Axiomathes
DOI 10.1007/s10516-007-9012-1

ORIGINAL PAPER

A Non-Abelian, Categorical Ontology of Spacetimes and Quantum Gravity

I. C. Baianu · R. Brown · J. F. Glazebrook

Received: 17 June 2007 / Accepted: 25 June 2007

Abstract A non-Abelian, Universal SpaceTime Ontology is introduced in terms of Categories, Functors, Natural Transformations, Higher Dimensional Algebra and the Theory of Levels. A Paradigm shift towards Non-Commutative Spacetime structures with remarkable asymmetries or broken symmetries, such as the CPT-symmetry violation, is proposed. This has the potential for novel applications of Higher Dimensional Algebra to SpaceTime structure determination in terms of universal, topological invariants of 'hidden' symmetry. Fundamental concepts of Quantum Algebra and Quantum Algebraic Topology, such as Quantum Groups, von Neumann and Hopf Algebras are first considered with a view to their possible extensions and future applications in Quantum Field theories. New, non-Abelian results may be obtained through Higher Homotopy, General van Kampen Theorems, Lie Groupoids/Algebroids and Groupoid Atlases, possibly with novel applications to Quantum Dynamics and Local-to-Global Problems, Quantum Logics and Logic Algebras. Many-valued Logics, Łukasiewicz–Moisil Logics lead to Generalized LM-Toposes as global representations of SpaceTime Structures in the presence of intense Quantum Gravitational Fields. Such novel representations have the potential to develop a Quantum/General Relativity Theory in the context of Supersymmetry, Supergravity, Supersymmetry Algebras and the Metric Superfield in the Planck

I. C. Baianu (✉)
FSHN and NPRE Departments, AFC-NMR and NIR Microspectroscopy Facility, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA
e-mail: ibaianu@uiuc.edu

R. Brown
School of Informatics, University of Wales, Dean St. Bangor, Gwynedd LL57 1UT, UK
e-mail: r.brown@bangor.ac.uk

J. F. Glazebrook
Department of Mathematics and Computer Science, Eastern Illinois University, 600 Lincoln Ave, Charleston, IL 61920-3099, USA
e-mail: jfglazebrook@eiu.edu

limit of spacetime. Quantum Gravity and Physical Cosmology issues are also considered here from the perspective of multiverses, thus leading also to novel types of Generalized, non-Abelian, Topological, Higher Homotopy Quantum Field Theories (HHQFT) and Non-Abelian Quantum Algebraic Topology (NA-QAT) theories.

Keywords
Universal SpaceTime Ontology, Axioms of Abelian Categories and Categorical Ontology/the Theory of Levels ·
The Paradigm Shift towards Non-Abelian Ontology, Relations and Abstract Structures ·
Commutativity vs. Non-Commutativity, Symmetry vs. Asymmetry, CPT Symmetry violation Quantum Field Theory and Quantum Dynamics ·
Quantum Logics and Logic Algebras, Many-valued Logics, Łukasiewicz–Moisil Logics and Generalized LM-Toposes (GLM) ·
Quantum Fields, General Relativity, Supersymmetry, Supergravity, and the Metric Superfield ·
Supersymmetry Algebras, Symmetry Breaking, and Quantum General Relativity ·
Quantum Gravity and Physical Cosmology, Non-Abelian Gauge Theories and Higgs bosons, Chronotopoids and Multiverses ·
Higher Dimensional Algebra (HDA) in SpaceTime Ontology, Higher Homotopy and General van Kampen Theorem (HHvKT) ·
Local-to-Global Problems and Combinations of Local Procedures (COLP), Lie Groupoids, Groupoid Atlases and Graded Lie Algebroids in Non-Linear Quantum Gravity ·
Fundamental Quantum Double Groupoids and Spacetime Topological Invariants

1 The Classification Problem for SpaceTime Structures

We shall consider first how the space and time concepts evolved, resulting in the joint concept of an objective *'spacetime'* in the physical Relativity theory, in spite of the distinct, human perceptions of space and time dimensions. Then, we shall proceed to define the role(s) played by the space, time and spacetime concepts in the broader context(s) of Categorical Ontology; this, in its turn, leads at a fundamental level to the consideration of basic, mathematical and physical, internal symmetries widely known as *'commutativity'* or *'naturality'*. Upon consideration of such basic, internal symmetry properties, it becomes apparent that a paradigm shift is now occurring in both mathematics and physics towards *non-commutative* concepts of space/spacetimes, that have also much wider implications for the complex systems encountered in biology, psychology, sociology and the environmental sciences. Such a paradigm shift has already begun as early as the birth of Quantum theories and Quantum Logic which are intrinsically non-commutative. Its implications are evident in the latest attempts in *'Quantum Gravity'* at unifying/reconciling Quantum Field theories with Relativistic theories of gravitation. It is here proposed that such theoretical developments of *non-commutative spacetime* concepts will also require a shift towards *non-commutative*, (or *non-Abelian*) extensions of Categorical Ontology.

1.1 The Evolution of the Space and Time Concepts

Already the computational machines of complexity serve as devices for the analytic study of space–time, matter and microsystems as the constituent parts of the universe. The subdivision of space is an ancient tool of mathematical thought which may be originally attributed to the early BC Greeks. More recent approaches seek out computational-logical mechanisms which fathom out deeper insight into the architecture of the universe.

As every algebraic topologist knows, the adding of cells to a space can radically alter its algebraic structure in terms of its homotopy type, its (co)homology groups, etc. Likewise a space is surgically treated by removal of some of its constituent parts. Spencer-Brown (1969) writes of 'universes' which result from the 'peeling off' and/or subdivision of spatial structures. Once achieved, there follows a system of logical associations framed under 'distinction' or other categorical attributes that crystallize into a Boolean-style logic enhanced by self-referencing (such as feedback) leading to analytical blueprints for various types of causal processes. It is a basic construction upon which more complex-interactive systems can be conceived. Relevant is an idea proposed in Quantum Gravity that the Big Bang, or Big-Bounce could have been an astronomically large phase transition by which a new region of space and time was created out of phase from that from which it originated (Smolin 2001). We shall discuss later how this opens up a pandora's box of logical questions.

How models of translated theories logically correspond to their originals, has been discussed by Manders (1982) who demonstrates an equivalence between postulating space–time as an infinite totality and formulations allowing only possible spatio-temporal relations of physical (point-) objects. On the basis that physical theories can be reformulated in terms of physically interpretable relational primitives, Manders (1982) proves that given a theory cast in terms of relational primitives, there is a translation into a theory about finite collections of objects (configurations) and relations among these, such that (i) any model of the translated theory determines a model of the original theory, and (ii) each model of the original theory is obtained in this way. Thus the translated theory is formally equivalent to the original theory.

Consider the causal structure of Minkowski space–time axiomatized in terms of the relation $\mathcal{S}$ on space–time points whereby a signal can be relayed from x_1 to x_2 at the velocity of light. In the translated model, individuals are *point events*, regarded as idealizations of ordinary physical events analogous to point-objects and ordinary objects ("items" in the sense of Baianu and Poli 2008). Configurations in this context are seen as finite collections of point-events with a corresponding relation to $\mathcal{S}$. The first interpretation conceives of configurations as possible states of affairs, in which $\mathcal{S}$ is valid, provided a signal could be relayed as above, $c_1 > c_2$ if c_1 contains the events in c_2 "at the same spatiotemporal location", so that $\mathcal{S}$ is valid of given events in c_2 if and only if it is valid between them in c_1. The second interpretation conceives of configurations as possible experimental outcomes or observations: $\mathcal{S}cxy$ is valid if in the experiment c a light signal is sent from x to y. For experiments c, c' with $c > c'$ if c contains all events in c', at the same spatiotemporal location, and for any x,y in c, if a light signal is observed from x to y in c', then it is also sent in c. But it is allowed that no signal from x to y be observed in c' and still one to be observed in c. Manders (1982) compares the examples closely to the Leibnitz construction of physical theories via physical (spatio-temporal) relations between physical objects, and thus avoiding the incorporation of infinite totalities of abstract entities such as Newtonian space.

For the purpose of this report these are patently separate issues peculiar to a different science. We seek a deeper meaning to Ontology by passing to geometry, topology and physics within the cosmological structure of space (and time) as governed by the strong/weak nuclear interactive forces, electromagnetism and gravity. As it is theoretically understood, gravitational fields can deflect light, thus creating a distortion of space along time and may cause the former to bend in on itself—a legacy of ongoing complexity afforded by the Big Bang theory. Since the speed of light surpasses any other known signal transmission, gravity thus determines the causal structure of the universe. Although enjoying such a grandiose property, gravity is in fact the weakest of the four forces (Hawking and Ellis 1973).

