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COVERINGS OF GROUPOIDS AND MAYER-VIETORIS TYPE SEQUENCES

Ronald Brown, Philip R. Heath, Klaus Heiner Kamps

Introduction

In [2] it was shown that a fibration of groupoids gives rise to a family of six-term exact sequences of groups and pointed sets. These exact sequences imply various exact sequences in homotopy theory, homological algebra and group theory (see [2],[3],[10],[11],[15],[16],[17],[18],[19],[24],[25]).

In the special case of a covering of groupoids, these six-term exact sequences reduce to five-term exact sequences. Here we exploit the latter sequences in applications to operations of groupoids, and to Mayer-Vietoris type sequences. In particular we generalize the results of [5], and give new applications. We also show how the exact sequence of a fibration can be deduced from the apparently less general sequence of a covering.

One of the reasons for emphasing coverings of groupoids is that the relation between these and operations of groupoids, which we recall in Section 1, is becoming more generally important. For example, the construction of a covering groupoid of a group G from a G-set gives one of the basic examples of groupoids, used in many applications (see [1],[2],[3],[11],[13],[14],[20],[21],[22]).

So we have further justification for regarding the extension of viewpoint from groups to groupoids as significant.

1. Covering morphisms and operations

A *groupoid* is a small category in which every morphism is invertible. A *morphism* of groupoids is simply a functor. So we have a category Gd of groupoids, which contains as full subcategory the category G of groups, where a group is considered as a groupoid with one object. On the other hand, any groupoid G contains a family $G(x) = G(x,x)$ of *vertex groups*, one for each $x \in Ob(G)$.

Thus the idea of a groupoid is a natural extension of the idea of a group. As shown in [1] and [13], this extension is useful. The reasons for considering a wider algebraic structure are the usual ones: The category Gd has nicer formal properties than G (it is cartesian closed, [2]); examples of groupoids occur in many different branches of mathematics; there are useful constructions leading from groups to groupoids; groupoids can be used in giving new proofs and in proving new theorems.

A by now classical construction of a groupoid is that arising from an operation of a group G on a set M , via $G \times M \xrightarrow{\cdot} M$, $(g,m) \mapsto g \cdot m = gm$, $g \in G$, $m \in M$. The groupoid $G \ltimes M$ has object set M , and morphisms $m \to m'$ the pairs $(g,m) \in G \times M$ such that $gm = m'$, with composition

$$(g',gm)(g,m) = (g'g,m) .$$

We call this groupoid the *semi-direct product* of G and M (it is called the *split extension* in [3]) because it is a special case of the semi-direct product construction for an action of a group on a groupoid, or more generally of a groupoid on a groupoid.

The set $\pi_0(G \ltimes M)$ of components of the groupoid $G \ltimes M$ is the orbit set M/G of M under the action of G. Let $p : G \ltimes M \to G$ be the projection morphism $(g,m) \mapsto g$. Then for any $m \in M$, p maps the vertex group $(G \ltimes M)(m)$ isomorphically to the stability (isotropy) group $G(m) = \{g \in G \mid gm = m\}$.

The morphism $p : G \ltimes M \to G$ is a *covering morphism* of groupoids. In general, a morphism $q : H' \to H$ of groupoids is a *fibration* if for each object y' of H' and morphism h of H starting at $q(y')$, there is a morphism h' of H' starting at y' and with $q(h') = h$, and q is a *covering* if the lift h' of h is uniquely determined by h and y'.

The above construction of $G \ltimes M$ gives an equivalence between the category of operations of the group G on sets and the category of covering morphisms of G. In fact a more general result is true. We recall the following definition due to Ehresmann ([7]).

1.1. Definition. An *operation* of a groupoid G on a set M consists of a function $v : M \to Ob(G)$ and a family of functions $g_* : v^{-1}(x) \to v^{-1}(y)$, one for each $g \in G(x,y)$ and $x,y \in Ob(G)$, such that $1_* = 1$ and $(g'g)_* = g'_* g_*$.

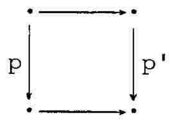
As usual, $g_*(m)$ is also written $g \cdot m$ or gm, $m \in v^{-1}(x)$. Such an operation is often written (G,M,v). Note that, for each g, g_* is a bijection.

