in 2.22, 2.23 are complete sets of relations, but we shall not need this fact.

(2.3) Let κ be a Künneth isomorphism of type $(L, A; B)$.
It is found, in describing κ on the generators of 2.22, 2.23, that the signs are more convenient (c.f. V. 1.61) if we take κ as mapping

$H(L\otimes K,A) \approx H(K,B)$, that is if we precede κ by the twist automorphism $T^{*}: H(L\otimes K,A) \approx H(K\otimes L,A)$ (VI.1.5). We shall accordingly, in this section, and in § 2,3 of Chapter XI, write κ for κT^{*}.

(2.4) Let L be free and finitely generated and let $H(L,Z) \approx \eta^{q}Z$ $(t > 0)$. If κ is a Kunneth isomorphism of type $(L,Z;\eta^{q}Z_t)$, then, for each free complex $K \in \mathcal{C}_0$ we have in dimension m

$$H^{m}(L\otimes K,Z) \xrightarrow{\kappa} H^{m}(K,\eta^{q}Z_t) = H^{m-q}(K,Z_t) .$$

Let $K \in \mathcal{C}_0$ be free and finitely generated.

Theorem 2.41 There is a Kunneth isomorphism of type $(L,Z;\eta^{q}Z_t)$ which on $H(L\otimes K, Z)$ is given by the formulae

(i) $\kappa(a_0 \times x_0) = x_t$,

(ii) $\kappa\,\delta_d(a_d \times x_d) = h_{t,d}\, x_d$.

The proof is tedious and is left to an Appendix.

(2.5) Let L be free and finitely generated and let $\vartheta : H(L,Z) \approx \eta^{q}Z_t$ $(t > 0)$. Let $n > 0$, and let $N = \eta^{q}Z_d + \eta^{q+1}Z_d$, where $d = (n,t)$; then $H(L,Z_n) \approx N$. If κ is a Kunneth isomorphism type $(L,Z_n;N)$, then for each free complex $K \in \mathcal{C}_0$ we have in dimension m the maps

$$\begin{array}{c} H^{m}(L\otimes K,Z_n) \\ \kappa \downarrow \approx \\ H^{m}(K, N) \\ \| \\ H^{m}(K,\eta^{q}Z_d) + H^{m}(K,\eta^{q+1}Z_d) \\ \| \\ H^{m-q}(K, Z_d) + H^{m-q-1}(K,Z_d) . \end{array}$$

The composite of these maps is also written κ .

Let $b_o \in H^q(L,Z)$ be such that $\vartheta(b_o) = 1 \in \eta^q Z_t$. Since $t > 0$, $\delta_t: H^{q+1}(L,Z_t) \approx H^q(L,Z)$; let a_t be the unique element such that $\delta_t a_t = b_o$.

<u>Theorem 2.51</u> Let α, β be integers such that $\alpha n + \beta t = (n,t) = d$. Let $K \in \mathcal{C}_o$ be free and finitely generated. There is a Kunneth isomorphism κ of type $(L, Z_n; N)$ which on $H(L \otimes K, Z_n)$ is given by

(i) $\kappa h_{n,i}(a_i \times x_i) = h_{d,i} x_i$,

(ii) $\kappa h_{n,i}(b_i \times x_i) = h_{d,n} h_{n,i} x_i + (-1)^q \beta h_{d,o} \delta_i x_i$,

(iii) $\kappa \delta_{n,n} h_{n,i}(a_i \times x_i) = h_{d,t} h_{t,i} x_i + (-1)^{q+1} \alpha h_{d,o} \delta_i x_i$.

We may also cover the case $t = 0$ by omitting (i) and (iii) above and taking $\beta = 0$ in (ii).

The proof is tedious and is left to an Appendix.

XI. k-INVARIANTS OF FUNCTION COMPLEXES (2).

§1. Determination of k^Y

Theorems A and B give the solutions to Problems 3.1, 3.2 of Chapter IV.

(1.1) Let $Y \in \mathcal{K}$, $A, B \in \mathcal{FD}$ and κ a Kunneth isomorphism of type $(Y, NA; NB)$ associated with $(F; f, g)$. We construct an equivalence $\lambda : A^Y \simeq B$ as follows : since $f : F \to C_N(Y) \pitchfork NA$, $g : F \to NB$ induce isomorphisms in homology, the maps $Rf : RF \to C(Y) \pitchfork A$, $Rg : RF \to B$ induce isomorphisms in homotopy and so are homotopy equivalences (but not necessarily FD-homotopy equivalences). So the composite

$$A^Y \xrightarrow{\hat{\Delta}} C(Y) \pitchfork A \xrightarrow{(Rf)'} RF \xrightarrow{(Rg)} B,$$

where we write ϕ' for a homotopy inverse of a map ϕ, is an equivalence $\lambda : A^Y \simeq B$. We say λ is associated with κ.

With these constructions given, we can now easily prove

Theorem A. Let $X \in \mathcal{K}$. There is a commutative diagram

$$\begin{array}{ccc} H^0(X, A) & \xrightarrow{\varepsilon^*} & H^0(X^Y \ast Y, A) \\ F^Y \downarrow & & \| \\ H^0(X^Y, A^Y) & & H^0(X^Y \ast Y, NA) \\ \lambda_* \downarrow \approx & & \approx \downarrow \kappa \\ H^0(X^Y, B) & = & H^0(X^Y, NB) . \end{array}$$

Proof. We consider the following diagram

$$\begin{array}{ccc}
H^0(X,A) & \xrightarrow{\ \varepsilon^*\ } & H^0(X^Y \divideontimes Y, A) \\
{\scriptstyle F^Y}\big\downarrow & & \big\| \\
H^0(X^Y, A^Y) & & H^0(X^Y \divideontimes Y, NA) \\
{\scriptstyle \hat{\Delta}_*}\big\downarrow{\scriptstyle \approx} & & {\scriptstyle \approx}\big\downarrow{\scriptstyle \theta} \\
H^0(X^Y, C(Y) \pitchfork A) & = & H^0(X^Y, C_N(Y) \pitchfork NA) \\
{\scriptstyle (Rf)_*}\big\uparrow & & {\scriptstyle \approx}\big\uparrow{\scriptstyle f_*} \\
H^0(X^Y, RF) & = & H^0(X^Y, F) \\
{\scriptstyle (Rg)_*}\big\downarrow & & {\scriptstyle \approx}\big\downarrow{\scriptstyle g_*} \\
H^0(X^Y, B) & = & H^0(X^Y, NB) .
\end{array}$$

The top rectangle is commutative by IX. 2.12, and the bottom squares are obviously commutative. The theorem follows immediately.

(1.2) Let $A_i, B_i \in \mathcal{FD}$, $Y \in \mathcal{X}$, and κ_i be a Kunneth isomorphism of type $(Y, A_i; B_i)$ with associated equivalence $\lambda_i : A_i^Y \simeq B_i$ $(i = 1,2)$.

A map $\kappa_{12} : H^0(A_1, A_2) \longrightarrow H^0(B_1, B_2)$ is defined in X.1.31.

__Theorem B.__ The following is a commutative diagram of homomorphisms

$$\begin{array}{ccc}
H^0(A_1, A_2) & \xrightarrow{\ F^Y\ } & H^0(A_1^Y, A_2^Y) \\
{\scriptstyle \kappa_{12}}\big\downarrow & & {\scriptstyle \approx}\big\downarrow{\scriptstyle \lambda_{2*}} \\
H^0(B_1, B_2) & \underset{\lambda_1^*}{\xrightarrow{\ \approx\ }} & H^0(A_1^Y, B_2)
\end{array}$$

__Proof.__ Let $k \in H^0(A_1, A_2)$. For any $X \in \mathcal{X}$, $\kappa_{12}(k)$ is defined as an operation by the following diagram, in which we assume κ_i is associated with $(F_i; f_i, g_i)$ $(i = 1,2)$.

$$
\begin{array}{ccc}
H^0(X \ast Y, A_1) & \xrightarrow{\quad k \quad} & H^0(X \ast Y, A_2) \\
\theta \downarrow \approx & & \approx \downarrow \theta \\
H^0(X, C_N(Y) \pitchfork NA_1) & \xrightarrow{\quad \phi \quad} & H^0(X, C_N(Y) \pitchfork NA_2) \\
f_{1*} \uparrow \approx & & \approx \uparrow f_{2*} \\
H^0(X, F_1) & & H^0(X, F_2) \\
g_{1*} \downarrow \approx & & \approx \downarrow g_{2*} \\
H^0(X, NB_1) & \xrightarrow[\kappa_{12}(k)]{} & H^0(X, NB_2)
\end{array}
$$

The map written ϕ is simply $G^Y(k)$ (IX. 2.3.).

In the following diagram

$$
\begin{array}{ccc}
A_1^Y & \xrightarrow{\quad F^Y(k) \quad} & A_2^Y \\
\hat{\Delta} \downarrow \simeq & & \simeq \downarrow \hat{\Delta} \\
C(Y) \pitchfork A_1 & \xrightarrow{\quad G^Y(k) \quad} & C(Y) \pitchfork A_2 \\
(Rf_1)' \downarrow \simeq & & \simeq \downarrow (Rf_2)' \\
RF_1 & & RF_2 \\
Rg_1 \downarrow \simeq & & \simeq \downarrow Rg_2 \\
B_1 & \xrightarrow[\kappa_{12}(k)]{} & B_2 ,
\end{array}
$$

the top square is homotopy commutative by IX.2.31, and the bottom rectangle is homotopy commutative by the definition of $\kappa_{12}(k)$. The theorem follows immediately from the definition of λ_1, λ_2.

§2. The evaluation class.

(2.1) In this section we are primarily concerned with computations using Theorem B; let us for the moment use the notation of this theorem.

In order to determine $\kappa_{12}(k) \in H^0(B_1, B_2)$, we calculate, using the definition of κ_{12}, the class $\kappa_{12}(k)\,\omega(B_1)$, where $\omega(B_1) \in H^0(B_1, B_1)$ is the fundamental class, that is, we determine the image of $\omega(B_1)$ under the maps

$$H^0(B_1,B_1) \xrightarrow[\approx]{\kappa_1^{-1}} H^0(B_1 \divideontimes Y, A_1) \xrightarrow{k} H^0(B^1 \divideontimes Y, A_2) \xrightarrow[\approx]{\kappa_2} H^0(B^1, B^2) \,.$$

Because of the following theorem, the class $\kappa_1^{-1}\omega(B_1)$ is called the *evaluation class*.

Theorem 2.11. Let $A, B \in \mathfrak{J}\mathcal{D}$, $Y \in \mathcal{K}$. Let κ be a Künneth isomorphism of type $(Y, A; B)$, and $\lambda : A^Y \simeq B$ the associated equivalence. The maps of the diagram

$$\begin{array}{ccc}
H^0(B \divideontimes Y, A) & \xrightarrow[\approx]{(\lambda \divideontimes 1)^*} & H^0(A^Y \divideontimes Y, A) \\
\kappa \downarrow \approx & & \uparrow \varepsilon^* \\
H^0(B, B) & & H^0(A, A)
\end{array}$$

satisfy $\varepsilon^*\omega(A) = (\lambda \divideontimes 1)^*\,\kappa^{-1}\,\omega(B)$.