Space is conceived in terms of dimensions. But from the latter part of the 20th century onwards physicists have considered the possibility of more than three. For instance, Chodos and Detweiler (1980) suggested that at the origin of the universe several dimensions prevailed, but three in particular super-dominated at the cost of others having been microscopically relegated such that their symmetries were absorbed as those of elementary particles (presumably a brain-child anticipating string theory). Why indeed should three domineering spatial dimensions be attributed to 'absolute space' in any sense? After all, beyond the ancient Medieval idea of the universe formed by concentric spherical layers, space has long since been considered infinite. One may question what happens to a flying object on the boundary of space, and to where does it pass? The Big Bang theory has suggested that space may in fact be 'finite' and that the three (dominating) dimensions may be modeled by manifolds such as the 3-torus $\mathbb{T}^3$, thus constituting a flat (finite) universe periodic in three directions (since $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$). Such a manifold appears a better candidate than say, the 3-sphere, S^3, [1] or a three dimensional hyperbolic space.

In counterpoise, one may ask if established physical (or intellectual) concepts actually impose their own informational limitations on understanding the exact nature of space–time? Quite often the answer is 'yes', and the culprits may be found in a less than judicious choice of laboratory frames and coordinate systems for managing the task. This is apart from what sense one can make of 'dimension' in the

[1] At the time of writing there is a general consensus that the famous *Poincaré conjecture* which states that every orientable, simply connected, compact 3-manifold, is the 3-sphere, has now been proved by G. Perelman to be true. In other words, any 3-manifold that is topologically like the 3-sphere, *is* the 3-sphere.

first place: perhaps just a theoretical benchmark for the size of a suitable set of spanning vectors that supports a particular mechanism.

To most physicists these issues by today's standards seem relatively benign compared to the mysterious nature of black hole information and thermodynamics (see e.g., Unruh 2001), particularly for those speculators who regard black holes as channels into other universes. A typical example is *the Beckenstein bound* (Smolin 2001) that claims limits on information passing from one universe to another as dependent upon the dividing surface area (surprisingly, not upon the volume). An upshot is that it might be impossible to 'see' a black-hole in comparison to how one may 'see' a person riding a bicycle, since in the former case conscious perception is sold short through paucity of information, even though to an extent its existence can be detected through radiation. Likewise the symmetries of elementary particles are theoretically representable but their parent *hidden* dimensions remain up to now, precisely that. But drawing upon the Kaluza–Klein theories, these hidden dimensions are indeed those that may be 'replaced' by non-commutative structures.

Our viewpoint is that models constructed from category theory and higher dimensional algebra have potential applications towards creating a higher science of analogies which, in a descriptive sense, is capable of mapping imaginative subjectivity beyond conventional relations of complex systems. Of these, one may strongly consider a *generalized chronoidal-topos* notion that transcends the concepts of spatial-temporal geometry by incorporating non-commutative multi-valued logic. Current trends in the fundamentally new areas of quantum-gravity and string theories appear to endorse taking such a direction. We aim further to discuss some pre-requisite algebraic-topological and categorical ontology tools for this endeavor, however relegating all rigorous mathematical definitions to Brown et al. (2007).

1.2 Horizons, Singularities, Boundaries and Universes

As discussed by Poli (2008), the concept of $\langle$*horizon*$\rangle$ cannot be considered as a type of system because its essence is the absence of any *boundary*, with the latter being essential to defining any system, either closed or open, simple or complex, etc.; not even a flexible or permeable boundary can be chosen in a manner consistent with any 'horizon'. The claim is made in black hole 'theories' that such a 'singularity', mostly derived from General Relativity (GR) and certain recent cosmological-physical theories of 'our' Universe, has a horizon; a singularity is thought as a 'point' (thus, of zero dimension), or as a 'region outside' spacetime. However, the claim of a black hole 'horizon' in the sense discussed above appears to contradict both logic and the known laws of gravity; furthermore, it is not a true horizon in the sense discussed by Poli (2008) because it can be crossed in one direction-towards the singularity, and also less frequently in the opposite direction by virtual photons, that according to Hawking become real in a process of gradual energy 'leakage' from, or 'evaporation' of, the black hole. Black holes are thus thought to exist in regions that classically appear to be excluded from our spacetime Universe, and are not described by GR, in the sense that such invisible items would no longer be subject to our 'spacetime' postulates and laws as established by either classical GR or special relativity (SR). In fact, such mathematical 'singularities' appear even more artificial, or 'non-physical', in the Euclidean space of SR.

A mathematical singularity and its corresponding black hole is thought as being surrounded by an event (spherical, so-called) 'horizon' beyond which one cannot 'see' anything (hence the qualifier 'black' in their name, whereas the *'hole inside'* of such a *black hole horizon* (which is totally invisible for black holes 'without hair') is assumed in GR to have infinite density of both matter particles and energy. Therefore, within such a black hole, the gravitational field must also be considered by GR to be infinite in amplitude and intensity, with the consequence of there being no time evolution within the black hole, and also a curving of spacetime on itself; this may perhaps also imply either a total absorption of photons, with/without their re-emission inside the black hole, or else some kind of perpetual circling of photons within the black hole. Last-but-not least, the region inside/beyond the horizon of the black hole would have totally suspended any form of classical causality-it would be acausal, except perhaps for the notable exception of the infinite gravitational fields (within) that are responsible for the very unusual characteristics of the black hole. However, from a strictly mathematical point of view, the philosophical or mathematical possibility of the existence of an infinite gravitational field within the black hole also implies the existence of such an infinite gravitational field even outside the 'horizon' of the black hole, therefore, making the very existence of such a 'horizon' logically and physically inconsistent with the presence of such infinite gravitational fields. At this stage, there remain four major alternatives or options:

(A) GR (and also SR) is an incomplete theory that does not have the correct formulation for very intense gravitational fields;
(B) The 'singularities' predicted by GR do not have any physical existence as precisely predicted by the classical GR theory (which is also consistent with A);
(C) The gravitational fields of the black holes have very large but *finite* intensity within the 'horizon' and they also decrease as $1/r^2$ in the Universe 'surrounding' the black hole horizon;
(D) If option C. were incorrect, and black holes do indeed possess *infinite* gravitational fields, then the 'horizon'—as it is currently described by either classical GR or quantum theories—cannot exist, and the collapse, or 'Big Crunch', of our Universe surrounding such a black hole is inevitable! This latter possibility is not supported by any observations and is the most unlikely alternative of the four major options considered above.

We note, here, however, that the current concept of a black hole 'horizon' in Physical Cosmology acts more like a *'boundary'* than a true horizon as it separates our Universe from the 'inside' (in an abstract topological sense) of the black hole 'region', and it seems that this important attribute is pointed out here for the first time. Very recent observations of supernovae and quasars, as well as computations of supernovae explosions, (Scientific American, December 2006, issue) clearly visualize the black hole *boundary* by the surrounding hot plasma and symmetric high-energy plasma jets that accompany the supernova and the ejected, spherical neutron star 'core'.

On the other hand, certain prominent quantum theoretical approaches to the black hole 'problem' of Physical Cosmology/Astrophysics (such as those advanced for example by Hawking 2006)—that are obviously inconsistent with any 'classical' formulation of GR, including that of Einstein—predict a somewhat 'visible horizon' only for black holes 'with hair' from which some photons escape into our Universe, perhaps through a virtual process allowed by the Heisenberg Uncertainty Principle of Quantum Mechanics (which is however incompatible, and also not accepted, by any 'classical' GR or SR theory). In several hypothetical versions of Cosmology, black holes may be part of 'wormholes' connecting our known Universe with 'other, yet unknown Universes' (Hawking 2004). Furthermore, the view of Hawking for our Universe is that of a *finite* spacetime ('system') *with a boundary* (universal, or 'global', envelope).

1.3 The Two-Cycle Origin of our Universe and Matter–Antimatter Asymmetry: A Non-Commutativity Conjecture or Metaphor?

The following is just a conjecture or a metaphor for what might have happened before the 'Big-Bang' start of our inflationary Universe, a question that Steven Hawking and his fellow cosmologists, as well as some religions, decline to answer as 'meaningless'. Alternative propositions based on Q-logics and quantum axiomatics also exist but for the sake of brevity such non-standard Q-logic conjectures will not be addressed in this essay.