A *morphism* $(G,M,v) \to (H,N,w)$ of such operations is a pair (ψ,κ) where $\psi : G \to H$ is a morphism of groupoids, $\kappa : M \to N$ is a function, and we have the rules $w\kappa = Ob(\psi)v$ and $\psi(g)\kappa(m) = \kappa(gm)$ whenever gm is defined. So we obtain a category *OpGd* of operations of groupoids on sets.

Alternatively,one can consider an operation of a groupoid G simply as a functor M from G to the category of sets, and consider a morphism $M \to N$ of such operations as a morphism $\psi : G \to H$ of groupoids and a natural transformation $\lambda : M \to N\psi$ of functors. This gives a category *Op'Gd*. But it is easy to construct an equivalence of categories *OpGd* $\to$ *Op'Gd* (see [2]).

A third way of considering operations arises from the following definition. The category *CovGd* has objects the covering morphisms of groupoids and has morphisms $p \to p'$

the commutative squares



of morphisms of groupoids, with the usual horizontal composition.

1.2 Proposition. *There is an equivalence of categories*

$$OpGd \to CovGd$$

given by a semi-direct product construction sending the operation (G,M,v) *to the covering morphism* $G \ltimes M \to G$. □

Here $G \ltimes M$ is the groupoid with object set M and morphisms $m \to m'$ the pairs (g,m) such that $g \in G(vm,vm')$ and $m' = gm$. The proof of the proposition is straightforward.

The canonical construction associating an operation to a covering morphism $q : H' \to H$ can be described as follows. If $h \in H(x,y)$, $x' \in Ob(H')$ such that $q(x') = x$, then $h \cdot x'$ is the final point of the unique lift of h starting at x' .

Thus we have three ways of considering operations of groupoids, corresponding to our three categories, each way being appropriate in particular circumstances. The covering approach is less well known. Its value comes from the relation between coverings and fibrations (discussed below) and the fact that covering morphisms of groupoids model covering maps of spaces (see [1] for a detailed discussion of this). Proposition 1.2 may be generalized to groupoid objects in arbitrary categories with finite limits - see [4] for the appropriate definition of covering. For relations between coverings of groupoids and coverings of categories, see [9],[23].

2. Exact sequences and fibrations

In this section we give the exact sequence associated with a covering of groupoids. We show how this applies to operations and also how the exact sequence of a fibration can be deduced from the exact sequence of a covering.

The importance of these considerations is that many situations in group theory lead to operations and so to groupoids. The following exact sequences link many apparently disparate facts.

2.1. Proposition. *If* $q : H' \to H$ *is a covering morphism of groupoids,* $y' \in \mathrm{Ob}(H')$ *and* $F'_{qy'}$ *is the discrete groupoid (or set)* $q^{-1}qy'$ *, then*

$$(2.2) \qquad 1 \to H'(y') \xrightarrow{q} H(qy') \xrightarrow{\partial} F'_{qy'} \xrightarrow{i} \pi_0 H' \xrightarrow{q} \pi_0 H$$

is an exact sequence of groups and pointed sets (exact covering sequence), where the base points are y' *in* $F'_{qy'}$*, the component* $\tilde{y}'$ *of* y' *in* $\pi_0 H'$ *, and the component* $(qy')^{\sim}$ *of* qy' *in* $\pi_0 H$ *. The connecting map* ∂ *is given by the operation of* $H(qy')$ *on the element* y' *of* $F'_{qy'}$: $\partial(h) = h\cdot y'$, $h \in H(qy')$.

Further:

(a) *If* $h,k \in H(qy')$ *, then* $\partial(h) = \partial(k)$ *if and only if there is an* $h' \in H'(y')$ *such that* $q(h') = k^{-1}h$.

(b) *If* $y,z \in F'_{qy'}$ *, then* $i(y) = i(z)$ *if and only if there is an* $h \in H(qy')$ *such that* $h\cdot y = z$. □

The proof of this proposition is straightforward (see [2] for the more general case of a fibration).