Proof. Let $\lambda = \vartheta\hat{\Delta}$, where $\vartheta : C(Y) \pitchfork A \simeq B$. We consider the following commutative diagram, in which all maps but ε^* are isomorphisms.

$$\begin{array}{ccccc}
H^0(A,A) & & & & \\
\varepsilon^* \downarrow & & & & \\
H^0(A^Y \divideontimes Y, A) & \xleftarrow{(\hat{\Delta} \divideontimes 1)^*} & H^0((C(Y) \pitchfork A) \divideontimes Y, A) & \xleftarrow{(\vartheta \divideontimes 1)^*} & H^0(B \divideontimes Y, A) \\
\theta \downarrow & & \downarrow \theta & & \downarrow \theta \\
H^0(A^Y, C(Y) \pitchfork A) & \xleftarrow{\hat{\Delta}^*} & H^0(C(Y) \pitchfork A, C(Y) \pitchfork A) & \xleftarrow{\vartheta^*} & H^0(B, C(Y) \pitchfork A) \\
\vartheta_* \downarrow & & \downarrow \vartheta_* & & \downarrow \vartheta_* \\
H^0(A^Y, B) & \xleftarrow[\hat{\Delta}^*]{} & H^0(C(Y) \pitchfork A, B) & \xleftarrow[\vartheta^*]{} & H^0(B, B)
\end{array}$$

According to IX. 2.22 , $\theta\varepsilon^*\omega(A) = \hat{\Delta}^*\omega(G(Y)\wedge A)$. Therefore

$$(\lambda \mathbin{\#} 1)^* \kappa^{-1} \omega(B) = (\lambda \mathbin{\#} 1)^* \theta^{-1} \nu_*^{-1}\omega(B)$$
$$= \theta^{-1}\hat{\Delta}^*\omega(G(Y)\wedge A)$$
$$= \varepsilon^*\omega(A) .$$

Using the results of X. § 2, we now give explicit formulae for the evaluation class when $H(Y,Z) \approx \eta^a Z_t$. It is clear from X. Lemma 1.1 that from these formulae we may determine the evaluation class for all finite Y .

For reasons of signs, in this and the next section we take the evaluation class to be in $H^0(Y \mathbin{\#} B,A)$ (using the notation of 2.11), so that $\kappa : H^0(Y \mathbin{\#} B,A) \approx H^0(B,B)$.

One trivial point must be made; if π finitely generated, but not finite, then $C_N(K(\pi,m))$ is not finitely generated. However the results of X. § 2 still apply since there is a free finitely generated chain complex $A(\pi,m)$ which is naturally chain equivalent to $C_N(K(\pi,m))$ [18].

(2.2) Let $Y \in \mathcal{K}$ be finite, and let $\nu : H(Y,Z) \approx \eta^{-r}Z$ $(t \geq 0)$. Let κ be the Kunneth isomorphism of type $(Y,\eta^m Z; \eta^{m-r} Z_t)$, associated with $\eta^m\nu : H(Y,\eta^m Z) \approx \eta^{m-r} Z_t$, which is given by X.2.41

Let $b_0 \in H^{-r}(Y,Z)$ be such that $\nu b_0 = 1 \in Z_t$. If $t > 0$, then $\delta_t : H^{-r+1}(Y,Z_t) \approx H^{-r}(Y,Z)$, and we let $a_t \in H^{-r+1}(Y,Z_t)$ be the unique element such that $\delta_t a_t = b_0$. Let $\omega \in H^{-m+r}(K(Z_t,m-r),Z_t)$ be the fundamental class; here $K(Z_t, m-r) \approx K(Z,m)^Y$.

<u>Theorem 2.21</u> The evaluation class $\underline{\xi} \in H^{-m}(K(Z_t, m-r) \bar{\wedge} Y, Z)$ is given by

(i) $\underline{\xi} = b_0 \times \omega$, if $t = 0$

(ii) $\underline{\xi} = \delta_t\,(a_t \times \omega)$, if $t > 0$.

<u>Proof.</u> We have simply to find $\underline{\xi}$ such that $\kappa(\xi) = \omega$. Since ω is a class mod t , the theorem follows at once from X. 2.41.

(2.3) Let $Y \in \mathcal{H}$ be finite, and let $\partial : H(Y,Z) \approx \eta^{-r}Z$ $(t > 0)$ Let $n > 0$, $d = (n,t)$, $N = \eta^{m-r}Z_d + \eta^{m-r+1}Z$; then $H(Y, \eta^{m}Z_n) \approx N$. Let b_0, a_t be as in 2.2 (and as in X.2.5) and let κ be the Kunneth isomorphism of X. 2.51.

Now $K(Z_n, m)^Y \simeq K(Z_d, m-r) \times K(Z_d, m-r+1) = Q$ say. Let $p_1 : Q \longrightarrow K(Z_d, m-r)$, $p_2 : Q \rightarrow K(Z_d, m-r+1)$ be the projections, and let $\omega^{m-r} \in H^{r-m}(Q, Z_d)$, $\omega^{m-r+1} \in H^{r-m-1}(Q, Z_d)$ be the images of the fundamental classes under the maps p_1^*, p_2^* .

<u>Theorem 2.31</u> The evaluation class $\underline{\xi} \in H^{-m}(Y \bar{\wedge} Q, Z_n)$ is given by

$$\underline{\xi} = h_{n,d}\,(a_d \times \omega^{m-r+1}) + \alpha\, h_{n,d}\,(b_d \times \omega^{m-r}) + \beta\, \delta_{n,n} h_{n,d}(a_d \times \omega^{m-r}) .$$

<u>Proof.</u> By X. 2.51, since $\omega^{m-r}, \omega^{m-r+1}$ are mod d classes, the given class $\underline{\xi}$ satisfies

$$\kappa(\underline{\xi}) = \omega^{m-r+1} + \alpha\, h_{d,n}\, h_{n,d}\, \omega^{m-r} + \beta\, h_{d,t}\, h_{t,d}\, \omega^{m-r}$$

$$= \omega^{m-r+1} + \omega^{m-r} \quad , \text{ since } \frac{\alpha n}{(n,t)} + \frac{\beta t}{(n,t)} = 1 .$$

§3. Examples.

These examples are entirely applications of Theorem B, using the results of the last section. All of these examples seem to be new, except for Example 3.3, in which the case $n = 2$ has been obtained by other methods of F.P.Peterson (private communication).

Example 3.1 Let $k = Sq^n: K(Z_2,m) \longrightarrow K(Z_2,m+n)$, and let $Y = S^{r-1} \cup_2 e^r$ $(r < m)$. Then k^Y is given by the diagram

$$\begin{array}{ccccc} K(Z_2,m)^Y & \simeq & K(Z_2,m-r) & \times & K(Z_2,m-r+1) \\ k^Y \downarrow & & Sq^n \downarrow & \overset{Sq^{n-1}}{\swarrow} & \downarrow Sq^n \\ K(Z_2,m+n)^Y & \simeq & K(Z_2,m+n-r) & \times & K(Z_2,m+n-r+1) \end{array}$$

Proof. The calculations are covered by 2.31 and X. 2.51 with $n = t = d = 2$, $\alpha = 1$, $\beta = 0$. The evaluation class is

$$\underline{\xi} = a_2 \times \omega^{m-r+1} + b_2 \times \omega^{m-r} \; ;$$

and $Sq^n \underline{\xi} = Sq^1 a_2 \times Sq^{n-1} \omega^{m-r+1} + a_2 \times Sq^n \omega^{m-r+1} + b_2 \times Sq^n \omega^{m-r}$,

by the Cartan formula. But $Sq^1 a_2 = b_2$; so by X. 2.51

$$\kappa\, Sq^n \underline{\xi} = Sq^n \omega^{m-r+1} + Sq^n \omega^{m-r} + Sq^{n-1} \omega^{m-r+1} .$$

Example 3.2 Let $k = Sq^n: K(Z,m) \rightarrow K(Z_2,m+n)$, and let $Y = S^{r-1} \cup_2 e^r (r < m)$. Then k^Y is given by the diagram

$$\begin{array}{ccccc} K(Z,m)^Y & \simeq & & & K(Z_2,m-r) \\ k^Y \downarrow & & \overset{Sq^n + Sq^{n-1}Sq^1}{\swarrow} & & \downarrow Sq^n Sq^1 \\ K(Z_2,m+n) & \simeq & K(Z_2,m+n-r) & \times & K(Z_2,m+n-r+1) \end{array}$$

Proof. The evaluation class is given by 2.21 with $t = 2$; thus

$$\xi = \delta_2(a_2 \times \omega^{m-r}) .$$

$$\begin{aligned}
\text{So } Sq^n \xi &= Sq^n \delta_2 (a_2 \times \omega^{m-r}) \\
&= Sq^n Sq^1 (a_2 \times \omega^{m-r}) \\
&= Sq^n (Sq^1 a_2 \times \omega^{m-r} + a_2 \times Sq^1 \omega^{m-r}) \\
&= Sq^n (b_2 \times \omega^{m-r} + a_2 \times Sq^1 \omega^{m-r}) \\
&= b_2 \times Sq^n \omega^{m-r} + b_2 \times Sq^{n-1} Sq^1 \omega^{m-r} + a_2 \times Sq^n Sq^1 \omega^{m-r} .
\end{aligned}$$

By X. 2.51 , $k\, Sq^n \xi = (Sq^n Sq^1 + Sq^n + Sq^{n-1} Sq^1)\omega^{m-r}$.

Example 3.3. Let $k = \delta_2 Sq^n : K(Z,m) \longrightarrow K(Z,m+n+1)$, and let

$Y = S^{r-1} \cup_2 e^r$ $(r < m)$. Then k^Y is given by the diagram

$$\begin{array}{ccc}
K(Z,m)^Y & \simeq & K(Z_2, m-r) \\
\Big\downarrow k^Y & & \Big\downarrow (n+1)(Sq^{n+1} + Sq^n Sq^1) \\
K(Z,m+n)^Y & \simeq & K(Z_2, m+n-r+1)
\end{array}$$

Proof. As in 3.2, $\xi = \delta_2(a_2 \times \omega^{m-r})$,

$$\begin{aligned}
\text{and } \delta_2 Sq^n \xi &= \delta_2(a_2 \times Sq^n Sq^1 \omega^{m-r} + b_2 \times (Sq^n + Sq^{n-1} Sq^1)\omega^{m-r}) \\
&= \delta_2(a_2 \times Sq^n Sq^1 \omega^{m-r}) \pm b_2 \times \delta_2 (Sq^n + Sq^{n-1} Sq^1)\omega^{m-r} \text{ by X.2.22(iii).}
\end{aligned}$$

By X. 2.11 $k\,\delta_2 Sq^n \xi = Sq^n Sq^1 \omega^{m-r} + h_{2,0}\,\delta_2(Sq^n + Sq^{n-1}Sq^1)\omega^{m-r}$

$$= Sq^n Sq^1 \omega^{m-r} + Sq^1(Sq^n + Sq^{n-1}Sq^1)\omega^{m-r}$$

$$= Sq^n Sq^1 \omega^{m-r} + (n+1)\, Sq^{n+1}\omega^{m-r} + n\, Sq^n Sq^1 \omega^{m-r}$$

$$= (n+1)\,(Sq^{n+1} + Sq^n Sq^1)\,\omega^{m-r} .$$

Example 3.4 Let $k = (\omega^m)^2 : K(Z,m) \longrightarrow K(Z,2m)$, and let $Y = S^2 \cup e^4 \cup e^6$ be complex projective 3-space. Then k^Y is given by the diagram

$$\begin{array}{ccccccc} K(Z,m)^Y & \simeq & K(Z,m-2) & \times & K(Z,m-4) & \times & K(Z,m-6) \\ k^Y \downarrow & & & & \downarrow (\omega^{m-2})^2 + \gamma\, \omega^{m-2} \circ \omega^{m-4} & & \\ K(Z,2m)^Y & \simeq & K(Z,2m-2) & \times & K(Z,2m-4) & \times & K(Z,2m-6) \end{array}$$

where $\gamma = 0$ or 2 according as m is odd or even.

Proof. Let a_i generate $H^i(Y,Z)$ for $i = -2, -4, -6$. Then the evaluation class $\xi = a_2 \times \omega^{m-2} + a_4 \times \omega^{m-4} + a_6 \times \omega^{m-6}$. So

$$k\,\xi = (\xi)^2$$

$$= a_4 \times (\omega^{m-2})^2 + a_6 \times (\omega^{m-2} \circ \omega^{m-4}) + a_6 \times (\omega^{m-4} \circ \omega^{m-2})$$

$$= a_4 \times (\omega^{m-2})^2 + \gamma\, a_6 \times (\omega^{m-2} \circ \omega^{m-4}) .$$

§ 4. The non-base point case.

(4.1) The preceding theory runs perfectly smoothly if we are in the css-category without base points, except on questions of suspensions. Also VII 5.22 is false. The evaluation class is slightly more complicated, but the extra terms added are always cartesian products.

The major points of translation are: (a) for $\divideontimes$ read $\times$ (b) $C(X)$ was defined in VII 1.2 to be $B(X)/B(*)$; we must now define $C(X) = B(X)$ (c) homotopy in the css-category now means free homotopy, and $[X,Y]$ means the set of free homotopy classes of maps $X \to Y$.