In the primeval, symmetric spacetime (perhaps a Kalutza-Klein 5D-ST) of the pre-universe—before a single time dimension could be at all preferred—there was *Commutativity*, maybe a 3D-sphere or tube with two added time dimensions/channels one for matter and the other for antimatter, generated by the quantum creation operators, C_M* for matter and C_{AM}* for antimatter, confined to a relatively small (but not point-like), real space region. Thus,

$$C_M^* \cdot C_{AM}^* = C_{AM}^* \cdot C_M^* \tag{1.1}$$

$$\nearrow \cdots \searrow$$

$$\nwarrow \cdots \swarrow$$

in this symmetric pre-Universe, equal amounts of matter attracted the same amounts of antimatter in the gravitationally curved, primeval, highly CPT symmetric, 5D spacetime region; they collided, annihilated, here, there, everywhere. This first cycle took virtually no time at all-as it run in loops, 'forward for matter' and backwards for antimatter in a very tiny space region—but generated lots of very short-wave energy. Thus 'ended' the first symmetric cycle and the existence of the symmetric pre-Universe through a terrific burst of energy from matter–antimatter annihilation. This huge energy started the second cycle with a huge 'Big-Bang', which unfolded the primeval small region spacetime dimensions of the curled up pre-Universe to four large/macroscopic dimensions of spacetime (and one 'hidden time dimension' of antimatter) which rapidly inflated (that keeps inflating) at the speed of light as the extremely hot energy bubble expanded in the ever decreasing curvature of spacetime. There was then asymmetry in spacetime and CPT symmetry-violation. This caused 'immediately' the basic *non-commutativity* of the matter–antimatter quantum creation operators:

$$C_M^* \cdot C_{AM}^* \neq C_{AM}^* \cdot C_M^* \tag{1.2}$$

$$\begin{array}{ccc} \nwarrow & \cdots\cdots\cdot & \nearrow \\ \\ \searrow & \cdots\cdots\cdot & \searrow \end{array}$$

which in its turn meant that more 'real matter' was being generated out of energy with time running 'forward' than antimatter with time running 'backwards' (as it was just in the beginning!).

To sum up this conjecture, a transition occurred from Cycle 1 to $\Longrightarrow$ Cycle 2:

$$\begin{array}{lcc} \nearrow & \cdots & \searrow \\ \\ \nwarrow & \cdots & \swarrow \\ \Downarrow & & \\ \nwarrow & \cdots\cdots\cdot & \nearrow \\ \\ \searrow & \cdots\cdots\cdot & \searrow \end{array}$$

Thus came into existence our *non-commutative, asymmetric spacetime Universe*, during the second cycle, with CPT symmetry violation and all that follows...

2 Categorical Ontology

General system analysis is currently leading to the development of a Categorical Ontology (Poli 2008, TAO-1). We shall therefore adopt here a categorical approach as we are looking for "*what is universal*" (in some domain, or in general), and that for simple systems this involves the consideration of commutative modeling diagrams and structures (Fig. 1 of Rosen 1987). Regarding the first property, the most universal feature of reality is that it is *temporal*, i.e. it changes, it is subject to countless transformations, movements, alterations. From the point of view of mathematical modeling, the mathematical theory of categories models the dynamical nature of reality by resorting to *variable* categories and *toposes*, including generalized ones such as those with many-valued Lukasiewicz–Moisil Logic Algebra as a Logic subobject. A previous claim advanced in several recent reports, such as those by Isham and Buttersham and Isham (2005) is that mainstream topos theory may also suit to a significant degree the needs of quantum gravity theory and other theoretical systems. These claims are discussed in further detail in our accompanying report on the ontology of space and time in complex systems (Baianu et al. 2007a, in this issue).

As *structures* and *relations* are present at the very core of mathematical developments (Ehresmann 1965, 1967), the theory of categories and toposes distinguishes two fundamental types of items: *objects* and *arrows* (also called suggestively *'morphisms'*). Thus, first-level arrows may represent mappings, relations, interactions, dynamic transformations, and so on, whereas categorical objects are usually endowed with a selected type of structure, unless they are simple sets (thus devoid of any internal structure). As explained next in Sect. 2.3, the second level arrows, or 2-arrows (*'functors'*) representing relations, or comparisons, between the first level categories do not 'look inside' the 1-objects, which may appear as a necessary limitation of the mathematical construction; however, the important ability to 'look inside' 1-objects at their structure, for example, is recovered by the third level arrows, or 3-arrows, called *'natural transformations'*. This seems also to provide an elegant formalization that matches the ontological theory of levels briefly described next in Sect. 2.3. The major restriction–as well as for some, attraction of the 3-level categorical construction outlined above seems to be its built-in *commutativity* (see also Sect. 2.3 for further details). Note also how 2-arrows become '3-objects' in the meta-category, or '3-category', of functors and natural transformations. This construction has already been considered to be suitable for representing dynamic processes in a generalized Quantum Field Theory.

2.1 Spacetime Structures as a Local-to-Global Problem: Homogeneity vs. Heterogeneity and Continuity vs. Discreteness

Summarizing in this paragraph the evolution of the physical concepts of space and time we are pointing out how the views changed from and homogeneity and continuity to inhomogeneity and discreteness. Physics, up to 1900's, involved a concept of both *continuous* and *homogeneous* space and time with strict causal (mechanistic) evolution of all physical processes ("*God does not play dice*", cf. Albert Einstein). Furthermore, up to the introduction of *quanta*–discrete portions, or packets–of energy by Ernst Planck (which was further elaborated by Einstein, Heisenberg, Dirac, Feynman, Weyl and other eminent physicists of the last century), energy was also considered to be a continuous function, though not homogeneously distributed in space and time. Einstein's Relativity theories joined together space and time into one 'new' entity–the concept of *spacetime*. Furthermore, in the improved form of General Relativity (GR), inhomogeneities were allowed to occur in spacetime caused by the presence of matter. Causality, however, remained strict, but also more complicated than in the Newtonian theories. Although Einstein's Relativity theories incorporate the concept of quantum of energy, or photon, into their basic structures, they also deny such discreteness to spacetime even though the discreteness of energy is obviously accepted within Relativity theories. The GR concept of spacetime being modified, or distorted, by matter goes further back to Riemann, but it was Einstein's GR theory that introduced the idea of representing gravitation as the result of spacetime distortion by matter. Implicitly, such spacetime distortions remained continuous even though the gravitational field energy—as all energy—was allowed to vary in discrete, albeit very tiny portions—the gravitational quanta. So far the detection of gravitons—the quanta of gravity—related to the spacetime distortions by matter—has been unsuccessful. Mathematically elegant/precise and physically 'validated' through several crucial experiments and astrophysical observations, Einstein's GR is obviously not reconcilable with Quantum theories. GR was designed as the large-scale theory of the Universe, whereas Quantum theories—at least in the beginning—were designed to address the problems of microphysical measurements at very tiny scales of space and time involving extremely small quanta of energy. Quantum theories were developed that are just as elegant mathematically as GR, and they were also physically 'validated' through numerous, extremely sensitive and carefully designed experiments. However, to date quantum theories have not been extended, or generalized, to a form capable of recovering the results of Einstein's GR.

2.2 Deterministic time—reversible versus probabilistic time—Irreversibility and its laws. Unitary or General Transformations?

A significant part of the scientific-philosophical work of Ilya Prigogine (see e.g. Prigogine 1980) has been devoted to the dynamical meaning of *irreversibility* expressed in terms of the second law of thermodynamics. For systems with strong enough instability of motion the concept of phase space trajectories is no longer meaningful and the dynamical description has to be replaced by the motion of distribution functions on the phase space. The viewpoint is that quantum theory produces a more coherent type of motion than in the classical setting and the quantum effects induce correlations between neighboring classical trajectories in phase space (which can be compared with the Bohr–Sommerfeld postulate of the image of phase cells having area $\hbar$). The idea of Prigogine (1980) is to associate a macroscopic entropy (or Lyapounov function) with a microscopic entropy super-operator M. Here the time–parametrized distribution function ρ_t are regarded as densities in phase space such that the inner product $\langle \rho_t, M\ \rho_t \rangle$ varies monotonously with t as the functions ρ_t evolve in accordance with Liouville's equation

$$\imath \frac{\partial \rho_t}{\partial t} = L \rho_t, \tag{2.1}$$

where L denotes *the Liouville (super) operator* (Prigogine 1980; Misra et al. 1979). In order to show that there are well defined systems for which the super-operators M exist, a time operator T ('age' or 'internal time') is introduced such that we have the 'uncertainty' relation

$$\imath [L, T] = \imath (LT - TL) = I. \tag{2.2}$$

The super-operators M may then be obtained as monotone positive operator functions of T, and under certain conditions may engender similarity transformations $\Lambda = M^{\frac{1}{2}}$ which convert the original deterministic evolution described by the Liouville equation into the stochastic evolution of a certain Markov process, and in this way the second law of thermodynamics can be expressed *via* the M super-operators (Misra et al. 1979). Furthermore, the equations of motion with randomness on the microscopic level then emerge as irreversibility on the macroscopic level. Unlike the usual quantum operators representing observables, the super-operators are non-Hermitian operators.