As an example of the above, let $i : G \to H$ be the inclusion of a subgroup G of a group H . Then H operates on the set H/G of left cosets. The semi-direct product $\tilde{H} = H \ltimes H/G$ is denoted by $\mathrm{Tr}(H : G)$ in [13]. The exact sequence of the covering $\tilde{H} \to H$ at the identity coset G is equivalent to

$$1 \longrightarrow G \longrightarrow H \xrightarrow{\partial} H/G \longrightarrow 1 \ ,$$

where H/G is of course a pointed set with base point the coset G. In effect, the properties of the exact sequence (2.2) imply Lagrange's Theorem.

Since the motivation for Proposition 2.1 comes from homotopy theory, we see here many useful and interesting analogies.

More generally, if (G,M,v) is an operation of a groupoid G on a set M, and $m \in M$, the exact sequence of the covering $G \ltimes M \to G$ at the object m is the sequence

(2.3) $$1 \to (G(vm))(m) \to G(vm) \to v^{-1}vm \to M/G \to \pi_0 G ,$$

where $(G(vm))(m)$ as above denotes the stability group of m under the induced action of $G(vm)$ on $v^{-1}vm$ and M/G is the set of equivalence classes of M under the relation $m \sim m'$ if there exists $g \in G(vm,vm')$ with $m' = gm$.

In the case G is a group, (2.3) reduces to the exact sequence

(2.4) $$1 \to G(m) \to G \to M \to M/G \to 1 .$$

The link between fibrations and coverings is given by the following construction. If $f : G \to H$ is a morphism of groupoids and $y \in \mathrm{Ob}(H)$, we denote the groupoid $f^{-1}(y)$ by F_y, and call it the *fibre* of f over y. Let $\ker f$ denote the sum of the fibres F_y for all objects y of H. It is pointed out in [2] that f has a factorization $G \xrightarrow{q'} G/\ker f \xrightarrow{q} H$, where $G/\ker f$ is the quotient groupoid, q' is the quotient morphism (see [13]), and q has discrete fibres with $q^{-1}(y)$ bijective with $\pi_0(F_y)$, $y \in \mathrm{Ob}(H)$. Further, f is a fibration if and only if q is a covering morphism.

Suppose now that $f : G \to H$ is a fibration of groupoids, and $x \in \mathrm{Ob}(G)$. Apply 2.1 to the covering q and the object $q'(x)$ in the above factorization $f = qq'$ of f. The resulting exact sequence (2.2) together with the exact sequence of groups

$$1 \to F_{fx}(x) \to G(x) \to H'(q'x) \to 1 \;,$$

where $H' = G/\ker f$, now gives the exact sequence of a fibration discussed in [2].

If $(\psi,\kappa) : (G,M,v) \to (H,N,w)$ is a morphism of operations, and $\psi : G \to H$ is a fibration of groupoids, then so also is $\psi \ltimes \kappa : G \ltimes M \to H \ltimes N$. The exact sequence of this fibration then relates the various orbit sets / stability subgroups of the operations involved. This idea is exploited in [11] for the case G and H are groups.

3. <u>Homotopy pullbacks and Mayer-Vietoris type sequences</u>

Groupoids have a 'model' of the unit interval, namely the groupoid I with two objects 0 and 1, and two non-identity morphisms $\iota : 0 \to 1$, $\iota^{-1} : 1 \to 0$. Many standard constructions of homotopy theory can therefore be imitated for groupoids (this was the origin of the fibration notion), but with a much simpler theory than for the usual homotopy theory. This fact has been exploited in [1],[2],[8],[13], and [18] and in a number of other papers. It shows one of the advantages groupoids have over groups.

In this section we borrow ideas on homotopy pullbacks (see [6] for the topological case) to obtain a Mayer-Vietoris sequence.

Throughout this section we consider a pullback square of groupoids

$$(3.1) \qquad \begin{array}{ccc} D & \xrightarrow{\bar{f}} & E \\ {\scriptstyle \bar{p}}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow[f]{} & B \end{array}$$

so that D is the subgroupoid of $A \times E$ whose elements are pairs (a,e) such that $f(a) = p(e)$, and $\bar{p},\bar{f}$ are given by projection onto the factors. We will also need the standard homotopy pullback which is the groupoid double mapping track Z of f and p defined as follows.