(4.2) The calculations of the previous section are not much changed in the non-base-point case provided we keep to additive operations. One such example was given as IV. 36. We give one example for a non-additive operation.

Example 4.21 Let $k = (\omega^m)^2 : K(Z,m) \longrightarrow K(Z, 2m)$, and let $Y = S^2 \cup e^4$ be the complex projective plane. Then k^Y is given by the diagram

$$\begin{array}{ccc} K(Z,m)^Y & \simeq & K(Z,m) \times K(Z,m-2) \times K(Z, m-4) \\ k^Y \downarrow & & k' \downarrow \\ K(Z,2m)^Y & \simeq & K(Z,2m) \times K(Z,2m-2) \times K(Z,2m-4) \end{array}$$

where $k' = (\omega^m)^2 + (\omega^{m-2})^2 + \gamma \omega^m . (\omega^{m-2} + \omega^{m-4})$, $\gamma = \begin{cases} 0 & m \text{ odd} \\ 2 & m \text{ even} \end{cases}$

Proof. This follows si ply from

$$\begin{aligned} k\,\xi &= k(a_0 \times \omega^m + a_2 \times \omega^{m-2} + a_4 \times \omega^{m-4}) \\ &= (a_0 \times \omega^m + a_2 \times \omega^{m-2} + a_4 \times \omega^{m-4})^2 \\ &= a_0 \times (\omega^m)^2 + a_4 \times (\omega^{m-2})^2 + \gamma\, a_2 \times (\omega^m . \omega^{m-2}) + a_4 \times (\omega^m . \omega^{m-4}) \end{aligned}$$

Remark 4.22 A result similar to that of 4.21, but for real cohomology and for $m = 6$, has been given by Thom in [51]. However there is a gap in his argument which seems difficult to fill except by the sort of method we have given here.

We say no more on the non-base point case.

XII. Homotopy Groups of Fibre Spaces and Track Groups.

In this chapter we shall apply the previous theory to obtain M.G.Barratt's results on track groups (c.f. Chapter III) except for the description of the extension in the low dimensional case.* This will also illustrate how other applications may be made.

It is clear from Chapter IV that we shall need to determine the homotopy groups of a fibre space whose fibre is a css-abelian group. This involves describing the homotopy transgression of the fibre space and then finding the extensions involved. The general procedure is due to G.W. Whitehead in an American Mathematical Society Notice [54]; as further details have not appeared, we describe the procedure in § 1,3. It is convenient at one point to use a generalisation of the Moore-Postnikov system of a fibre map due to M.G.Barratt (unpublished). This generalisation is of independent interest, and I am grateful to Dr. Barratt for permission to give his results here (c.f. § 2).

§1. The homotopy transgression.

(1.1) We recall that there is a canonical identification, for any $A \in \mathfrak{J}\mathcal{D}$, I $: \pi_*(A) \approx H_*(NA)$ [37]. The identity $1 : A \to A$ induces $\iota = D(1) : C(A) \longrightarrow A$ (VII. 2.21), and so $\iota' = (ND1)_* : H_*(C_N(A)) \longrightarrow H_*(NA)$. Let $\omega : \pi_*(A) \longrightarrow H_*(A) = H_*(C_N(A))$ be the Hurewicz map. The following proposition is easily verified.

* I have not yet completed the calculations in this case.

Proposition 1.11 The following diagram is commutative

$$\begin{array}{ccc} & \pi_*(A) & \\ \omega \swarrow & & \searrow I \ (\approx) \\ H_*(A) & \xrightarrow{\iota'} & H_*(NA) \end{array}$$

An immediate corollary is the following proposition.

Proposition 1.12 Let $X \in \mathfrak{X}$, $A \in \mathfrak{F}\mathfrak{D}$ and $k : X \longrightarrow A$. Then the following diagram is commutative

$$\begin{array}{ccc} \pi_*(X) & \xrightarrow{k_*} & \pi_*(A) \\ \omega \big\downarrow & & \big\downarrow I \ (\approx) \\ H_*(X) & \xrightarrow{k'} & H_*(NA) \end{array}$$

where $k' = (NDk)_*$.

In particular, let $A = K(\pi, m)$, so that $NA = \eta^m \pi$, and let $k : X \longrightarrow A$ be regarded as a cohomology class in $H^{-m}(X, \pi)$. Then the map sending $k \longrightarrow k'$ is the projection $H^{-m}(X, \pi) \longrightarrow \mathrm{Hom}\,(H_m(X), \pi)$ of the Universal Coefficient Theorem. So 1.12 implies the well-known proposition :

Proposition 1.13 Let $X \in \mathfrak{X}$ and $k : X \longrightarrow K(\pi, m)$ a map. Then $k_* : \pi_m(X) \longrightarrow \pi_m(K(\pi, m)) = \pi$ is the composition

$$\pi_m(X) \xrightarrow{\omega} H_m(X) \xrightarrow{k'} \pi .$$

(1.2) These propositions apply immediately to give the homotopy transgression Δ of a bundle $A \longrightarrow E \longrightarrow X$ induced by a map $k : X \longrightarrow \overline{W}A$. ($A \in \mathfrak{F}\mathfrak{D}$, $X \in \mathfrak{X}$). For the transgression of the bundle $A \longrightarrow WA \longrightarrow \overline{W}A$ is an isomorphism $\pi_*(\overline{W}A) \approx \eta\, \pi_*(A)$, and Δ is the composition

$$\pi_*(X) \xrightarrow{k_*} \pi_*(\overline{W}A) \xrightarrow[\approx]{} \eta\, \pi_*(A) .$$

§2. The Barratt-Moore Postnikov System of a fibre map.

The results of this section are due to M.G.Barratt.

(2.1) Definition 2.1. Let $E, B \in \mathcal{K}$ and let $p : E \longrightarrow B$ be a map. A css-equivalence relation $\overset{m,n}{\sim}$ is defined in E as follows : if $x,y \in E_q$, then $x \overset{m,n}{\sim} y \Longleftrightarrow x,y$ have the same m-sections and px, py have the same n-sections.

Let $E^{m,n} = E/\overset{m,n}{\sim}$; if it is necessary that the map p should be referred to, we write ${}_pE^{m,n}$ rather than $E^{m,n}$. We write $\tilde{p}$ for any projection $E \longrightarrow E^{m,n}$.

If $x,y \in E_q$, then $x \overset{m',n'}{\sim} y \Longrightarrow x \overset{m,n}{\sim} y$ for any $0 \leqslant m \leqslant m'$, $0 \leqslant n \leqslant n'$. So if $m \leqslant m'$, $n \leqslant n'$ there are canonical projections $\tilde{p} : E^{m',n'} \longrightarrow E^{m,n}$. The whole collection of these projections for $0 \leqslant m \leqslant m' \leqslant \infty$, $0 \leqslant n \leqslant n' \leqslant \infty$ is called the Barratt-Moore Postnikov System of the map p.

If X is a css-complex, let $p : X \longrightarrow *$ be the unique map. The Moore-Postnikov system of X [37] consists of the complexes $X^{(m)} = {}_pX^{m,m}$ and the projections $X^{(m')} \longrightarrow X^{(m)}$ $(0 \leqslant m \leqslant m' \leqslant \infty)$.

If $p : E \rightarrow B$ is a css-map, then the system of projections $E^{m',\infty} \longrightarrow E^{m,\infty}$ $(0 \leqslant m \leqslant m' \leqslant \infty)$ constitutes the Moore-Postnikov system of the map p [38].

(2.2.) The following theorem may be proved by the methods of [37,38].

Theorem 2.2 Let $F \longrightarrow E \xrightarrow{p} B$ be a css-fibration with p onto and E, B Kan. Let $\Delta_m : \pi_m(B) \longrightarrow \pi_{m-1}(F)$ be the homotopy transgression of the fibration.

(2.21) The projections $E^{m',n'} \longrightarrow E^{m,n}$ $(0 \leqslant m \leqslant m' \leqslant \infty,\ 0 \leqslant n \leqslant n' \leqslant \infty)$ are fibre maps.

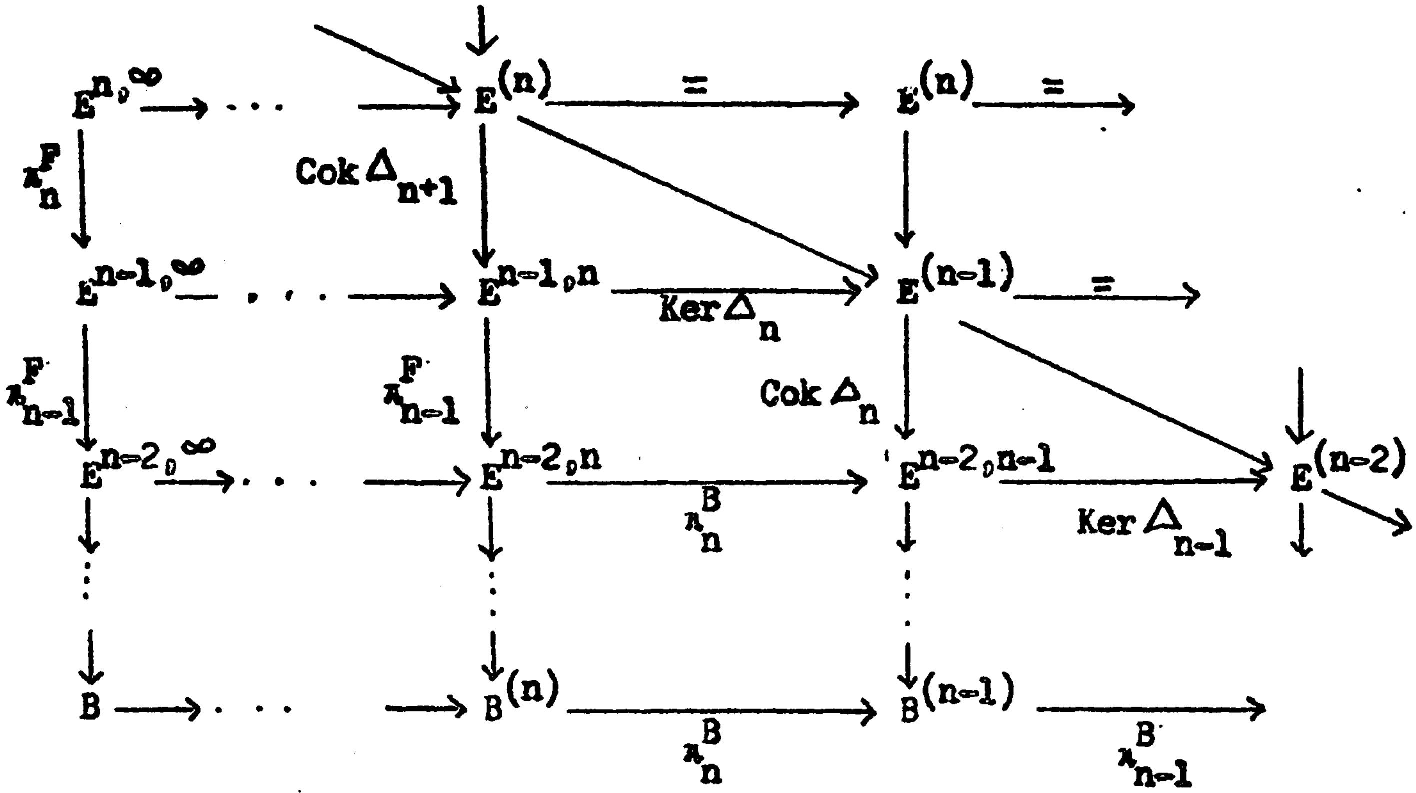
(2.22) If $m < n$, the fibre of $E^{m,n} \longrightarrow E^{m-1,n}$ is of type $(\pi_m(F), m)$, while the fibre of $E^{n,n} \longrightarrow E^{n-1,n}$ is of type $(\mathrm{Cok}\,\Delta_{n+1}, n)$.