One also notes the possibility of 'contingent universes' with this 'probabilistic time' paradigm.

Now the requirement that the super-operator M increases monotonically with time is given by the following relation with the Hamiltonian of the system

$$\imath [H, M] = D \geq 0, \tag{2.3}$$

where D denotes an *(micro)-entropy operator* whose measurement is compatible with M, which implies the further (commutativity) equation $[M, D] = 0$. However, there are certain provisions have to be made in terms of the spectrum of the Hamiltonian H: if H has a pure point spectrum, then M does not exist, and likewise, if H has a continuous but bounded spectrum then M cannot exist. Thus, the super-operator M cannot exist in the case of only finitely extended systems containing only a finite number of particles. Furthermore, M does not admit a factorization in terms of self-adjoint operators A_1, A_2, or in other words $M\rho \neq A_1 \rho A_2$. Thus the super-operator M cannot preserve the class of 'pure states' since it is non-factorizable. The distinction between pure states (represented by vectors in a Hilbert space) and mixed states (represented by density operators) is thus lost in the process of measurement. In other words, the distinction between pure and mixed states is lost in a quantum system for which the algebra of observables can be extended to include a new dynamical variable representing the non-equilibrium entropy. In this way, one may formulate the second law of thermodynamics in terms of M for quantum mechanical systems.

Let us mention that the time operator T represents 'internal time' and the usual, 'secondary' time in quantum dynamics is regarded as an average over T. When T reduces to a trivial operator the usual concept of time is recovered $T \cdot \rho(x,y,z,t) = t \cdot \rho(x,y,z,t)$, and thus time in the usual sense is conceived as an average of the individual times as registered by the observer. Given the latter's ability to distinguish between future and past, a self-consistent scheme may be summarized in the following diagram (Prigogine 1980):

$$\begin{tikzcd} \text{Observer} \arrow[r] & \text{Dynamics} \arrow[d] \\ \text{Broken time symmetry} \arrow[u] & \text{Dissipative structures} \arrow[l] \end{tikzcd} \tag{2.4}$$

for which ‘irreversibility’ occurs as the intermediary stage in the sequence:

$$\text{Dynamics} \Rightarrow \text{Irreversibility} \Rightarrow \text{Dissipative structures}$$

Note however, that certain quantum theorists, as well as Einstein, regarded irreversibility of time as an ‘illusion’. Others—operating with minimal representations in quantum logic for finite quantum systems—go further still by denying that there is any need for real time to appear in the formulation of quantum theory.

2.3 Categorical Theory Levels vs. Generalized Non-Commutative Structures

One could formalize—for example as in Poli (2008)—the hierarchy of multiple-level relations and structures that are present in many types of systems in terms of the mathematical Theory of Categories, Functors and Natural Transformations (TC-FNT, see MacLane 2000). On the first level of such a hierarchy are the links between the system components represented as *‘morphisms’* of a structured category which are subject to several axioms/restrictions of Category Theory. Then, on the next, second level of the hierarchy one considers *‘functors’*, or links between such first level categories, that compare categories without ‘looking inside’ their objects/ system components. On the third level, one compares, or links, functors using *‘natural transformations’* in a 3-category (meta-category) of functors and natural transformations. At this level, natural transformations not only compare functors but also look inside the first level objects (system components) thus ‘closing’ the structure and establishing ‘the universal links’ between items as an integration of both first and second level links between items. The advantages of this constructive approach in the mathematical theory of categories, functors and natural transformations have been recognized since the beginnings of this mathematical theory in the seminal paper of Eilenberg and MacLane (1945). Such examples would be those of ‘*supergroups*’ in quantum field theory, 2-groupoids, or double groupoids of groups of items.

The hierarchy constructed above, up to level 3, can be further extended to higher, n-levels, always in a consistent, natural manner–which is *commutative on all levels*. This type of global, natural hierachy of items inspired by the mathematical TC-FNT has a kind of *internal symmetry* because at all levels, the link compositions are *natural*, that is, all link compositions that exist are strictly ordered, i.e., $x < y$ and $y < z \Longrightarrow x < z$, or $f : x \longrightarrow y$ and $g : y \longrightarrow z \Longrightarrow h : x \longrightarrow z$, yielding a composition $h = g \circ f$. The general property of such link composition chains or diagrams involving any number of sequential links on different paths to the same object is called *commutativity*, or the *naturality condition*. This key mathematical property also includes the mirror-like symmetry $x \star y = y \star x$ when x and y are operators and the symbol '$\star$' represents the operator multiplication. Then, the equality of $x \star y$ with $x \star y$ implies that the x and y operators *commute*; in the case of an eigenvalue problem involving such commuting operators in quantum theories, the two operators would share the same system of eigenvalues, thus leading to 'equivalent' numerical results. This is very convenient for both mathematical and physical applications (such as those encountered in quantum mechanics). Unfortunately, not all operators 'commute', and not all mathematical structures are commutative.

The more general case is, however, the *non-commutative* one. Moreover, one is used to considering– not only in the sciences but also in the visual arts–that things which are 'beautiful' must be *symmetric*, perhaps with the possible exception of certain abstract paintings. Furthermore, the high school and college educational systems have over-emphasized/are emphasizing in both mathematics and physics the older concepts of space, either Euclidean or the simplest Riemannian, that have associated with them *commutative* algebraic structures, specific *Abelian* groups. The theory of *Lie groups* (Chevalley 1946) provides some necessary insight. The spaces Euclidean $\mathbb{R}^n$ and the n-torus $\mathbb{T}^n$ are 'commutative' spaces in the sense that they are Abelian Lie groups, and each acts upon itself in a commutative way (by definition). Whereas the Abelian Lie groups can be considered as 'flat', non-Abelian Lie groups can be viewed as the most basic Riemannian manifolds having non-trivial curvature properties. Other standard space forms are representable in the quotient form G/K where G is a Lie group and $K \subset G$ is a closed subgroup, that is, as *homogeneous spaces* usually with the extra property of symmetry (thus *symmetric spaces*). The n–sphere S^n, for instance is such a symmetric space, but in the traditional Riemannian–geometric sense it is not normally considered as a 'non-commutative space' unless it is 'quantized' by some means (à la Connes 1994), and that is indeed a separate matter which we shall bring to the fore later in this report.

One is thus often prejudiced quite heavily in favor of commutative structures and the 'working mathematician' only deals with Abelian theories that rely heavily on 'pictorial' representations which are either attractive, seductive, or 'beautiful', but not necessarily true to the subject of such representations– the real spacetime in our universe. Not so, however, is the case of theoretical physicists developing quantum gravity involving *non-Abelian* gauge formulations. An example of a widely known *non-commutative* structure relevant to Quantum Theory is that of the *Clifford algebra* of quantum observable operators (Dirac 1962; see also the precise definition of the Clifford algebra in Brown et al. 2007); yet another, more recent and popular, example is that of C*-algebras of (quantum) Hilbert spaces. Last-but-not least, there are the interesting mathematical constructions of non-commutative 'geometric spaces' obtained by 'deformation' introduced by Connes (1994) as possible models for the physical, quantum space–time which will be further discussed in Sect. 6.1. Thus, the microscopic, or quantum, 'first' level of physical reality does not appear to be subject to the categorical naturality conditions of Abelian TC-FNT—the 'standard' mathematical theory of categories (functors and natural transformations). It would seem therefore that the commutative hierarchy discussed above is not sufficient for the purpose of a General, Categorical Ontology which considers all items, at all levels of reality, including those on the 'first', quantum level, which is non-commutative. On the other hand, the mathematical, Non-Abelian Algebraic Topology (Brown et al. 2008), the Non-Abelian Quantum Algebraic Topology (NA-QAT; Baianu et al. 2004; Brown et al. 2007), and physical, Non-Abelian Gauge theories (NAGTs) may provide the ingredients for a proper foundation for Non-Abelian, hierarchical multi-level theories of super-complex system dynamics in a General Categorical Ontology (GCO). Furthermore, it was recently pointed out (Baianu et al. 2004, 2006; Brown et al. 2007) that the current and future development of NA-QAT involve *a fortiori non-commutative*, many-valued logics of quantum events, such as the Łukasiewicz–Moisil (LMV) logic algebra, complete with a fully-developed, novel probability measure theory grounded in the LMV-logic algebra (Georgescu 2006). The latter paves the way to a new projection operator theory founded upon the (non-commutative) quantum logic of events, or dynamic processes, thus opening the possibility of a complete, Non-Abelian Quantum theory. Furthermore, such recent developments point towards a paradigm shift in Categorical Ontology and to its extension to more general, Non-Abelian theories, well beyond the bounds of commutative structures/spaces and also free from the restrictions and limitations imposed by the Axiom of Choice to Set Theory.