The objects of Z are triples (a,β,e), where $a \in \mathrm{Ob}(A)$, $e \in \mathrm{Ob}(E)$ and $\beta : p(e) \to f(a)$ is a morphism of B. The morphisms $(\alpha,\varepsilon) : (a,\beta,e) \to (a',\beta',e')$ of Z are pairs of morphisms $\alpha : a \to a'$ of A, $\varepsilon : e \to e'$ of E, such that

$$\begin{array}{ccc} \cdot & \xrightarrow{f(\alpha)} & \cdot \\ \beta \uparrow & & \uparrow \beta' \\ \cdot & \xrightarrow[p(\varepsilon)]{} & \cdot \end{array}$$

is a commutative square in B. Let $q : Z \to A \times E$ be defined by $q(a,\beta,e) = (a,e)$ on objects and $q(\alpha,\varepsilon) = (\alpha,\varepsilon)$ on morphisms.

3.2. <u>Proposition.</u> $q : Z \to A \times E$ *is a covering of groupoids.*

<u>Proof.</u> Let (a,β,e) be an object of Z and $(\alpha,\varepsilon) : (a,e) \to (a',e')$ a morphism of $A \times E$. Then $(\alpha,\varepsilon) : (a,\beta,e) \to (a',\beta',e')$ with $\beta' = f(\alpha)\beta p(\varepsilon)^{-1}$ is the unique lifting. □

3.3. <u>Theorem.</u> *For each object* $d_o = (a_o,e_o)$ *of* D *and each element* β_o *of* $B(b_o)$ *where* $b_o = f(a_o) = p(e_o)$ *there is an exact (Mayer-Vietoris) sequence of groups and based sets*

$$(3.4) \quad 1 \to Z(z_o) \xrightarrow{q} A(a_o) \times E(e_o) \xrightarrow{\partial z_o} B(b_o) \xrightarrow{\Delta} \pi_o Z \xrightarrow{(\bar{p},\bar{f})} \pi_o A \sqcap \pi_o E \to 1,$$

where z_o *is the object* (a_o,β_o,e_o) *of* Z, *and* $\pi_o A \sqcap \pi_o E$ *denotes the pullback of* $\pi_o A \xrightarrow{f} \pi_o B \xleftarrow{p} \pi_o E$. *The base points are* $\beta_o \in B(b_o)$ *and the components* $\tilde{z}_o$ *in* $\pi_o Z$ *and* $(\tilde{a}_o,\tilde{e}_o)$ *in* $\pi_o A \sqcap \pi_o E$.

The boundary Δ *is given by* $\Delta(\beta) = (a_o,\beta,e_o)^{\sim}$ *for* $\beta \in B(b_o)$, *while* ∂_{z_o} *is the restriction of the operation of* $A(a_o) \times E(e_o)$ *on* $B(b_o)$, *given by*

$(\alpha,\varepsilon)\cdot\beta = f(\alpha)\beta p(\varepsilon)^{-1}$ *for* $\alpha \in A(a_o)$, $\varepsilon \in E(e_o)$, $\beta \in B(b_o)$, *to the base point* β_o.

Furthermore:

if $\beta, \beta' \in B(b_0)$ *, then* $\Delta(\beta) = \Delta(\beta')$ *if and only if there are* $\alpha \in A(a_0)$, $\varepsilon \in E(e_0)$ *such that* $\beta' = f(\alpha)\beta p(\varepsilon)$.

Proof. We apply 2.1 to the covering $q : Z \to A \times E$ of 3.2 and the object $z_0 = (a_0, \beta_0, e_0)$ of Z . Note that the fibre of q over (a_0, e_0) has $B(b_0)$ as its set of objects. We obtain the sequence above with the extreme right hand terms replaced by $\pi_0 A \times \pi_0 E$. The operation is determined by the unique lifting property of a covering (cf. the proof of 3.2). To see that the induced map $\pi_0 Z \to \pi_0 A \sqcap \pi_0 E$ is surjective let $a \in \mathrm{Ob}(A)$, $e \in \mathrm{Ob}(E)$ such that the components in B of $f(a)$ and $p(e)$ coincide. Thus we have a morphism $\beta : p(e) \to f(a)$ in B . Then (a, β, e) is an object of Z whose component is mapped to $(\tilde{a}, \tilde{e})$. □

Next we relate the pullback to the homotopy pullback. Let $\phi : D \to Z$ be the canonical functor defined by $\phi(a,e) = (a, 1_{f(a)}, e)$ on objects and $\phi(\alpha, \varepsilon) = (\alpha, \varepsilon)$ on morphisms.