(2.23) If $m < n-1$, the fibre of $E^{m,n} \longrightarrow E^{m,n-1}$ is of type $(\pi_m(B), m)$; the fibre of $E^{n-1,n} \longrightarrow E^{n-1,n-1}$ is of type $(\mathrm{Ker}\,\Delta_n, n)$; the fibre of $E^{n,n} \longrightarrow E^{n,n-k}$ $(k > 0)$ is trivial.

(2.24) There are natural identifications $E^{0,n} = B^{(n)}$, $E^{n,n} = E^{(n)}$. The set of maps $E^{n',n'} \longrightarrow E^{n,n}$ $(0 \leqslant n \leqslant n' \leqslant \infty)$ constitutes the Moore-Postnikov system of E. The set of maps $E^{0,n'} \longrightarrow E^{0,n}$ $(0 \leqslant n \leqslant n' \leqslant \infty)$ constitutes the Moore-Postnikov system of B.

(2.25) If $p : E \longrightarrow B$ is a minimal fibre map, then so is $\tilde{p} : E^{m',n'} \longrightarrow E^{m,n}$ $(0 \leqslant m \leqslant m' \leqslant \infty,\ 0 \leqslant n \leqslant n' \leqslant \infty)$.

This theorem is conveniently represented by the following diagram.



Here $\pi_r^X = \pi_r(X)$, and we have written in the non-zero homotopy groups of the fibre of each map rather than the fibre itself.

What this theorem gives is a method of building up the fibre space E by putting in the homotopy groups of the fibre and base one at a time.

(2.3) We shall wish to apply these results to principal bundles ([5], [10; Exposé 1, §4]).

By a map $\Phi: \underset{\sim}{B} \to \underset{\sim}{B}'$ of principal bundles $\underset{\sim}{B}: \Gamma \to E \to B$, $\underset{\sim}{B}': \Gamma' \to E' \to B'$ is meant a commutative diagram of maps

$$\begin{array}{ccccc} \Gamma & \longrightarrow & E & \longrightarrow & B \\ \varphi\downarrow & & \psi\downarrow & & \downarrow\chi \\ \Gamma' & \longrightarrow & E' & \longrightarrow & B' \end{array}$$

where φ is a css-homomorphism, and, if Γ acts on E' through φ , then ψ is a Γ -map ; i.e. the following diagram commutes

$$\begin{array}{ccc} \Gamma \times E & \xrightarrow{\ \pi\ } & E \\ \varphi\times\psi\downarrow & & \downarrow\psi \\ \Gamma' \times E' & \xrightarrow{\ \pi'\ } & E' \end{array} ,$$

where π, π' are the maps giving the actions of Γ, Γ' on E, E' respectively. Let $k : B \to \bar{W}\Gamma$, $k' : B' \to \bar{W}\Gamma'$ be classifying maps for B, B'. Let $\{k\}$ denote the homotopy class of a map k. The following proposition is readily verified.

<u>Proposition 2.31</u> If $\Phi : \underset{\sim}{B} \to \underset{\sim}{B}'$ is as above, then $\chi^*\{k\} = (\bar{W}\varphi)_*\{k'\}$.

Let $\underset{\sim}{B} : \Gamma \longrightarrow E \xrightarrow{p} B$ be a principal bundle, and let $\pi : \Gamma \times E \longrightarrow E$ give the action of Γ on E. Now $\Gamma^{(n)}$ is a group: for $(\Gamma \times \Gamma)^{(n)} = \Gamma^{(n)} \times \Gamma^{(n)}$, and so the group multiplication determines a group multiplication $\Gamma^{(n)} \times \Gamma^{(n)} \longrightarrow \Gamma^{(n)}$. Similarly the action of Γ on E determines an action of $\Gamma^{(n)}$ on $E^{(n)}$. Let $\Gamma^{\langle n\rangle}$ be the quotient of $\Gamma^{(n)}$ by the subgroup of $\Gamma^{(n)}$ acting trivially on $E^{(n)}$.

<u>Proposition 2.32</u> The action of $\Gamma^{\langle n\rangle}$ on $E^{(n)}$ determines a principal bundle. Also $E^{(n)}/\Gamma^{\langle n\rangle}$ is naturally isomorphic to $B^{(n)}$ in such a way that the projection $E^{(n)} \longrightarrow E^{(n)}/\Gamma^{\langle n\rangle}$ corresponds to the projection $E^{(n)} \longrightarrow B^{(n)}$ of the Barratt-Moore-Postnikov system of E.

Proposition 2.32 implies that we have a map of bundles

$$\begin{array}{ccccc} \Gamma & \longrightarrow & E & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma^{\langle n\rangle} & \longrightarrow & E^{(n)} & \longrightarrow & B^{(n)} \end{array} \qquad (2.33)$$

Let $k^{\langle n\rangle} : B^{(n)} \longrightarrow \bar{W}\Gamma^{\langle n\rangle}$ be a classifying map for the bundle $\underset{\sim}{B}^{(n)} : \Gamma^{\langle n\rangle} \longrightarrow E^{(n)} \longrightarrow B^{(n)}$. Then by 2.31, there is a diagram commutative up to homotopy

$$\begin{array}{ccc} B & \xrightarrow{k} & \bar{W}\Gamma \\ \downarrow & & \downarrow \\ B^{(n)} & \xrightarrow[k^{\langle n\rangle}]{} & \bar{W}\Gamma^{\langle n\rangle} \end{array} \qquad (2.34)$$

in which the vertical maps are the natural projections.

§3. Determination of certain extensions.

(3.1) Let $\underset{\sim}{B} : \Gamma \longrightarrow E \longrightarrow B$ be a principal bundle, with classifying map $k : B \longrightarrow \bar{W}\Gamma$. The homotopy sequence of B breaks up into short exact sequences

$$\underset{\sim}{B}_n \; : \; 0 \rightarrow \mathrm{Cok}\,\Delta_{n+1} \longrightarrow \pi_n(E) \longrightarrow \mathrm{Ker}\,\Delta_n \longrightarrow 0$$

where $\Delta_n : \pi_n(B) \longrightarrow \pi_{n-1}(\Gamma)$ is the homotopy transgression, which has been described in terms of k for abelian Γ in §1. We shall show how the extension $\underset{\sim}{B}_n$ may also, for abelian Γ, be described in terms of k.

It is clear from (2.33) that the extension $\underset{\sim}{B}_n$ is also derived from the fibration $\underset{\sim}{B}^{(n)}$ induced by $k^{\langle n\rangle} : B^{(n)} \longrightarrow \bar{W}\Gamma^{\langle n\rangle}$. Let us assume B is minimal. Then the fibre of the projection $B^{(n)} \longrightarrow B^{(n-1)}$ is a complex $K(\pi_n(B),n)$, and so there is a natural inclusion $i : K(\pi_n(B),n) \longrightarrow B^{(n)}$.

There is also a natural inclusion $i' : K(\mathrm{Ker}\,\Delta_n, n) \longrightarrow K(\pi_n(B),n)$. Let $\Gamma^{\langle n\rangle} \longrightarrow E' \longrightarrow K(\mathrm{Ker}\,\Delta_n, n)$ be the bundle induced by $k^{\langle n\rangle}\, i\, i'$.

Suppose now Γ is abelian, so that $\Gamma^{\langle n\rangle}$ is abelian. There is a map $h : \bar{W}\Gamma^{\langle n\rangle} \longrightarrow K(\mathrm{Cok}\,\Delta_{n+1}, n+1)$ inducing an isomorphism of π_{n+1}. Let $K(\mathrm{Cok}\,\Delta_{n+1}, n+1) \longrightarrow E'' \longrightarrow K(\mathrm{Ker}\,\Delta_n, n)$ be the bundle induced by $h\, k^{\langle n\rangle}\, i\, i'$.

There are maps of bundles

$$\begin{array}{ccccc}
\Gamma^{\langle n\rangle} & \longrightarrow & E^{(n)} & \longrightarrow & B^{(n)} \\
\| & & \uparrow & & \uparrow\, ii' \\
\Gamma^{\langle n\rangle} & \longrightarrow & E' & \longrightarrow & K(\mathrm{Ker}\Delta_n, n) \\
h \downarrow & & \downarrow & & \| \\
K(\mathrm{Cok}\Delta_{n+1}, n+1) & \rightarrow & E'' & \longrightarrow & K(\mathrm{Ker}\Delta_n, n)
\end{array}$$

A check of the maps of homotopy exact sequences shows that we have maps of extensions

$$\begin{array}{ccccc}
 & \nearrow & \pi_n(E) & \searrow & \\
0 \rightarrow \mathrm{Cok}\,\Delta_{n+1} & \longrightarrow & \pi_n(E')\ \ (\uparrow,\downarrow) & \longrightarrow & \mathrm{Ker}\,\Delta_n \rightarrow 0 \\
 & \searrow & \pi_n(E'') & \nearrow &
\end{array}$$

So it suffices to determine the extension for the bundle $E'' \rightarrow K(\mathrm{Ker}\Delta_n, n)$. The extension in this case is given by classical theorems, (3.21, 3.22).

Before giving these theorems, we note one useful fact. The map $k^{\langle n\rangle}\ i : K(\pi_n(B), n) \longrightarrow \overline{W}\,\Gamma^{\langle n\rangle}$ induces a bundle over $K(\pi_n(B), n)$, and it is obvious that the homotopy transgression of this bundle $\pi_n(B) \rightarrow \pi_{n-1}(\Gamma^{\langle n\rangle})$ is the same as Δ_n . Hence we have

Proposition 3.11 The homotopy transgression Δ_n is determined by $k^{\langle n\rangle} \mid K(\pi_n(B), n)$.

(3.20) Let $k : K(A,n) \longrightarrow K(B,n+1)$ $(n \geq 1)$ correspond to an element $\underline{k} \in H^{n+1}(K(A,n), B)$. Let $K(B,n) \longrightarrow E \longrightarrow K(A,n)$ be the principal bundle

induced by k and let $G = \pi_n(B)$, so that G is an extension
$\underset{\sim}{G} : 0 \to B \to G \to A \to 0$.

When $n = 1$, G and A may be non-abelian. However, the bundle is principal, so that A operates trivially on B , and hence the extension $\underset{\sim}{G}$ is a central extension.

Theorem 3.21 (Eilenberg - Maclane). The equivalence classes of central extensions of B by A are in 1 - 1 correspondence with $H^{-2}(K(A,1),B)$ in such a way that the class of the extension $\underset{\sim}{G}$ corresponds to $\underline{k}$.

Let $n > 1$. The Universal Coefficient Theorem and the Hurewicz theorem imply that there are isomorphisms

$$H^{-n-1}(K(A,n),B) \approx \mathrm{Ext}(H_n(K(A,n)),B) \approx \mathrm{Ext}(A,B)$$

The group Ext(A,B) is naturally isomorphic to the group Extabel (A,B) of equivalence classes of abelian extensions of B by A. So we have a natural 1 - 1 correspondence between Extabel (A,B) and $H^{-n-1}(K(A,n),B)$.

Theorem 3.22 (G.W.Whitehead; [54]) . Under the above correspondence, the class of the extension $\underset{\sim}{G}$ corresponds to $\underline{k}$.

§4. Function complexes of bundles.

(4.1) Let $X \in \mathfrak{X}$, $A \in \mathfrak{JD}$ and $k : X \longrightarrow \bar{W}A$ a map inducing a bundle $A \to E \xrightarrow{p} X$. Let $Y \in \mathfrak{X}$ and let $X' \subset X^Y$ be the image of $p^Y: E^Y \longrightarrow X$. The following is the css-analogue of a proposition well-known for topological spaces.

Proposition 4.11. E^Y is a principal bundle over X' with fibre A^Y .

Proof The action $A \times E \longrightarrow E$ of A on E determines an action $A^Y \times E^Y \longrightarrow E^Y$ of A^Y on E^Y . Obviously X' is the orbit space of E^Y

under this action. The only extra condition necessary ([10;p.1.10]) is that if $f \in A^Y$, $x \in E^Y$, then $f.x = x$ implies $f = 0$. Since this condition holds for the action of A on E, it obviously holds for the action of A^Y on E^Y.