3 Measurement Theories for Quantum Systems

3.1 Measurements and Phase–Space

We have already mentioned the issue of quantum measurement and now we offer a sketch of the background to its origins and where it may lead. Firstly, the question of measurement in quantum mechanics (QM) and quantum field theory (QFT) has flourished for about 75 years. The intellectual stakes have been dramatically high, and the problem rattled the development of 20th (and 21st) century physics at the foundations. Up to 1955, Bohr's Copenhagen school dominated the terms and practice of quantum mechanics having reached (partially) eye-to-eye with Heisenberg on empirical grounds, although not the case with Einstein who was firmly opposed on grounds of incompleteness with respect to physical reality. Even to the present day, the hard philosophy of this school is respected throughout most of theoretical physics. On the other hand, post 1955, the measurement problem received a precise form when von Neumann's beautifully formulated QM in the mathematically rigorous context of Hilbert spaces of states. As Birkhoff and von Neumann (1936) remark:

> There is one concept which quantum theory shares alike with classical mechanics and classical electrodynamics. This is the concept of a mathematical "phase–space". According to this concept, any physical system $\mathfrak{C}$ is at each instant hypothetically associated with a "point" in a fixed phase–space Σ; this point is supposed to represent mathematically, the "state" of $\mathfrak{C}$, and the "state" of $\mathfrak{C}$ is supposed to be ascertained by "maximal" observations.

In this respect, *pure states* are considered as maximal amounts of information about the system, such as in standard representations using *position–momenta* coordinates (Dalla Chiara et al. 2004).

The concept of 'measurement' has been argued to involve the influence of the Schrödinger equation for time evolution of the wave function ψ, so leading to the notion of entanglement of states and the indeterministic reduction of the wave packet. Once ψ is determined it is possible to compute the probability of measurable outcomes; at the same time, modifying ψ relative to the probabilities of outcomes and observations eventually causes the 'collapse of the wave function'. The well–known paradox of Schrödinger's cat and the Einstein–Podolsky–Rosen (EPR) 'experiment' are questions mooted once dependence on reduction of the wave packet is jettisoned, but then other interesting paradoxes have shown their faces. Consequently, QM opened the door to other interpretations such as 'the hidden variables' and the Everett–Wheeler assigned measurement within different worlds, theories not without their respective shortcomings. In recent years some countenance has been shown towards Cramer's 'advanced-retarded waves' transactional formulation (Cramer 1980) where $\psi\psi^*$ corresponds to a probability that a wave transaction has been finalized ('the quantum handshake').

Let us now turn to another facet of quantum measurement. Note firstly that QFT pure states resist description in terms of field configurations since the former are not always physically interpretable. Algebraic quantum field theory (AQFT) as expounded by Roberts (2004) points to various questions raised by considering theories of (unbounded) operator-valued distributions and nets of von Neumann algebras. Using in part a gauge theoretic approach, the idea is to regard two field theories as equivalent when their associated nets of observables are isomorphic. More specifically, AQFT considers taking (additive) nets of field algebras $\mathcal{O} \longrightarrow \mathcal{F}(\mathcal{O})$ over subsets of Minkowski space, which among other properties, enjoy Bose–Fermi commutation relations. Although at first glances there may be analogies with sheaf theory, these analogies are severely limited. The typical net does not give rise to a presheaf because the relevant morphisms are in reverse. Closer then, is to regard a net as a precosheaf, but then the additivity does not allow proceeding to a cosheaf structure. This may reflect upon some incompatibility of AQFT with those aspects of quantum gravity (QG) where for example sheaf-theoretic/topos approaches are advocated (as in e.g. Butterfield and Isham 1999, 2004).

3.2 The Kochen–Specker (KS) Theorem

Arm-in-arm with the measurement problem goes a problem of 'the right logic', for quantum mechanical/complex biological systems and quantum gravity. It is well–known that classical Boolean truth-valued logics are patently inadequate for quantum theory. Logical theories founded on projections and self-adjoint operators on Hilbert space H do run into certain problems. One 'no-go' theorem is that of *Kochen–Specker* (KS) which for dim $H > 2$, does not permit a global evaluation on a Boolean system of 'truth values'. In Butterfield and Isham (1999)–(2004), self-adjoint operators on *H* with purely discrete spectrum were considered. The KS theorem is then interpreted as saying that a particular presheaf does not admit a global section. Partial valuations corresponding to local sections of this presheaf are introduced, and then generalized evaluations are defined. The latter enjoy the structure of a Heyting algebra and so comprise an intuitionistic logic. Truth values are describable in terms of sieve–valued maps, and the *generalized evaluations* are identified as *subobjects in a topos*. The further relationship with interval valuations motivates associating to the presheaf a von Neumann algebra where the supports of states on the algebra determines this relationship. Unfortunately, the Heyting logic algebra is *commutative* and thus inconsistent with Quantum Logics.

The above considerations lead directly to the organization of the next four sections which proceeds from linking quantum measurements with *Quantum Logics*, and then to the *construction* of spacetime structures on the basis of Quantum Algebra/Algebraic Quantum Field Theory (AQFT) concepts presented in Sect. 4; such constructions of QST representations in Sects. 4 and 7 are based on the existing QA, AQFT and Algebraic Topology concepts, as well as several new QAT concepts that are being developed in this paper. The quantum algebras that are precisely defined in Sect. 4 have corresponding, 'dual' quantum state spaces that are concisely discussed in Sect. 7. (For the QSS detailed properties, and also the rigorous proofs of such properties, the reader is referred to the recent book by Alfsen and Schultz 2003). Then, we utilize in Sects. 7–9 a significant amount of recently developed results in Algebraic Topology (AT), such as for example, the Generalized van Kampen theorem (GvKT) (see the relevant subsection in Brown et al. (2007) for further mathematical details) to illustrate how constructions of QSS and QST, *non-Abelian* representations can be either generalized or extended on the basis of GvKT. We also employ the categorical form of the CW–complex Approximation (CWA) theorem) in Sect. 7 to both systematically construct such generalized representations of quantum space–time and provide, together with GvKT, the principle methods for determining the general form of the fundamental *algebraic invariants* of their *local or global*, topological structures. The algebraic invariant of Quantum Loop (such as, the graviton) Topology in QST is defined in Sect. 8.7 as the Quantum Fundamental Groupoid (QFG) of QST which can be then calculated– at least in principle – with the help of AT fundamental theorems, such as GvKT, especially for the relevant case of spacetime representations in *non-commutative* algebraic topology.

Several competing, tentative but promising, frameworks were recently proposed in terms of categories and the 'standard' topos for Quantum, Classical and Relativistic observation processes. These represent important steps towards developing a Unified Theory of Quantum Gravity, especially in the context-dependent measurement approach to Quantum Gravity (Isham 1998; Isham and Butterfield, 1999, Isham 2003). The possibility of a unified theory of measurement was suggested in the context of both classical, Newtonian systems and quantum gravity (Isham 1998; Isham and Butterfield 1999; Butterfield and Isham 1999). From this standpoint, Isham and Butterfield (1997, 1999) proposed to utilize the concept of 'standard' topos (MacLane and Moerdijk 1996) for further development of an unified measurement theory and quantum gravity (see also, Butterfield and Isham 1999 for the broader aspects of this approach). Previous and current approaches to quantum gravity in terms of categories and higher dimensional algebra (especially, 2-categories) by John Baez (1998, 2000, 2002) should also be mentioned in this context. Furthermore, time—as in Minkowski 'spacetime'—is not included in this mathematical concept of "most general space" and, therefore, from the beginning such quantum gravity theories appear to be heavily skewed in favor of the quantum aspects, at the expense of time as considered in the space–time of general relativity theory.

The first choice of logic in such a general framework for quantum gravity and context-dependent measurement theories was intuitionistic related to the set-theoretic and presheaf constructions utilized for a context-dependent valuation theory (Isham 1998, 2003). The attraction, of course, comes from the fact that a topos is arguably a very general, mathematical model of a 'generalized space' that involves an intuitionistic logic algebra in the form of a special distributive lattice called a *Heyting Logic Algebra*, as further discussed in the next section.