3.5. Proposition. *For each object* d *of* D, $\phi : D(d) \to Z(\phi(d))$ *is an isomorphism; the induced map* $\phi : \pi_0 D \to \pi_0 Z$ *is injective.*

Proof. We have $Z(\phi(d)) = D(d)$, and ϕ is the identity map. Suppose we are given objects $d = (a,e)$, $d' = (a',e')$ of D and a morphism $(\alpha, \varepsilon) : (a, 1_{f(a)}, e) \to (a', 1_{f(a')}, e')$ in Z . Then we have $f(\alpha) = p(\varepsilon)$. Thus $(\alpha, \varepsilon) : d \to d'$ is a morphism in D . Hence $\phi : \pi_0 D \to \pi_0 Z$ is injective. □

3.6. Definition. A pair of morphisms (f,p) of groupoids with common target

$$A \xrightarrow{f} B \xleftarrow{p} E$$

is called *surjective* if for each object (a, β, e) of Z there are morphisms $\varepsilon : e \to e'$ in E and $\alpha : a' \to a$ in A where $p(e') = f(a')$ with $\beta = f(\alpha) p(\varepsilon)$.

3.7. Examples. (i) If f or p is a fibration, then the pair (f,p) is surjective.
(ii) If f and p are group homomorphisms, then the pair (f,p) is surjective if and only if B is the product set $f(A)p(E)$. In the case that f and p are inclusions of subgroups G_1, G_2 into a group G this condition reduces to $G = G_1 G_2$.

3.8. Proposition. *The pair* (f,p) *is surjective if and only if the induced map* $\phi : \pi_o D \to \pi_o Z$ *is surjective.*

Proof. Assume that (f,p) is surjective. Let $(a,\beta,e) \in Ob(Z)$. Then $\beta = f(\alpha)p(\varepsilon)$ for appropriate $\alpha : a' \to a$, $\varepsilon : e \to e'$ with $p(e') = f(a')$. Then $(\alpha^{-1},\varepsilon) : (a,\beta,e) \to (a', 1_{f(a')}, e')$ is a morphism of Z into the image of ϕ, and conversely. □

3.9. Remarks. (i) Note that the following Whitehead-type lemma holds for groupoids. A morphism $f : G \to H$ is a homotopy equivalence if and only if the induced map $f : \pi_o G \to \pi_o H$ is bijective and for each $a \in Ob(G)$ $f : G(a) \to H(fa)$ is an isomorphism of groups.
(ii) In view of (i), 3.5 and 3.8, the pair (f,p) is surjective if and only if $\phi : D \to Z$ is a homotopy equivalence which means that the pullback (3.1) is a homotopy pullback in the category of groupoids.

3.10. Corollary. *If in diagram* (3.1) *the pair* (f,p) *is surjective, then for each object* $d_o = (a_o, e_o)$ *of* D *and* $b_o = f(a_o) = p(e_o)$ *there is an exact (Mayer-Vietoris) sequence of groups and based sets*

$$(3.11) \quad 1 \to D(d_o) \xrightarrow{(\bar{p},\bar{f})} A(a_o) \times E(e_o) \xrightarrow{fp^{-1}} B(b_o) \xrightarrow{\Delta} \pi_o D \xrightarrow{(\bar{p},\bar{f})} \pi_o A \sqcap \pi_o E \to 1 ,$$

where $(fp^{-1})(\alpha,\varepsilon) = f(\alpha)p(\varepsilon)^{-1}$ *for* $\alpha \in A(a_o)$, $\varepsilon \in E(e_o)$. *The base points are* $1 \in B(b_o)$ *and the components* $\tilde{d}_o$ *in* $\pi_o D$ *and* $(\tilde{a}_o, \tilde{e}_o)$ *in* $\pi_o A \sqcap \pi_o E$.

Proof. We apply 3.3 for $\beta_o = 1$. Then 3.5 and 3.8 give the result. □

Note that, in general, fp^{-1} in (3.11) is not a group homomorphism.