From the bundle $A \longrightarrow WA \longrightarrow \overline{W}A$ we obtain a bundle $A^Y \longrightarrow (WA)^Y \longrightarrow V$, where V is the image of $w^Y: (WA)^Y \longrightarrow ((\overline{W}A)^Y)$. Since $(WA)^Y$ is contractible, V is simply the identity component of $((\overline{W}A)^Y)$. Clearly we have a map of bundles

$$\begin{array}{ccccc} A^Y & \longrightarrow & E^Y & \longrightarrow & X' \\ \downarrow & & \downarrow & & \downarrow k' \\ A^Y & \longrightarrow & (WA)^Y & \longrightarrow & V \end{array}$$

where $k' = k^Y | X'$.

<u>Proposition 4.12</u> $E^Y \longrightarrow X'$ is the bundle induced from $(WA)^Y \longrightarrow V$ by k'.

<u>Proof</u> We may represent E as the subcomplex of the product $X \times WA$ consisting of elements $(x, \underline{a})$ such that $kx = w\underline{a}$. Hence E^Y is the subcomplex of the product $X' \times (WA)^Y$ consisting of elements (f,h) such that $k^Y(f) = w^Y(h)$.

(4.2) We now wish to replace A^Y by a minimal complex. Let $T = R\pi_*(V)$, and let $f : V \simeq T$ be a homotopy equivalence. Then T is connected and $T \approx \overline{W}U$, where $U = R\pi_*(A^Y) = R\eta^- \pi_*(T)$. The map $fk' : X' \longrightarrow T$ induces a bundle $U \to E' \to X'$, and the map $E' \longrightarrow X'$ is of the same homotopy type as the map $E^Y \longrightarrow X'$.

To determine the homotopy type of E' it is necessary to determine X' and $fk' : X' \longrightarrow \overline{W}U$. The elements of $(X')_q$ are those maps

$g: \Delta * Y \longrightarrow X$ such that $kg \simeq *$; so the determination of X' is essentially an obstruction problem. The theory of the previous chapters enables us to choose f so that fk' may be evaluated.

In § 5 we shall restrict attention to the component X^* of the base point of X^Y ; clearly $X^* \subset X'$.

(4.3) Let now $X = B \in \mathcal{J}\mathcal{D}$. Then we may choose equivalences $f : V \simeq T$, and an equivalence $f' : R\pi_*(B^Y) \simeq B^Y$, such that $fk\, f' = G^Y(k) : R\pi_*(B\ ) \to T = R\eta\pi_*(A^Y)$, where G^Y is defined in Chapter X. By X. 1.44, the induced map of homotopy $G^Y(k)_*: \pi_i(B^Y) \longrightarrow \pi_{i-1}(A^Y)$ is the i^{th} suspension, $\sigma^{-i}k$, of the operation k . (This result is due to Thom. [51] when B, A are Eilenberg-Maclane complexes).

The extensions which give $\pi_i(E)$ are determined by the operations

$$p_{i,i+1}G^Y(k) \in \mathcal{O}(H^i(Y,A),\ H^{i+1}(Y,\overline{W}B)) = \mathcal{O}(H^0(Y,\eta^{-i}NA),\ H^0(Y,\eta^{-i}NB)) .$$

We have no general formula for these operations, although they may be determined in particular cases.

§ 5. Applications to track groups.

(5.1) We now consider the problem of determining the track group $\pi_1^Y(X) = \pi_1(X^Y,*)$, when Y is an A_2^n -complex.

We proceed by induction on the Postnikov system of X . Clearly $K(\pi,r)^Y = *$ for $r < n$, and $K(\pi,n)^Y$ is a set of points. So if $X_{(n)}$ is the fibre of the projection $X \to X^{(n)}$ of the Moore-Postnikov system of X , the injection $X_{(n)} \longrightarrow X$ induces a map $(X_{(n)})^Y \longrightarrow X^Y$

which is a homotopy equivalence of components of the trivial maps. So we may suppose from the start that X is n-connected.

Let X have Postnikov system

$$\mathcal{P}(\pi_{n+1}, n+1; \pi_{n+2}, n+2, k; \pi_{n+3}, n+3, \ell; \dots).$$

Here k, ℓ are maps $k : K(\pi_{n+1}, n+1) \to K(\pi_{n+2}, n+3)$, $\ell : X^1 \to K(\pi_{n+3}, n+4)$, where $X^1 = X^{(n+2)}$. Let us write $H^r(s)$ for $H^{-r}(Y, \pi_s)$. Then k^Y is given by a diagram

$$(5.11)\qquad \begin{array}{ccc} K(\pi_{n+1}, n+1)^Y \simeq & & K(H^n(n+1), 1) \\ \downarrow k^Y & & k_3 \swarrow \quad \downarrow k_2 \quad \searrow k_1 \\ K(\pi_{n+2}, n+3)^Y & \simeq & K(H^n(n+2), 3) \times K(H^{n+1}(n+2), 2) \times K(H^{n+2}(n+2), 1) \end{array}$$

where in the right-hand side we have written in only the identity component of the complexes.

The maps $k_i\ (i = 1,2,3)$ determine a bundle $A \to E' \to K(H^n(n+1), 1)$ with homotopy sequence

$$0 \to H^n(n+2) \to \pi_2(E') \to 0 \to H^{n+1}(n+2) \to \pi_1(E') \to H^n(n+1) \xrightarrow{\Delta} H^{n+2}(n+2) \to \pi_0(E') \to 0 .$$

By 4.3, the transgression Δ is $\sigma(k)$. Hence

(5.12) $\pi_2(E') = H^{-n}(Y, \pi_{n+2})$, and $\pi_1(E')$ is an extension

$$0 \to H^{-n-1}(Y, \pi_{n+2}) \longrightarrow \pi_1(E') \to \operatorname{Ker} \sigma(k) \to 0$$

(5.2) The map $\ell^Y : (X^1)^Y \to K(\pi_{n+3}, n+4)^Y$ is determined by a diagram

$$\begin{array}{ccc} (X')^Y \simeq & & E' \\ \ell^Y \downarrow & & \ell_4 \swarrow \quad \ell_3 \downarrow \quad \searrow \ell_2 \\ K(\pi_{n+3}, n+4)^Y & \simeq & K(H^n(n+3), 4) \times K(H^{n+1}(n+3), 3) \times K(H^{n+2}(n+3), 2) . \end{array}$$

The maps ℓ_i $(i = 2,3,4)$ induce a bundle $B \to E^2 \to E^1$ part of whose homotopy sequence is

$$\cdots \to \pi_2(E^2) \to H^n(n+2) \xrightarrow{\Delta'} H^{n+2}(n+3) \to \pi_1(E^2) \to \pi_1(E^1) \to 0 .$$

Let $h = \ell \,|\, K(\pi_{n+2}, n+2) : K(\pi_{n+2}, n+2) \to K(\pi_{n+3}, n+4)$, where $K(\pi_{n+2}, n+2)$ is, as the fibre of the projection $X^1 \to K(\pi_{n+1}, n+1)$, a subcomplex of X^1. The transgression Δ' is then given by the following proposition.

<u>Proposition 5.21</u> $\Delta' = \sigma^{-2}(h)$.

<u>Proof</u> By 3.11 the transgression Δ' is determined by the restriction $\ell_2 \,|\, K(H^n(n+2), 2)$. We consider the following diagram

$$\begin{array}{ccccc}
A \circledast Y & \xrightarrow{\lambda^1 \circledast 1} & K(\pi_{n+2}, n+2)^Y \circledast Y & \xrightarrow{\varepsilon^1} & K(\pi_{n+2}, n+2) \\
\downarrow{\scriptstyle i'' \circledast 1} & & \downarrow{\scriptstyle i' \circledast 1} & & \downarrow{\scriptstyle i} \\
E^1 \circledast Y & \xrightarrow{\lambda^2 \circledast 1} & (X^1)^Y \circledast Y & \xrightarrow{\varepsilon^2} & X^1 \\
\downarrow & & \downarrow & & \downarrow \\
K(H^n(n+1), 1) \circledast Y & \xrightarrow{\lambda^3 \circledast 1} & K(\pi_{n+1}, n+1)^Y \circledast Y & \xrightarrow{\varepsilon^3} & K(\pi_{n+1}, n+1)
\end{array} \qquad (*)$$

where ε^i is an evaluation map and λ^i a homotopy equivalence as given in Chapter XI. $(i = 1,2,3)$. Let i, i', i'' be the injections shown in (*) . Let κ be a Künneth isomorphism of type $(Y, \eta^{n+4}\pi_{n+3} \,;\, H^*(Y, \eta^{n+4}\pi_{n+3}))$.

The class we require is (by XI. Theorem A)

$$\begin{aligned}
i''^{*} \kappa (\lambda^2 \circledast 1)^* \varepsilon^{2*}(\ell) &= \kappa (i'' \circledast 1)^* \ (\lambda^2 \circledast 1)^* \ \varepsilon^{2*}(\ell) \quad \text{by naturality of } \kappa \\
&= \kappa (\lambda^1 \circledast 1)^* \ (i' \circledast 1)^* \ \varepsilon^{2*}(\ell) \\
&= \kappa (\lambda^1 \circledast 1)^* \varepsilon^{1*} \ i^* (\ell) \\
&= \kappa (\lambda^1 \circledast 1)^* \varepsilon^{1*} \ (h) .
\end{aligned}$$

Hence $\ell_2 | K(H^n(n+2),2)$ is the map of π_2 induced by h . So, by X. 1.44, $\Delta' = \sigma^{-2}(h)$.

Remark 5.22 Although we do not know how to determine $\ell_2, \ell_3, \ell_4,$ in general, in particular cases information may be obtained from the diagram (*) .

(5.3) The above results can be expressed when Y is finite in terms of squaring operations.

Let $\eta \in \pi_{n+2}(S^{n+1})$ be a non-zero element containing maps of Hopf invariant one if $n = 1$; the composition $\eta \circ \alpha$ defines for $n > 1$ a homomorphism $\gamma^*: \pi_{n+1}(X) \to \pi_{n+2}(X)$ such that $2\gamma^* = 0$, and for $n = 1$ it defines a transformation $\gamma^*: \pi_2(X) \longrightarrow \pi_3(X)$ such that

$$\gamma^*(\alpha+\beta) - \gamma^*(\alpha) - \gamma^*(\beta) = [\alpha,\beta] \qquad \alpha, \beta \in \pi_2(X) ,$$

where $[\alpha,\beta]$ is the Whitehead product of α and β (c.f. [59]). Thus γ^* defines homomorphisms

$$i_*: \pi_{n+1}(X) \otimes Z_2 \longrightarrow \pi_{n+2}(X) \qquad n > 1 ,$$

$$i_*: \Gamma(\pi_2(X)) \longrightarrow \pi_3(X) ,$$

where $\Gamma(G)$ is defined in [59] for any abelian group G .

It is well-known that there are factorisations

$$\begin{array}{ccc} K(\pi_{n+1}, n+1) & \xrightarrow{k} & K(\pi_{n+2}, n+3) \\ & {\scriptstyle Sq^2} \searrow \quad \nearrow {\scriptstyle i_{**}} & \\ & K(\pi_{n+1} \otimes Z_2, n+3) & \end{array} \qquad \begin{array}{ccc} K(\pi_2, 2) & \xrightarrow{k} & K(\pi_3, 4) \\ & {\scriptstyle \mathcal{P}_1} \searrow \quad \nearrow {\scriptstyle i_{**}} & \\ & K(\Gamma(\pi_2), 4) & \end{array}$$

where Sq^2, $\mathcal{P}_1$ are respectively the Steenrod and Pontrjagin squares. The first of these factorisations is essentially due to Steenrod [49a], the second to J.H.Whitehead [60].

If we use these determinations of k (and also of h) in the discussion of (5.1), (5.2) we obtain the following theorem, which is due to M.G.Barratt [3].

Theorem 5.31 (Barratt) If Y is an A_2^n-complex, then $\pi_1^Y(X)$ is given by a diagram of exact sequences

$$\begin{array}{ccccccc} & & & & & & 0 \\ & & & & & & \downarrow \\ & & & & & & H^{n+1}(n+2) \\ & & & & & & \downarrow \\ H^n(n+2) & \xrightarrow{i_*Sq^2} & H^{n+2}(n+3) & \longrightarrow & \pi_1^Y(X) & \longrightarrow & G \to 0 \\ & & & & & & \downarrow \\ & & & & & & H^n(n+1) \\ & & & & & & \downarrow k' \\ & & & & & & H^{n+2}(n+2) \end{array}$$

in which $H^r(s) = H^{-r}(Y,\pi_s(X))$, and

$$k' = \begin{cases} i_*Sq^2 & n>1 \\ i_*\mathfrak{P}_0 & n=1 \end{cases}$$

where $\mathfrak{P}_0 = \sigma(\mathfrak{P}_1)$ is the Postnikov square.