4 Quantum Algebra and Quantum Algebraic Topology: C*-Convolution Algebroids, von Neumann Algebra, Symmetry and Quantum Groups

4.1 Quantum Effects

Let $\mathcal{H}$ be a (complex) Hilbert space (with inner product denoted $\langle , \rangle$) and $\mathcal{L}(\mathcal{H})$ the bounded linear operators on $\mathcal{H}$. We place a natural *partial ordering* "$\leq$" on $\mathcal{L}(\mathcal{H})$ by $S \leq T$ if

$$\langle S\psi, \psi \rangle \leq \langle T\psi, \psi \rangle, \text{ for all } \psi \in \mathcal{H}.$$

In the terminology of Gudder (2004), an operator $A \in \mathcal{H}$ is said to represent a *quantum effect* if $0 \leq A \leq I$. Let $\mathcal{E}(\mathcal{H})$ denote the set of quantum effects on $\mathcal{H}$. Next, let

$$P(\mathcal{H}) = \{P \in \mathcal{L}(\mathcal{H}) : P^2 = P, \, P = P^*\},$$

denote the space of projection operators on $\mathcal{H}$. The space $P(\mathcal{H}) \subseteq \mathcal{E}(\mathcal{H})$ constitute the *sharp quantum effects* on $\mathcal{H}$. Likewise a natural partial ordering "$\leq$" can be placed on $P(\mathcal{H})$ by defining $P \leq Q$ if $PQ = P$.

A *quantum state* is specified in terms of a probability measure $m : P(\mathcal{H}) \longrightarrow [0, 1]$, where $m(I) = 1$ and if P_i are mutually orthogonal, then $m(\sum P_i) = \sum m(P_i)$. The corresponding quantum probabilities and stochastic processes, may be either "sharp" or "fuzzy". A brief mathematical formulation following Gudder (2004) accounts for these distinctions as it will be explained next.

Let $\mathcal{A}(\mathcal{H})$ be a σ–algebra generated by open sets and consider the *pure states* as denoted by $\Omega(\mathcal{H}) = \{\omega \in \mathcal{H} : \|\omega\| = 1\}$. We have then relative to the latter an *effects space* $\mathcal{E}(\Omega(\mathcal{H}), \mathcal{A}(\mathcal{H}))$ less "sharp" than the space of projections $P(\mathcal{H})$ and thus comprising an entity which is "*fuzzy*" in nature. For a given *unitary operator* $U : \mathcal{H} \longrightarrow \mathcal{H}$, a *sharp observable* X_U is expressed abstractly by a map

$$X_U : \mathcal{A}(\mathcal{H}) \longrightarrow \mathcal{E}(\Omega(\mathcal{H}), \mathcal{A}(\mathcal{H})),$$

for which $X_U(A) = I_{U^{-1}(A)}$.

Suppose then we have a *dynamical group* $(t \in \mathbb{R})$ satisfying $U(s + t) = U(s)\,U(t)$, such as in the case $U(t) = \exp(-it\,H)$ where H denotes the energy operator of Schrödinger's equation. Such a group of operators extends X_U as above to a *fuzzy (quantum) stochastic process*

$$\tilde{X}_{U(t)} : \mathcal{A}(\mathcal{H}) \longrightarrow \mathcal{E}(\Omega(\mathcal{H}), \mathcal{A}(\mathcal{H})).$$

One can thus define classes of *analogous* quantum processes with 'similar' dynamic behaviors (see also our discussion in Sect. 7) by employing dynamical group *isomorphisms*, whereas comparisons between dissimilar quantum processes could be represented by dynamical group *homomorphisms*. In particular, the interactions between a classical dynamic system–the measurement instrument plus the observer, and the observed quantum dynamic system poses a special problem in quantum theories that cannot be resolved without the consideration of the Quantum Logic (QL) of events. As discussed in Sect. 3, for example, in connection with the K–S theorem, the need for a logical re-evaluation exists both for ontological and operational reasons. Such a logical formulation for quantum measurements is discussed next.

4.2 Multiple Quantum Symmetries and Representations by C*-Convolution Algebroids

Formulations involving symmetry in Quantum Mechanics (QM) and Quantum Field Theory (QFT) have been extremely fruitful in solving a wide range of important microphysical problems. As a direct, novel application of extended symmetry to microphysics we shall consider its implications for the general theory of scattering by partially ordered, atomic or molecular, structures which was first formulated in a complete, analytical form by Hosemann and Bagchi (1964) in terms of *'paracrystals'* and *lattice convolutions*. Thus, a natural generalization of such extended symmetries and their corresponding analytical version in this previous convolution algebra–based theory will be here presented in terms of our more general, novel concept of a *convolution-algebroid* of an extended symmetry groupoid of a paracrystalline lattice, or indeed, of any molecular system that has a partially disordered/ordered structure. The notion of the C^*-algebra of a (discrete) group is well known. The underlying vector space is that of complex valued functions with finite support, and the multiplication of the algebra is the so called *convolution product* which it is convenient for our purposes to write slightly differently from the common formula as

$$(f*g)(z) = \sum_{xy=z} f(x)g(y). \tag{4.1}$$

and *-operation

$$f^*(x) = \overline{f(x^{-1})}. \tag{4.2}$$

(The more usual expression of the formula (4.1) has a sum over the elements of the group.) For *topological* groups, where the underlying vector space is of continuous complex valued functions, this *convolution product* requires the availability of some structure of *measure* and of *measurable functions*, with the sum replaced by an integral. (Notice that this algebra has an identity, the function δ_1, which has value 1 on the identity 1 of the group, and has zero value elsewhere.)

On the other hand, post 1955, quantum theories adopted a new lease of life when von Neumann's beautifully formulated QM in the mathematically rigorous context of Hilbert spaces. We shall first recall the basic definitions of *von Neumann Algebras*, *Quantum Groups* and *(quantum) Hopf Algebras*. Then, we shall proceed to relate these mainly algebraic concepts to symmetry and also consider their extensions in the context of *Local Quantum Physics, broken symmetries* and *Quantum Field Theory*. The extension to *supersymmetry* leads then to superalgebra, superfield symmetries and their involvement in supergravity or Quantum Gravity theories for intense gravitational fields in fluctuating, quantized spacetimes.

4.3 Von Neumann Algebra

Definition 1.1 Let $\boldsymbol{H}$ be a complex Hilbert space. A *von Neumann algebra V* acting on $\boldsymbol{H}$ is a subset of the algebra $\mathcal{L}(\boldsymbol{H})$ of all bounded operators on $\boldsymbol{H}$ such that:

1. V is closed under the adjoint operation (with the adjoint of A denoted by A*);
2. V equals its *bicommutant*, namely:

$$V'' = \{A \in \mathcal{L}(H) : \forall \mathbf{B} \in \mathcal{L}(H) \& \forall C \in V, (BC = CB) \Rightarrow (AB = BA)\} = V.$$

If one calls a *commutant* of a set V the special set of bounded operators on $\mathcal{L}(\boldsymbol{H})$which commute with all elements in V, then this second condition implies that the commutant of the commutant of V is still V.

On the other hand, a von Neumann algebra V inherits a *unital* subalgebra from ***B(H)***, and according to the first condition in its definition V does indeed inherit a *-*subalgebra* structure, as further explained in the next section on C*-algebras. Furthermore, one can prove the famous, Von Neumann's *bicommutant theorem* which states that:

*V is a von Neumann algebra if and only if V is a *-subalgebra of* ***B(H)***, *closed for the smallest topology defined by continuous maps* $(\xi, \eta) \longmapsto (A\xi, \eta)$ *for all* $< A\xi,\eta) >$ *in* $\boldsymbol{H}^2$, *where* $< .,. >$ *is the inner product defined on* $\boldsymbol{H}$.

There are several types of *quantization* procedures that lead to distinct algebraic structures and approaches to extending either 'standard' Quantum Mechanics (QM) or Quantum Field Theory (QFT). We shall begin by considering quantum systems with a finite number of degrees of freedom (QM), and then proceed to consider quantum systems with an infinite number of degrees of freedom (QFT). QM utilizes mostly *Type II* subfactors of von Neumann algebras, whereas the *Type* III_1 subfactors are those encountered in QFT and Local Quantum Physics which are subject to 'extended' symmetries that require more general, geometric, topological and algebraic structures, such as those of quantum groupoids and their C*-convolution algebroids (introduced here for the first time).

4.4 Quantum Groups and Hopf Algebras

We shall consider first the relationship between quantum groups and their Hopf algebra representations.

A *quantum group* is often realized as an *automorphism group* for a quantum space, that is, an object in a suitable category of generally *non-commutative* algebras. The most frequent representation of a quantum group is as the *dual* of a non-commutative, non-associative Hopf algebra. Therefore, we commence here by establishing the concept of *Hopf algebras* as essential building blocks in Quantum Mechanics. For further details we also refer to Chaician and Demichev (1996), and Magid (1995).

4.4.1 Hopf Algebras

Our development here follows Alfsen and Schultz (2003), and Landsman (1998). Firstly, an *algebra* consists of a vector space E over a ground field (typically $\mathbb{R}$ or $\mathbb{C}$) equipped with a bilinear and distributive multiplication $\circ$. Note that E is not necessarily commutative or associative.