If p is a fibration, then 3.10 is essentially 2.4 of [5]. Using the methods of [5] Mayer-Vietoris type sequences for homotopy pullbacks have been obtained in [26].

4. Applications to group theory

We start with an improved version of an example from elementary group theory taken from [25].

4.1. Example. Let G be a group and G_1 and G_2 subgroups of G. Consider the diagram of inclusions

$$(4.2) \qquad \begin{array}{ccc} & & G_2 \\ & & \downarrow \cap \\ G_1 & \xrightarrow[\subset]{} & G \end{array}$$

in the category of groupoids.

Let $q : Z \to G_1 \times G_2$ be the covering constructed as in 3.2. Let 1 denote the unique object of a group considered as a groupoid. Then $Ob(Z) = \{(1,g,1) \mid g \in G\} \cong G$ and $\tilde{g} = \tilde{g}'$ in $\pi_o Z$ if and only if there are elements $g_1 \in G_1$ and $g_2 \in G_2$ with $g = g_1^{-1} g' g_2$, showing $\pi_o Z$ is the set $G/G_1 \times G_2$ of double cosets. Finally $(g_1, g_2) \in Z(g)$ if and only if $g_1 = g g_2 g^{-1}$ and so the homomorphism $Z(g) \to G_2 \cap g^{-1} G_1 g$ given by $(g_1, g_2) \mapsto g_2$ is an isomorphism. Putting all this together with 3.3, for each $g \in G$ we obtain an exact sequence

$$1 \to G_2 \cap g^{-1} G_1 g \xrightarrow{\kappa_g} G_1 \times G_2 \xrightarrow{\partial_g} G \longrightarrow G/G_1 \times G_2 \to 1 ,$$

where $\kappa_g(g_2) = (g g_2 g^{-1}, g_2)$ and where $\partial_g(g_1, g_2) = g_1 g g_2^{-1}$.

We obtain the classification

$$G = \bigsqcup_g (G_1 \times G_2)/\kappa_g(G_2 \cap g^{-1}G_1 g) ,$$

where the disjoint union is taken over a complete set of representatives of the double cosets $G_1 g G_2$.

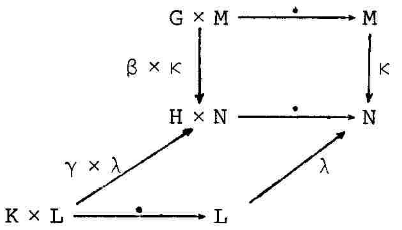
For finite G as in [25], we obtain the counting formula

$$o(G) = o(G_1)o(G_2) \sum_g \frac{1}{o(G_2 \cap g^{-1}G_1 g)} ,$$

where $o(G)$ denotes the order of G . □

We consider next operations of groups on sets. Consider the commutative diagram

(4.3)



of group actions, where $\beta : G \to H$ and $\gamma : K \to H$ are homomorphisms and $\kappa : M \to N$ and $\lambda : L \to N$ are maps.

Let $K \sqcap G$ and $L \sqcap M$ denote the pullbacks of $K \xrightarrow{\gamma} H \xleftarrow{\beta} G$ and $L \xrightarrow{\lambda} N \xleftarrow{\kappa} M$ respectively. We have generalizing 2.3 of [11]:

4.4. <u>Proposition.</u> (i) *The group* $K \sqcap G$ *operates on* $L \sqcap M$ by

$$(k,g) \cdot (l,m) = (k \cdot l, g \cdot m) .$$

(ii) *The pullback of the induced diagram of groupoid morphisms*

$$K \ltimes L \xrightarrow{\gamma \ltimes \lambda} H \ltimes N \xleftarrow{\beta \ltimes \kappa} G \ltimes M$$

is $(K \sqcap G) \ltimes (L \sqcap M)$. □

Here, $\beta \ltimes \kappa$, for example, is given by the map κ on objects and by $(\beta \ltimes \kappa)(g,m) = (\beta g, \kappa m)$ on morphisms.

4.5. Lemma. *If in the situation* (4.3) *the pair* (γ,β) *is surjective, then the pair* $(\gamma \ltimes \lambda, \beta \ltimes \kappa)$ *is surjective and hence* $(K \sqcap G) \ltimes (L \sqcap M)$ *is homotopy equivalent to the standard homotopy pullback of* $\gamma \ltimes \lambda$ *and* $\beta \ltimes \kappa$.