The extension giving G of 5.31 is determined by the map k_2 of 5.11. We have

Theorem 5.32 If $n>1$, Y is finite and π_{n+1}, π_{n+2} are finitely generated, then

(a) k_2 is the composition

$$K(H^{-1}(Y,\pi_{n+1}),1) \xrightarrow{Sq^1} K(H^{-n}(Y,\pi_{n+1}),2) \xrightarrow{(Sq^1)_*} K(H^{-n-1}(Y,\pi_{n+1}),2)$$
$$\xrightarrow{(\gamma'')_{**}} K(H^{-n-1}(Y,\pi_{n+2}),2)$$

(b) $k_3 = 0$.

The proof is left to an Appendix.

Remark 5.33 (a) The determination of the extension giving the group G of 5.31 in terms of Sq^1 is due to M.G.Barratt [3].

(b) The fact that $k_3 = 0$ in 5.32 is because k_3 is Sq^2 (with the correct pairing) and Sq^2 is zero on 1-dimensional classes. That k_3 is really Sq^2 will be clear from the proof of 5.32.

The calculations of k_2 and k_3 for the case $n = 1$ are more complicated and are not yet complete.

The main outstanding problem is now the description of the extension giving $\pi_1^Y(X)$. This description must involve secondary operations, both in cohomology and homotopy.

Appendix 1. Proof of V. 3.6

We recall the notions of [33]. The topological space $|\Delta^q|$ is the space of the standard q-simplex: some convention is made so that the spaces $|\Delta^q|$ $(q = 0,1,\ldots)$ are disjoint. Any css-map $\phi^* : \Delta^r \longrightarrow \Delta^q$ induces a continuous map $\phi^* : |\Delta^r| \longrightarrow |\Delta^q|$.

If $K \in \mathcal{K}$, a point of $|K|$ is an equivalence class $|k_q, x_q|$ of pairs (k_q, x_q) such that $k_q \in K_q$, $x_q \in |\Delta^q|$, the equivalence relation being

$$(\phi k_r, x_q) \sim (k_r, \phi^* x_q) \quad k_r \in K_r,\ x_q \in |\Delta^q|,\ \phi^* : \Delta^q \to \Delta^r$$

The homeomorphism $\eta : |K \times K'| \longrightarrow |K| \times_W |K'|$ is given by

$$\eta\,|k_q, k'_q, x_q| = (|k_q, x_q|, |k'_q, x_q|) \quad k_q \in K_q,\ k'_q \in K'_q,\ x_q \in |\Delta^q|.$$

The isomorphism $\Phi : \mathrm{Map}(|K|, X) \approx \mathrm{Map}(K, S(X))$ is given by

$$\Phi(f)(k_q)(x_q) = f\,|k_q, x_q|,\ f \in \mathrm{Map}(|K|, X),\ k_q \in K_q, x_q \in |\Delta^q|$$

Lemma The isomorphism $\lambda : S(|K| \pitchfork X) \approx S(X)^K$ is given by

$$\lambda(f)(\phi\,\delta^q, k_p)(x_p) = f(\phi^* x_p)(|k_p, x_p|) \qquad \begin{cases} f : |\Delta^q| \longrightarrow |K| \pitchfork X \\ \phi\,\delta^q \in (\Delta^q)_p, \\ k_p \in K_p,\ x_p \in |\Delta^p|. \end{cases}$$

Proof. In dimension q, λ is the composition

$$\mathrm{Map}(|\Delta^q|, |K| \pitchfork X) \xrightarrow{\mu^{-1}} \mathrm{Map}(|\Delta^q| \times_W |K|, X) \xrightarrow{\eta^*} \mathrm{Map}(|\Delta^q \times K|, X) \xrightarrow{\Phi} \mathrm{Map}(\Delta^q \times K, SX).$$

Therefore

$$\lambda(f)(\phi\delta^q, k_p)(x_p) = (\eta^* \mu^{-1} f)(|(\phi\delta^q, k_p), x_p|)$$
$$= (\mu^{-1} f)(|\phi\delta^q, x_p|, |k_p, x_p|)$$

$$= f(|\phi \delta^q, x_p|)(|k_p, x_p|)$$
$$= f(\phi^* x_p)(|k_p, x_p|) .$$

To prove V.36, let $f : |\Delta^q| \longrightarrow (|K| \times_W |L|) \pitchfork X$. Then, with the obvious notation

$$(\mu \lambda f)(\phi \delta^q, k_p)(\psi \delta^p, \ell_r)(x_r)$$
$$= (\lambda f)(\psi \phi \delta^q, k_p \ , \ell_r)(x_r)$$
$$= f(\phi^* \psi^* x_r)(|\psi k_p, x_r|, |\ell_r, x_r|)$$

On the other hand

$$\lambda^2 S(\mu)\ (f)\ (\phi \delta^q, k_p)(\psi \delta^p, \ell_r)(x_r)$$
$$= \{\lambda S(\mu)(f)(\phi \delta^q, k_p)\}(\psi^* x_r)(|\ell_r, x_r|)$$
$$= S(\mu)(f)(\phi^* \psi^* x_r)(|k_p, \psi^* x_r|)(|\ell_r, x_r|)$$
$$= f(\phi^* \psi^* x_r)(|k_p, \psi^* x_r|, |\ell_r, x_r|) .$$

Appendix 2; Proof of IX. 1.22

We recall that if $K \in \mathcal{C}$, then

$$(RK)_q = \mathcal{J}(N(q), K) \qquad q = 0, 1, \ldots$$

and the isomorphism

$$\Psi : N R K \longrightarrow S K$$

is given by $\Psi(f) = f(\delta^q) \qquad f \in (RK)_q$.

For any $k \in (SK)_q$, $\Psi^{-1}(k)$ is a map $N(q) \longrightarrow K$ which we write $\hat{k}$.

Thus $\hat{k}$ is characterised (as an element of $(NRK)_q$) by the equation

$$\hat{k}(\delta^q) = k .$$

With this notation, the isomorphism $\lambda^1 : A \pitchfork B \longrightarrow A \pitchfork' B$ $(A, B \in \mathcal{F}\mathcal{D})$ is given by

$$\lambda^1(f)(\phi\delta^q \otimes a) = N f\widehat{(\phi\delta^q \otimes a)}\ , \quad f \in (A \pitchfork B)_q, \quad \phi\delta^q \otimes a \in N(q) \otimes NA\ .$$

The isomorphism $\lambda^2 : A \pitchfork' B \longrightarrow A \pitchfork'' B$ is given by

$$\lambda^2(g)(\phi\delta^q)(a) = g(\phi\delta^q \otimes a) \quad g \in (A \pitchfork' B)_q, \quad \phi\delta^q \in N(q), \quad a \in NA\ .$$

Let maps $\mu^1, \mu^2, \mu^3, \mu^4$, be defined by the diagram

$$\begin{array}{ccc}
\mathcal{F}(A \otimes B, C) & \xrightarrow{\ \mu\ } & \mathcal{F}(A, B \pitchfork C) \\
\| & \searrow^{\mu^1} & \approx\downarrow\lambda^1 \\
\| & & \mathcal{F}(A, B \pitchfork' C) \\
\| & & \approx\downarrow\lambda^2 \\
\mathcal{F}(A \otimes B, C) & \xrightarrow{\ \mu^2\ } & \mathcal{F}(A, B \pitchfork'' C) \\
N\downarrow\approx & & \approx\downarrow N \\
\mathcal{F}(N(A \otimes B), NC) & \xrightarrow{\ \mu^3\ } & \mathcal{F}(NA, N(B \pitchfork'' C)) \\
\Psi^{\cdot}\uparrow\approx & & \approx\downarrow\Psi_{\cdot} \\
\mathcal{F}(NA \otimes NB, NC) & \xrightarrow{\ \mu^4\ } & \mathcal{F}(NA, NB \pitchfork NC)\ ,
\end{array}$$

where μ is defined in 1.13 by

$$\mu(f)(a) = f(\hat{a} \otimes 1)\ , \qquad f \in \mathcal{F}(A \otimes B, C) \quad a \in A\ .$$

We calculate in turn $\mu^1, \mu^2, \mu^3, \mu^4$.

$$\left\{ \begin{array}{l} f \in \mathcal{F}(A \otimes B, C) \\ a \in A, \ \phi\delta^q \otimes b \in N(q) \otimes N \end{array} \right.$$

(1)
$$\begin{aligned}
\mu^1(f)(a)(\phi\delta^q \otimes b) &= \lambda^1\{(\mu f)(a)\}(\phi\delta^q \otimes b) \\
&= \lambda^1\{f(\hat{a} \otimes 1)\}(\phi\delta^q \otimes b) \\
&= N(f)\, N(\hat{a} \otimes 1)\, \Psi^{-1}(\phi\delta^q \otimes b) \\
&= N(f)\Psi^{-1}(N\hat{a} \otimes 1)(\phi\delta^q \otimes b) \quad \text{by naturality of } \Psi \\
&= N(f)\Psi^{-1}(\phi a^N \otimes b) \qquad \text{where } a^N = (N\hat{a})(\delta^q) \\
&= N(f)\,\widehat{(\phi a^N \otimes b)}
\end{aligned}$$

(2) $$\mu^2(f)(a)(\phi\delta^q)(b) = \mu^1(f)(a)\ (\phi\delta^q\otimes b) = N(f)\ \widehat{(\phi a^N\otimes b)}$$

(3) $$\mu^3(g)(a)\ (\phi\delta^q)(b) = g\,\widehat{(\phi\, a\otimes b)} \qquad \begin{cases} g\in \mathfrak{F}(N(A\otimes B),\ NC) \\ a\in NA \end{cases}$$

(4) $$\begin{aligned}\mu^4(h)(a)(b) &= \Psi.\mu^3\Psi'(h)(a)(b) \qquad \begin{cases} h\in \mathfrak{F}(NA\otimes NB,\ NC) \\ a\in NA_q,\ b\in NB \end{cases} \\ &= \mu^3\Psi'(h)(a)(\delta^q)(b) \\ &= \Psi'(h)\ \widehat{(a\otimes b)} \\ &= h(a\otimes b)\ .\end{aligned}$$

This last formula is that for the exponential map for chain complexes. So the theorem is proved.

<u>Appendix 3.</u> <u>Proof of IX. 1.51.</u>

In the following diagram

$$\begin{array}{ccccc} \mathfrak{F}(K(q)\times A\times B, C) & \xrightarrow{\Delta'} & \mathfrak{F}((K(q)\times A)\otimes B, C) & \xrightarrow{(\Delta\otimes 1)'} & \mathfrak{F}(K(q)\otimes A\otimes B,\ C) \\ \mu\downarrow & & \downarrow\mu & & \downarrow\mu \\ \mathfrak{F}(K(q)\times A,\ B\,\underline{\pitchfork}\,C) & \xrightarrow[\hat{\Delta}.]{} & \mathfrak{F}(K(q)\times A,\ B\pitchfork C) & \xrightarrow[\Delta']{} & \mathfrak{F}(K(q)\otimes A,\ B\pitchfork C) \end{array}$$

the left-hand square is commutative by 1.41 and the right-hand square is commutative by naturality of μ . We define $\Delta^1 : (A\times B)\,\underline{\pitchfork}\,C \to (A\otimes B)\pitchfork C$, $\Delta^2 : A\,\underline{\pitchfork}\,(B\,\underline{\pitchfork}\,C) \longrightarrow A\pitchfork(B\pitchfork C)$ to be in dimension q respectively $(\Delta\otimes 1)'\Delta'$, $\Delta'\hat{\Delta}$. . Since Δ , $\hat{\Delta}$ are natural, Δ^1, Δ^2 are FD-maps. That Δ^1, Δ^2 commute with μ is obvious, so we have only to prove that Δ^1, Δ^2 are FD-homotopy equivalences.