Secondly, a unital associative algebra consists of a linear space A together with two linear maps

$$
\begin{aligned}
m &: A \otimes A \longrightarrow A, \ (\textit{multiplication}) \\
\eta &: \mathbb{C} \longrightarrow A, \ (\textit{unity})
\end{aligned}
$$

satisfying the conditions

$$
\begin{aligned}
m(m \otimes \mathbf{1}) &= m(\mathbf{1} \otimes \mathbf{m}) \\
m(\mathbf{1} \otimes \eta) &= m(\eta \otimes \mathbf{1}) = \mathrm{id}.
\end{aligned}
$$

This first condition can be seen in terms of a commuting diagram:

$$
\begin{array}{ccc}
A \otimes A \otimes A & \xrightarrow{m \otimes \mathrm{id}} & A \otimes A \\
{\scriptstyle \mathrm{id} \otimes m} \big\downarrow & & \big\downarrow {\scriptstyle m} \\
A \otimes A & \xrightarrow{m} & A
\end{array}
$$

Next suppose we consider 'reversing the arrows', and take an algebra A equipped with a linear homorphisms $\Delta : A \longrightarrow A \otimes A$, satisfying, for $a,b \in A$:

$$\Delta(ab) = \Delta(a)\Delta(b)$$
$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta.$$

We call Δ a *comultiplication*, which is said to be *coassociative* in so far that the diagram

$$\begin{tikzcd}
A \otimes A \otimes A & A \otimes A \arrow[l, "\Delta\otimes\mathrm{id}"'] \\
A \otimes A \arrow[u, "\mathrm{id}\otimes\Delta"] & A \arrow[l, "\Delta"'] \arrow[u, "\Delta"']
\end{tikzcd}$$

commutes. There is also a counterpart to η, the *counity* map $\varepsilon : A \longrightarrow \mathbb{C}$ satisfying

$$(\mathrm{id} \otimes \varepsilon) \circ \Delta = (\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id}.$$

A *bialgebra* $(A, m, \Delta, \eta, \varepsilon)$ is a linear space A with maps m, Δ, η, ε satisfying the above properties.

Now to recover anything resembling a group structure, we must append such a bialgebra with an antihomomorphism $S : A \longrightarrow A$, satisfying $S(ab) = S(b)\, S(a)$, for $a,b \in A$. This map is defined implicitly via the property:

$$m(S \otimes \mathrm{id}) \circ \Delta = m(\mathrm{id} \otimes S) \circ \Delta = \eta \circ \varepsilon.$$

We call S the *antipode map*.

A *Hopf algebra* is a bialgebra $(A, m, \eta, \Delta, \varepsilon)$ equipped with an antipode map S.

Non-commutative Hopf algebras are representations of quantum groups which are essential to the generalizations of the concept of symmetry. Indeed, in many respects a quantum group is implicit in a Hopf algebra structure. When such algebras are associated to matrix groups there is considerable scope for representations on both finite and infinite dimensional Hilbert spaces. Analogous to how quantum mechanics relates to the classical limit, quantum groups can be seen to be related to (classical) Lie groups. Their mainstream applications are directed towards such areas as statistical mechanics, conformal field theory, the theory of knots and braids.

4.5 Jordan–Banach and JBL Algebras

A *Jordan algebra* (over $\mathbb{R}$), is an algebra over $\mathbb{R}$ for which

$$S \circ T = T \circ S$$
$$S \circ (T \circ S^2) = (S \circ T) \circ S^2$$

for all elements S, T of the algebra.

It is worthwhile remarking now that in the algebraic theory of Jordan algebras, an important role is played by the *Jordan triple product* $\{STW\}$ as defined by

$$\{STW\} = (S \circ T) \circ W + (T \circ W) \circ S - (S \circ W) \circ T,$$

which is linear in each factor and for which $\{STW\} = \{WTS\}$. Certain examples entail setting $\{STW\} = \frac{1}{2}\{STW + WTS\}$.

A *Jordan–Lie algebra* is a real vector space $\mathfrak{A}_{\mathbb{R}}$ together with a *Jordan product* $\circ$ and *Poisson bracket*, satisfying:

1. for all $S, T \in \mathfrak{A}_{\mathbb{R}}$,

$$S \circ T = T \circ S$$
$$\{S, T\} = -\{T, S\}$$

2. the *Leibniz rule* holds

$$\{S, T \circ W\} = \{S, T\} \circ W + T \circ \{S, W\}$$

for all $S, T, W \in \mathfrak{A}_{\mathbb{R}}$, along with

3. the *Jacobi identity*:

$$\{S, \{T, W\}\} = \{\{S, T\}, W\} + \{T, \{S, W\}\}$$

4. for some $\hbar^2 \in \mathbb{R}$, there is the *associator identity*:

$$(S \circ T) \circ W - S \circ (T \circ W) = \frac{1}{4}\hbar^2\{\{S, W\}, T\}.$$

A *Jordan–Banach algebra* (a JB–algebra for short) is both a real Jordan algebra and a Banach space, where for all $S, T \in \mathfrak{A}_{\mathbb{R}}$, we have

$$\|S \circ T\| \leq \|S\|\|T\|$$
$$\|T\|^2 \leq \|S^2 + T^2\|.$$

A *JLB–algebra* is a JB–algebra $\mathfrak{A}_{\mathbb{R}}$ together with a Poisson bracket for which it becomes a Jordan–Lie algebra for some $\hbar^2 \geq 0$. Such JLB–algebras often constitute the real part of several widely studied complex associative algebras.

4.6 Poisson Algebra

By a *Poisson algebra* we mean a Jordan algebra in which $\circ$ is associative. The usual algebraic types of morphisms (automorphism, isomorphism, etc.) apply to Jordan–Lie (Poisson) algebras (see Landsmann 1998).

Consider the classical configuration space $Q = \mathbb{R}^3$ of a moving particle whose phase space is the cotangent bundle $T^*\mathbb{R}^3 \cong \mathbb{R}^6$, and for which the space of (classical) observables is taken to be the real vector space of smooth functions $\mathfrak{A}^0_{\mathbb{R}} = C^\infty(T^*R^3, \mathbb{R})$. The usual pointwise multiplication of functions fg defines a bilinear map on $\mathfrak{A}_{\mathbb{R}}$, which is seen to be commutative and associative. Further, the Poisson bracket on functions is defined as:

$$\{f, g\} := \frac{\partial f}{\partial p^i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p^i},$$

which can be easily seen to satisfy the Liebniz rule above. The axioms above then set the stage of passage to quantum mechanical systems which the parameter $\hbar^2$ suggests.

4.7 C*-algebras (C*-A), JBW- and JC- Algebras

An *involution* on a complex algebra $\mathfrak{A}$ is a real-linear map $T \mapsto T^*$ such that for all $S, T \in \mathfrak{A}$ and $\lambda \in \mathbb{C}$, we have

$$T^{**} = T, (ST)^* = T^*S^*, (\lambda T)^* = \bar{\lambda}T^*.$$

A **-algebra* is said to be a complex associative algebra together with an involution *.

A *C*-algebra* is a simultaneously a *-algebra and a Banach space $\mathfrak{A}$, satisfying for all $S, T \in \mathfrak{A}$

$$\|S \circ T\| \leq \|S\| \, \|T\|$$
$$\|T^*T\|^2 = \|T\|^2.$$

We can easily see that $\|A^*\| = \|A\|$. By the above axioms a C*-algebra is a special case of a Banach algebra where the latter requires the above norm property but not the involution (*) property. Given Banach spaces E, F the space $\mathcal{L}(E, F)$ of (bounded) linear operators from E to F forms a Banach space, where for $E = F$, the space $\mathcal{L}(E) = \mathcal{L}(E, E)$ is a Banach algebra with respect to the norm

$$\|T\| := \sup\{\|Tu\| : u \in E, \ \|u\| = 1\}.$$

In quantum field theory one may start with a Hilbert space H, and consider the Banach algebra of bounded linear operators $\mathcal{L}(H)$ which given to be closed under the usual algebraic operations and taking adjoints, forms a *-algebra of bounded operators, where the adjoint operation functions as the involution, and for $T \in \mathcal{L}(H)$ we have:

$$\|T\| := \sup\{(Tu, Tu) : u \in H, \ (u, u) = 1\},$$

and

$$\|Tu\|^2 = (Tu, Tu) = (u, T^*Tu) \leq \|T^*T\| \|u\|^2.$$

By a morphism between C*-algebras $\mathfrak{A}, \mathfrak{B}$ we mean a linear map $\phi : \mathfrak{A} \longrightarrow \mathfrak{B}$, such that for all $S, T \in \mathfrak{A}$, the following hold:

$$\phi(ST) = \phi(S)\phi(T), \ \phi(T^*) = \phi(T)^*,$$

where a bijective morphism is said to be an isomorphism (in which case it is then an isometry). A fundamental relation is that any norm-closed *-algebra $\mathcal{A}$ in $\mathcal{L}(H)$ is a C*-algebra, and conversely, any C*-algebra is isomorphic to a norm-closed *-algebra in $\mathcal{L}(H)$ for some Hilbert space H.