Proof. Let (l,h,m) be an object of the standard homotopy pullback for $\gamma \ltimes \lambda$ and $\beta \ltimes \kappa$. Thus $l \in L$, $h \in H$, and $m \in M$ with $h \cdot \kappa(m) = \lambda(l)$. Since $H = \gamma(K)\beta(G)$ then $h = \gamma(k)\beta(g)$ for some $k \in K$, $g \in G$. Now $\varepsilon = g : m \to g \cdot m$ and $\alpha = k : k^{-1} \cdot l \to l$ do the trick. □

4.6. Corollary. *If in the situation* (4.3) *the group* H *is the product set* $\gamma(K)\beta(G)$, *then for any* $(l,m) \in L \sqcap M$ *we have an exact sequence*

$$1 \to (K \sqcap G)(l,m) \to K(l) \times G(m) \xrightarrow{\gamma\beta^{-1}} H(n) \to L \sqcap M/K \sqcap G \to L/K \sqcap M/G \to 1 \ .$$

4.7. Example. Let G_1 and G_2 be subgroups of a group G such that $G = G_1G_2$. Let each group act on itself by conjugation. Then for each $g \in G_1 \cap G_2$ we have an exact sequence

$$1 \to C_{G_1 \cap G_2}(g) \to C_{G_1}(g) \times C_{G_2}(g) \to C_G(g) \to [G_1 \cap G_2] \to [G_1] \sqcap [G_2] \to 1,$$

where $C_G(g)$ is the centralizer of g in G and [ ] denotes the set of conjugacy classes. This sequence is of the type used in [24] for rationalization problems in group theory. □

We conclude with a special case of the main algebraic result of [12].

Recall that for a homomorphism $f : G \to G$ of a group G the *Reidemeister number* $R(f)$ of f is the number of equivalence classes of G under the relation $g \sim g'$ if and only if there is $g_1 \in G$ with $g = g_1 g' f(g_1)^{-1}$. As was shown in [10] $R(f)$ can be defined as the cardinality of the orbit set of the operation $G \times G \to G$ given by

$$(\alpha,\beta) \mapsto \alpha\beta f(\alpha)^{-1} \ .$$

The stability subgroup of G at $1 \in G$ is written Fix f .

Let $\beta : G \to H$ be a surjective homomorphism and $\gamma : K \to H$

any homomorphism. Suppose further $f_K : K \to K$, $f_H : H \to H$ and $f_G : G \to G$ are homomorphisms. Let $f_K \sqcap f_G$ denote the induced homomorphism on $K \sqcap G$.

4.8. Proposition. *If in the above situation* K,H *and* G *are abelian then the Reidemeister numbers of* f_K, f_H, f_G *and* $f_K \sqcap f_G$ *are related by the formula*

$$[\mathrm{Fix}\, f_H : (\gamma \mathrm{Fix}\, f_K)(\beta \mathrm{Fix}\, f_G)]\, R(f_K) R(f_G) = R(f_K \sqcap f_G) R(f_H) ,$$

where the first number denotes the number of double cosets of $\gamma \mathrm{Fix}\, f_K$ *and* $\beta \mathrm{Fix}\, f_G$ *in* $\mathrm{Fix}\, f_H$.

Proof. This follows from the interpretation of the exact sequence of groups (see [10],1.3)

$$1 \to ((K \sqcap G) \ltimes (K \sqcap G))(1) \to ((K \ltimes K) \times (G \ltimes G))(1) \to (H \ltimes H)(1) \to$$
$$\to \pi_0((K \sqcap G) \ltimes (K \sqcap G)) \to \pi_0(K \ltimes K) \sqcap \pi_0(G \ltimes G) \to 1$$

and easy calculation on this last set. □

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Ronald Brown

Department of Pure Mathematics
University College of North Wales
Bangor, Gwynedd, LL57 2UW

Philip R. Heath

Department of Mathematics
Memorial University of Newfoundland
St. John's, Newfoundland
Canada, A1C 5S7

Klaus Heiner Kamps

Fachbereich Mathematik
und Informatik
Fernuniversität
Postfach 940
D-5800 Hagen