Let $\xi^1 : (A\otimes B)\pitchfork C \longrightarrow (A\times B)\,\underline{\pitchfork}\,C$, $\xi^2 : A\pitchfork(B\pitchfork C) \longrightarrow A\,\underline{\pitchfork}\,(B\,\underline{\pitchfork}\,C)$

be defined in dimension q in a similar way to Δ^1, Δ^2 by

$$\xi^1 = \xi^{\cdot}(\xi \otimes 1), \quad \xi^2 = \hat{\xi} \circ \xi^{\cdot}.$$

Then $\xi\Delta = 1$, $\hat{\Delta}\hat{\xi} = 1$ implies $\Delta^1\xi^1 = 1$, $\Delta^2\xi^2 = 1$, while $\Delta\xi \simeq 1$, $\hat{\xi}\hat{\Delta} \simeq 1$ implies that if $f \in \mathfrak{J}(K(q) \underline{\times} A \underline{\times} B, C)$, $g \in \mathfrak{J}(K(q) \underline{\times} A, B \underline{\wedge} C)$, then $\xi^1\Delta^1(f) \simeq f : K(q) \underline{\times} A \underline{\times} B \rightarrow C$, $\xi^2\Delta^2(g) \simeq g : K(q) \underline{\times} A \rightarrow B \underline{\wedge} C$. Let us write $D^1(f)$, $D^2(g)$ respectively for these homotopies.

Then maps

$$\underline{D}^1_q : K(1)_q \otimes \mathfrak{J}(K(q) \underline{\times} A \underline{\times} B, C) \longrightarrow \mathfrak{J}(K(q) \underline{\times} A \underline{\times} B, C)$$

$$\underline{D}^2_q : K(1)_q \otimes \mathfrak{J}(K(q) \underline{\times} A, B \underline{\wedge} C) \longrightarrow \mathfrak{J}(K(q) \underline{\times} A, B \underline{\wedge} C)$$

are defined by

$$\left.\begin{aligned} \underline{D}^1_q(\phi\delta^1, f)(\psi\delta^q \otimes a \otimes b) &= D^1(f)(\psi\phi\delta^1 \otimes \psi\delta^q \otimes a \otimes b) \\ \underline{D}^2_q(\phi\delta^1, g)(\psi\delta^q \otimes a) &= D^2(g)(\psi\phi\delta^1 \otimes \psi\delta^q \otimes a) \end{aligned}\right\} \begin{cases} a \in A_p, \ b \in B_p \\ (\psi\delta^q) \in (\Delta^q)_p \\ (\phi\delta^1) \in (\Delta^1)_q \end{cases}$$

Since the homotopies D^1, D^2 are natural, $\underline{D}^1_q$, $\underline{D}^2_q$ $(q = 0, 1, \ldots)$ define natural FD-homotopies

$$\underline{D}^1 : \xi\Delta \simeq 1 : (A \underline{\times} B) \underline{\wedge} C \longrightarrow (A \underline{\times} B) \underline{\wedge} C$$
$$\underline{D}^2 : \xi\Delta \simeq 1 : A \underline{\wedge} (B \underline{\wedge} C) \longrightarrow A \underline{\wedge} (B \underline{\wedge} C).$$

Appendix 4. Proof of Theorem X. 2.41

By X. 2.1, if $t > 0$, it may be assumed in constructing κ that $L \underline{\wedge} Z$ has only two independent generators a, b in dimensions $q+1$, q respectively with boundary $\delta a = tb$. The case $t = 0$ is also covered in what follows simply by omitting mention of a.

Let $h : L \pitchfork Z \longrightarrow \eta^q Z_t$ be defined by $h(a) = 0$, $h(b) = 1$; then h is a chain map inducing an isomorphism in homology. Let κ be the associated Künneth isomorphism of type $(L, Z; \eta^q Z_t)$; that is, let κ be, for each free $K \in \mathcal{C}_0$, the composite

$$H^m(L \otimes K, Z) \xrightarrow{\mu\Gamma} H^m(K, L \pitchfork Z) \xrightarrow{(1 \pitchfork h)_*} H^m(K, \eta^q Z_t) = H^{m-q}(K, Z_t) .$$

Since κ is natural with respect to maps of K , which is given to be a free, finitely generated complex, it is sufficient to prove the theorem when $K \pitchfork Z$ has only two independent generators x, y in dimensions $p + 1$, p respectively, with boundary $\delta x = sy$ $(s > 0)$. In this case the group $H(L \otimes K, Z)$ is zero except in dimensions $p + q + 1$, $p + q$, when it is given by

$$H^{p+q}(L \otimes K, Z) = Z_{(s,t)}[b_0 \times y_0] ,$$
$$H^{p+q+1}(L \otimes K, Z) = Z_{(s,t)}\left[\left(\frac{1}{(s,t)} \delta(a \times x)\right)_0\right].$$

(Here $Z_\lambda[u]$ denotes a cyclic group of order λ generated by u).

Let x', y' denote the unique elements of K such that respectively $x(x') = 1$, $y(y') = 1$. Then, by VI. Lemma 1.61,

$$\{(1 \pitchfork h)\mu T'(b \times y)\}(y') = h(b) = 1 \quad (1 \in Z_t)$$

$$\left\{(1 \pitchfork h)\mu T'\left(\frac{1}{(s,t)}\delta(a \times x)\right)\right\}(y') = \frac{s}{(s,t)} h(a) = 0 ,$$

$$\left\{(1 \pitchfork h)\mu T'\left(\frac{1}{(s,t)} \delta(a \times x)\right)\right\}(x') = \frac{t}{(s,t)} h(b) = \frac{t}{(s,t)} \in Z_t$$

Clearly $(b \times y)_0 = b_0 \times y_0$, $\left(\frac{1}{(s,t)} \delta(a \times x)\right)_0 = \delta_{(s,t)}(a_{(s,t)} \times x_{(s,t)})$.

Therefore $\kappa(b_o \times y_o) = y_t$, $\kappa\,\delta_{(s,t)}(a_{(s,t)} \times x_{(s,t)}) = (\frac{t}{(s,t)}x)_t = h_{t,(s,t)}x_{(s,t)}$

This proves 2.41 (ii) for the case $d = (s,t)$.

In order that a be a cycle mod d , it is necessary that $d \mid t$. In this case $d = (d,t)$. So 2.41 (ii) is proved for all d .

Appendix 5. Proof of X. 2.51

It is given that $L \pitchfork Z$ has two generators a, b in dimensions $q+1$, q respectively. with boundary $\delta a = tb$ $(t \geq 0)$. The elements $h_n a$, $h_n b \in L \pitchfork Z_n$ are also written a, b .

The first step in the proof of X. 2.51 is the construction of a Künneth isomorphism of type $(L, Z_n : N)$ where $N = \eta^q Z_{(n,t)} + \eta^{q+1} Z_{(n,t)}$. To this end, let F be the free complex, whose generators and boundary are given by the top part of the following table.

dimension	q	$q+1$	$q+2$
generators	u	v, w	z
boundary	$\partial u = 0$	$\partial v = tu$, $\partial w = nu$	$\partial z = nv - tw$
cycles	u	$\frac{1}{(n,t)}\partial z$	-
homology	$Z_{(n,t)}[u]$	$Z_{(n,t)}[\frac{1}{(n,t)}\partial z]$	-
$f: F \to L \pitchfork Z_n$	$fu = b$	$fv = a$, $fw = 0$	$fz = 0$
$g: F \to N$	$gu = 1_q$	$gv = \alpha 1_{q+1}$, $gw = -\beta 1_{q+1}$	$gz = 0$

The bottom part of the table defines maps $f: F \longrightarrow L \pitchfork Z_n$, $g: F \longrightarrow N$; in these definitions, $1_q, 1_{q+1}$ are the units of N_q, N_{q+1} respectively, and α, β are integers such that $\alpha n + \beta t = (n,t)$. It is easily checked that both f and g are chain maps inducing isomorphisms in homology. Let κ be the associated Künneth isomorphism of type $(L, Z_n : N)$.

Since κ is natural with respect to maps of K, it is sufficient to prove the theorem when $K \pitchfork Z$ has two generators, x, y in dimensions $p+1$, p respectively with boundary $\delta x = sy$ $(s \geqslant 0)$.

Let $x', y' \in K$ be the unique elements of K such that $x(x') = 1$, $y(y') = 1$. The relation $\delta x = sy$ implies $\partial y' = (-1)^p s x'$.

We use the following notation : for any complex C, and any $\sigma \in C$, the elements $(x'\sigma), (y'\sigma) \in K \pitchfork C$ are the unique maps such that

$$(x'\sigma)\,(x') = \sigma \;,\quad (x'\sigma)\,(y') = 0$$
$$(y'\sigma)\,(y') = 0 \;,\quad (y'\sigma)\,(y') = \sigma$$

The homology $H(L \otimes K, Z_n)$ is zero except in dimensions $p+q$, $p+q+1$, $p+q+2$, where it is given by

$$H^{p+q}(L \otimes K, Z_n) = Z_c[(b \times y)_n] \;, \qquad c = (n,s,t)$$
$$H^{p+q+2}(L \otimes K, Z_n) = Z_c[(\tfrac{n}{c}\, a \times x)_n] \;.$$
$$H^{p+q+1}(L \otimes K, Z_n) = Z_c + Z_c \;.$$

This last group has as generators the elements

$$P = \left(\frac{1}{(s,t)} \delta(a \times x)\right)_n , \qquad Q = \left(\frac{n}{(n,t)} \; a \times y\right)_n$$

$$R = \left(\frac{n}{(n,s)} \; b \times x\right)_n .$$

In terms of the generators of X. 2.23 these elements are given by

$$\left(\frac{n}{c} \, a \times x\right)_n = h_{n,c} \, (a_c \times x_c) , \qquad (b \times y)_n = b_n \times y_n ,$$

$$Q = h_{n,(n,t)} (a_{(n,t)} \times y_{(n,t)}), \qquad R = h_{n,(n,s)} \, (b_{(n,s)} \times x_{(n,s)}) ,$$

$$\frac{(s,t)}{c} P = \delta_{n,n} \, h_{n,c} \, (a_c \times x_c) .$$

The following table gives the generators and boundary of $K \pitchfork F$, and also the values of the maps $1 \pitchfork g$, $1 \pitchfork f$, $(\mu T')^{-1} (1 \pitchfork f)$. It should be noted that the elements in column 4 are all mod (n,t), and those in columns 5 and 6 are mod n. We write $1_q \in (\eta^{\,q} Z_d)_q$, $1_{q+1} \in (\eta^{\,q+1} Z_d)_{q+1}$ for the units of these groups $((n,t) = d)$.

dimension	generators	boundary	$1 \pitchfork g$	$1 \pitchfork f$	$(\mu T')^{-1}(1 \pitchfork f)$
p+q	$(y' u)$	0	$(y' 1_q)$	$(y' b)$	$b \times y$
p+q+1	$(y' v)$	$t(y' u)$	$\alpha(y' 1_{q+1})$	$(y' a)$	$a \times y$
	$(y' w)$	$n(y' u)$	$-\beta(x' 1_{q+1})$	0	0
	$(x' u)$	$(-1)^q s(y' u)$	$(x' 1_q)$	$(x' b)$	$b \times x$
p+q+2	$(x' v)$	$t(x' u) + (-1)^{q+1} s(y' v)$	$\alpha(x' 1_{q+1})$	$(x' a)$	$a \times x$
	$(x' w)$	$n(x' u) + (-1)^{q+1} s(y' w)$	$-\beta(x' 1_{q+1})$	0	0
	$(y' z)$	$n(y' v) - t(y' w)$	0	0	0
p+q+3	$(x' z)$	$n(x' v) - t(x' w) + (-1)^q s(y' z)$	0	0	0
			mod d	mod n	mod n

The values of κ on the generators of $H(L\otimes K, Z_n)$ can now be checked. In the following table column I gives cycles of $(L\otimes K)\pitchfork Z_n$, Column II gives cycles of $K\pitchfork F$ which map to the cycles of column I under $(1\pitchfork i)^{-1}(1\pitchfork f)$ while column III gives the images of the cycles of Column II under $1\pitchfork g$. Clearly we may identify y with $(y'1_q)$ and $(y'1_{q+1})$, x with $(x'1_q)$ and $(x'1_{q+1})$. No confusion results from this, since a count of dimension shows in which group elements lie.

I cycles of $(L\otimes K)\pitchfork Z_n$	II cycles of $K\pitchfork F$	III cycles of $K\pitchfork Z_d$
$b\times y$	$(y'u)$	y
$\frac{n}{(n,t)}\, a\times y$	$\frac{n}{(n,t)}(y'v) - \frac{t}{(n,t)}(y'w)$	y
$\frac{n}{(n,s)}\, b\times x$	$\frac{n}{(n,s)}(x'u) + (-1)^{q+1}\frac{s}{(n,s)}(y'w)$	$\frac{n}{(n,s)}\, x + (-1)^q\beta\frac{s}{(n,s)}\, y$
$\frac{1}{(s,t)}\,\delta\,(a\times x)$	$\frac{t}{(s,t)}(x'u) + (-1)^{q+1}\frac{s}{(s,t)}(y'v)$	$\frac{t}{(s,t)}\, x + (-1)^{q+1}\alpha\frac{s}{(s,t)}\, y$
$\frac{n}{c}(a\times x)$	$\frac{1}{c}\,\delta\,(x'z)$	$\alpha\frac{n}{c}\, x + \beta\frac{t}{c}\, x = \frac{(n,t)}{c}\, x$

From this table we deduce

$$\kappa h_{n,c}(a_c \times x_c) = \left(\frac{(n,t)}{c}\, x\right)_d = h_{d,c}\, x_c,$$

$$\kappa h_{n,d}(a_d \times y_d) = y_d\,,$$

$$\kappa(b_n \times y_n) = y_d = h_{d,n} h_{n,n}\, y_n + (-1)^q \beta\, h_{d,o}\, \delta_n y_n \quad (\text{since } \delta_n y_n = 0),$$

$$\kappa h_{n,(n,s)}(b_{(n,s)} \times x_{(n,s)}) = \left(\frac{n}{(n,s)}\, x\right)_d + (-1)^q \beta \left(\frac{s}{(n,s)}\, y\right)_d$$

$$= h_{d,n} h_{n,(n,s)} x_{(n,s)} + (-1)^q \beta\, h_{d,o}\, \delta_{(n,s)} x_{(n,s)}$$

$$\kappa \delta_{n,n} h_{n,c}(a_c \times x_c) = \left(\frac{t}{c}\, x\right)_d + (-1)^{q+1} \alpha \left(\frac{s}{c}\, y\right)_d$$

$$= h_{d,t} h_{t,c} x_c + (-1)^{q+1} \alpha\, h_{d,o}\, \delta_c\, x_c\,.$$

These formulae confirm 2.51 for particular values of i. But for $(a_i \times x_i)$, $(b_i \times x_i)$ to be defined we must have $i \mid n,t$ (for the case of $a_i \times x_i$) or $i \mid n$ (for the case of $b_i \times x_i$). So the cases we have covered are in fact sufficient for the theorem.

Appendix 6. Proof of XII. 5.32

Since the operations considered are additive, and by the "additivity lemma" X.21, it is sufficient to consider the case $\pi_{n+1} = Z_p$ $(p \geqslant 0)$, $\pi_{n+2} = Z_q$ $(q \geqslant 0)$. Now the theorem is trivial if $\gamma^* = 0$. Let us suppose then $\gamma^* \neq 0$; since γ^* is a homomorphism such that $2\gamma^* = 0$, this implies that $p, q \equiv 0(2)$, that $\gamma^*(1) = q/2 \in Z_q$, and so that $i_* = h_{q,2}$:

$H^{-n-1}(Y, \pi_{n+1} \otimes Z_2) \longrightarrow H^{-n-1}(Y, \pi_{n+2})$.

We choose a canonical basis for $(C_N(Y) \pitchfork Z)_{-n}$ consisting of elements a' such that $\delta a' = 0$ and elements a such that $\delta a = ub$ $(u > 0)$, where the elements b form part of a canonical basis for $(C_N(Y) \pitchfork Z)_{-n-1}$. We write $r = (u, p)$.

To the elements a', a correspond fundamental classes $\omega' \in H^{-1}(K(Z_p, 1), Z_p)$, $\omega \in H^{-1}(K(Z_r, 1), Z_r)$ By XI 2.31, the part of the evaluation class in the component of the base point is

$$\xi = \sum_{a'} a'_p \times \omega' + \sum_a h_{p,r}(a_r \times \omega).$$

We consider first the term $a'_p \times \omega'$. Now $Sq^1 a'_p = 0$, since $\delta a' = 0$, and $Sq^2 \omega' = 0$, since $\dim \omega' = -1$. Hence the Cartan formula implies $Sq^2(a'_p \times \omega') = Sq^2 a'_p \times h_{2,p}\omega'$; this term contributes only to k_1, which has already been determined.

We now consider

$$i_* Sq^2 h_{p,r}(a_r \times \omega) = h_{q,2} Sq^2 (a_r \times \omega)$$

$$= h_{q,2}(Sq^1 a_r \times Sq^1 \omega + Sq^2 a_r \times h_{2,r}\omega) \quad \text{since } Sq^2\omega = 0$$

The term $h_{q,2}(Sq^2 a_r \times h_{2,r}\,\omega)$ contributes only to k_1, which has already been determined. The term $h_{q,2}(Sq^1 a_r \times Sq^1 \omega)$ determines k_2.

It is clearly sufficient to prove the theorem when $C_N(Y) \pitchfork Z$ has only two (independent) generators a, b with $\delta a = ub$ $(u > 0)$. If $u \not\equiv 2(4)$, then $Sq^1 a_r = 0$, and the theorem is trivially proved. If $u \equiv 2(4)$, then $Sq^1 a_r = b_2$, and (by X. 2.51) we may choose a Künneth isomorphism κ such that

$$\kappa\, h_{q,2}(b_2 \times Sq^1\omega) = h_{r,q} h_{q,2}\, Sq^1 \omega \quad (\text{Since } \delta_2 Sq^1 \omega = 0)$$

Clearly $h_{r,q} h_{q,2} = i_{**}$, and so k_2 is the composition

$$K(Z_r,1) \xrightarrow{Sq^1} K(Z_2,2) \xrightarrow{i_{**}} K(Z_q 2) .$$

In the above Sq^2 is taken as an operation of type $(\eta^r G, \eta^{r+1} G \otimes Z_2)$, for G an abelian group, whereas for the statement of the Theorem in general it is more convenient to take Sq^1 as an operation of type $(\eta^r G, \eta^{r+1} G)$. If this is done, then the theorem follows immediately.

Appendix 7. A new product topology.

In this appendix we introduce a new product topology which seems to have many advantages over the weak product considered in Chapter I.

Definition. Let X, Y be spaces * and let $X \times Y$ be the (usual) topological product of X and Y. Let $X \underline{\times} Y$ be the set $X \times Y$ with the topology that a set $C \subset X \underline{\times} Y$ is closed in $X \underline{\times} Y$ if and only if $C \cap A \times Y$, $C \cap X \times B$ are closed in $A \times Y$, $X \times B$ respectively, for all compact subsets A of X, B of Y.

Obviously if one of X, Y is compact, then $X \underline{\times} Y = X \times Y$, and in fact the advantage of $X \underline{\times} Y$ over $X \times_w Y$ stems precisely from this fact. More generally, we have

Proposition 1. $X \underline{\times} Y = X \times Y$ if

(a) one of X, Y is locally compact,

or (b) both X and Y satisfy the first axiom of countability,

or (c) X and Y are CW-complexes such that $X \times Y$ is a CW-complex.

Proof. (a) Suppose Y is locally compact. Let W be a set open in $X \underline{\times} Y$ and let $(x,y) \in W$. Let $K \subset Y$ be a compact neighbourhood of y. Then $X \underline{\times} K = X \times K$, and so $W \cap X \times K$ is open in $X \times K$. Hence there are sets U, V open in X, Y respectively such that $(x,y) \in U \times V \subset W \cap X \times K$. Hence W is open in $X \times Y$.

(b) Since X, Y satisfy the first axiom of countability, so also does $X \times Y$. Hence $X \times Y$ is a k-space, and so $X \times Y = X \times_W Y$. Since the topology of $X \underline{\times} Y$ lies between those of $X \times Y$, $X \times_W Y$, we must have $X \underline{\times} Y = X \times Y$.

* In this appendix, the term space will always mean Hausdorff topological space.

(c) In this case also $X \times Y = X \times_W Y$, and hence $X \mathbin{\underline{\times}} Y = X \times Y$.

Let X^Y be the (usual) function space of all continued functions $Y \to X$ with the compact-open topology.

<u>Theorem 1.</u> The exponential map $\mu : X^{Z \mathbin{\underline{\times}} Y} \longrightarrow (X^Y)^Z$ is a homeomorphism (onto).

The proof is similar to that of I. 2.37, and is omitted.

<u>Corollary 1.1</u> The exponential map $\mu : X^{Z \times Y} \longrightarrow (X^Y)^Z$ (with the classical product) is a homeomorphism (onto) if

(a) [22] one of Z, Y is locally-compact,

or (b) (Fox; [22]) both Z and Y satisfy the first axiom of countability,

or (c) (Barcus-Barratt ; [2a]) Z and Y are CW-complexes such that $Z \times Y$ is a CW-complex.

<u>Proof.</u> The corollary follows immediately from Theorem 1 and Proposition 1.

<u>Corollary 1.2</u> (Jackson; [29a]) For all X, Y, Z, the exponential map $\mu : X^{Z \times Y} \longrightarrow (X^Y)^Z$ (with the classical product) is a homeomorphism into.

<u>Proof.</u> Since $Z \mathbin{\underline{\times}} Y$ has a larger topology than $Z \times Y$, $X^{Z \times Y}$ is a subspace of $X^{Z \mathbin{\underline{\times}} Y}$. So 1.2 follows from Theorem 1.

<u>Theorem 2.</u> The product of $f \mathbin{\underline{\times}} g$ of identification maps f, g is an identification map.

The proof is similar to that of I.3.32 and is omitted.

<u>Corollary 2.1</u> Let $f : P \to X$, $g : Q \to Y$ be identification maps. Then $f \mathbin{\underline{\times}} g : P \mathbin{\underline{\times}} Q \longrightarrow X \mathbin{\underline{\times}} Y$ is an identification map if

(a) (Cohen; [14]) one of P, Q, and one of X, Y , are locally compact.

(b) each of P, Q, X, Y satisfy the first axiom of countability.

Proof. The corollary follows immediately from Theorem 2 and Proposition 1.

Corollary 2.2. The product $X \times_k Y$ of CW-complexes X, Y is again a CW-complex.

Proof. This follows from Theorem 3 in the same way as H of [57] follows from [55; Lemma 4].

Corollary 2.3 (C.H.Dowker) If X, Y are locally countable CW-complexes, then $X \times Y$ is a CW-complex.

Proof. Since X, Y are locally countable CW-complexes, they satisfy the first axiom of countability. Hence by Proposition 1. $X \times Y = X \times_k Y$, and so, by Corollary 2.2, $X \times Y$ is a CW-complex.

Remark. Certain results may be obtained with a smaller topology than that of $X \times_k Y$. Thus let $X \times_R Y$ be the set $X \times Y$ with the topology that a set $C \subset X \times_R Y$ is closed in $X \times_R Y$ if and only if $C \cap X \times B$ is closed in $X \times B$ for all compact subsets B of Y. Then from the proofs of the theorems of Chapter I we may abstract the following results.

(a) if $f : Z \times_R Y \rightarrow X$ is continuous, then so is the map $\mu f : Z \rightarrow X^Y$.

(b) if $g : Z \rightarrow X^Y$ is continuous, then so is $\mu^{-1} g : Z \times_R Y \rightarrow X$.

(c) if $f : P \rightarrow X$ is an identification map, then so, for any Y, is $f \times_R 1 : P \times_R Y \rightarrow X \times_R Y$.

Of course, $\times_R$ is not an associative product, so the proof of the exponential law for $\times_k$ does not apply to $\times_R$.

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