For a C*-algebra $\mathfrak{A}$, we say that $T \in \mathfrak{A}$ is *self-adjoint* if $T = T^*$. Accordingly, the self-adjoint part $\mathfrak{A}^{sa}$ of $\mathfrak{A}$ is a real vector space since we can decompose $T \in \mathfrak{A}^{sa}$ as:

$$T = T' + T'' := \frac{1}{2}(T + T^*) + \imath(\frac{-\imath}{2})(T - T^*).$$

A *commutative* C*-algebra is one for which the associative multiplication is commutative. Given a commutative C*-algebra $\mathfrak{A}$, we have $\mathfrak{A} \cong \mathfrak{C}(\mathfrak{Y})$, the algebra of continuous functions on a compact Hausdorff space Y.

For the purpose of quantization, there are fundamental relations between $\mathfrak{A}^{sa}$, JLB and Poisson algebras. In fact, if $\mathfrak{A}$ is a C*-algebra and $\hbar \in \mathbb{R}/0$, then $\mathfrak{A}^{sa}$ is a JLB-algebra when it takes its norm from $\mathfrak{A}$ and is equipped with the operations:

$$S \circ T := \frac{1}{2}(ST + TS)$$
$$\{S, T\}_\hbar := \frac{\imath}{\hbar}[S, T].$$

Conversely, given a JLB-algebra $\mathfrak{A}_{\mathbb{R}}$ with $\hbar^2 \geq 0$, its complexification $\mathfrak{A}$ is a C*-algebra under the operations:

$$ST := S \circ T - \frac{\imath}{2}\hbar\{S, T\}$$
$$(S + \imath T)^* := S - \imath T.$$

For further details see Landsmann (1998) (Thm. 1.1.9).

A JB-algebra which is monotone complete and admits a separating set of normal sets is called a JBW-algebra. These appeared in the work of von Neumann who developed a (orthomodular) lattice theory of projections on $\mathcal{L}(H)$ on which to study quantum logic (see later). BW-algebras have the following property: whereas $\mathfrak{A}^{sa}$ is a J(L)B-algebra, the self adjoint part of a von Neumann algebra is a JBW-algebra.

A JC-algebra is a norm closed real linear subspace of $\mathcal{L}(H)^{sa}$ which is closed under the bilinear product $S \circ T = \frac{1}{2}(ST + TS)$ (noncommutative and nonassociative). Since any norm closed Jordan subalgebra of $\mathcal{L}(H)^{sa}$ is a JB-algebra, it is natural to specify the exact relationship between JB and JC-algebras, at least in finite dimensions. In order to do this, one introduces the 'exceptional' algebra $H_3(\mathbb{O})$, the algebra of 3×3 Hermitian matrices with values in the octonians $\mathbb{O}$. Then a finite dimensional JB-algebra is a JC-algebra if and only if it does not contain $H_3(\mathbb{O})$ as a (direct) summand (Alfsen and Schultz (2003)).

5 Quantum Logics in Categorical Ontology

5.1 Quantum Logics (QL) and Logical Algebras (LA)

As pointed out by Birkhoff and von Neumann (1936), a logical foundation of quantum mechanics consistent with quantum algebra is essential for both the completeness and mathematical validity of the theory. With the exception of the Isham and Butterfield framework in terms of the 'standard' Topos (Mac Lane and Moerdijk 1992), and the 2-category approach by John Baez (2000, 2002), quantum algebra and topological approaches are ultimately based on set-theoretical concepts and differentiable spaces (manifolds). Since it has been shown that standard set theory which is subject to the axiom of choice relies on Boolean logic (Diaconescu 1976; cited in Mac Lane and Moerdijk 1992), there appears to exist a basic logical inconsistency between the quantum logic—which is not Boolean—and the Boolean logic underlying all differentiable manifold approaches that rely on continuous spaces of points, or certain specialized sets of elements. A possible solution to such inconsistencies is the definition of a generalized Topos concept, and more specifically, of a Quantum Topos concept which is consistent with both Quantum Logic and Quantum Algebras, being thus suitable as a framework for unifying quantum field theories and physical modeling of complex systems and systems biology. The problem of logical consistency between the quantum algebra and the Heyting logic algebra as a candidate for quantum logic is here discussed next. The development of Quantum Mechanics from its very beginnings both inspired and required the consideration of specialized logics compatible with a new theory of measurements for microphysical systems. Such a specialized logic was initially formulated by Birkhoff and von Neumann (1936) and called 'Quantum Logic'. Subsequent research on Quantum Logics (Chang 1958; Dalla Chiara 2004) resulted in several approaches that involve several types of non-distributive lattice (algebra) for n-valued quantum logics. Thus, modifications of the Łukasiewicz Logic Algebras that were introduced in the context of algebraic categories by Georgescu and Vraciu (1973), can provide an appropriate framework for representing quantum systems, or—in their unmodified form—for describing the activities of complex networks in categories of Łukasiewicz Logic Algebras (Baianu 1977). (The reader who has only a philosophical interest in quantum spacetimes may wish to ommit at a first reading the Sects. 5.2–5.5 on logic algebras without losing continuity of our ontological 'thread' of physical spacetimes and 'simple' system dynamics that we are pursuing here.)

5.2 Lattices and Von Neumann–Birkhoff (VNB) Quantum Logic: Definitions and Logical Properties

5.2.1 Categorical Definition of a Lattice

Utilizing the category theory concepts defined in the next paper in this issue, we introduce a categorical definition of the concept of lattice that need be *'set-free'* in order to maintain logical consistency with the algebraic foundation of Quantum Logics and relativistic spacetime geometry. Such category-theoretical concepts provide the tools for deriving general results that link Quantum Logics with Quantum theories, and also pave the way towards a universal theory applicable also to semi-classical, or mixed, systems. Furthermore, such concepts are indeed applicable to measurements in complex biological networks, as it will be shown in considerable detail in a recent paper (Baianu and Poli 2008).

A *lattice* is defined as a category (see, for example: Lawvere, 1966; Baianu 1970; Baianu et al. 2004b) subject to all ETAC axioms, (but not subject, in general, to the Axiom of Choice usually encountered with sets relying on (distributive) Boolean Logic), that has all binary products and all binary coproducts, as well as the following 'partial ordering' properties:

(i) when unique arrows $X \longrightarrow Y$ exist between objects X and Y in $\mathbf{L}$ such arrows will be labeled by '' $\preceq$ '', as in ''$X \preceq Y$'';
(ii) *the coproduct* of X and Y, written as ''$X \bigvee Y$'' will be called the "*sup object*, or "*the least upper bound*", whereas the product of X and Y will be written as ''$X \bigwedge Y$'', and it will be called an *inf object*, or "*the greatest lower bound*";
(iii) *the partial order* defined by $\preceq$ holds in $\mathbf{L}$, as $X \preceq Y$ if and only if $X = X \bigwedge Y$ (or equivalently, $Y = X \bigvee Y$ (p. 49 of MacLane and Moerdijk, 1992).

If a lattice $\mathbf{L}$ has $\mathbf{0}$ and $\mathbf{1}$ as objects, such that $0 \longrightarrow X \longrightarrow 1$ (or equivalently, such that $0 \preceq X \preceq 1$) for all objects X in the lattice $\mathbf{L}$ viewed as a category, then $\mathbf{0}$ and $\mathbf{1}$ are the unique, initial, and respectively, terminal objects of this concrete category $\mathbf{L}$. Therefore, $\mathbf{L}$ has all finite limits and all finite colimits (p. 49 of MacLane and Moerdijk 1992), and is said to be *finitely complete and co-complete*. Alternatively, the lattice 'operations' can be defined via functors in a 2-category (for definitions of functors and 2-categories see, for example, p. 50 of MacLane 2000, p. 121 of Brown 1998, or Sect. 9 of Baianu et al. 2004b), as follows:

$$\bigwedge : L \times L \longrightarrow L, \qquad \bigvee : L \times L \to L \tag{5.1}$$

and 0,1: $1 \to L$ as a "lattice object" in a 2-category with finite products.

A lattice is called *distributive* if the following identity:

$$X \bigwedge (Y \bigvee Z) = (X \bigwedge Y) \bigvee (X \bigwedge Z). \tag{5.2}$$

holds for all X, Y, and Z objects in $\mathbf{L}$. Such an identity also implies the dual distributive lattice law:

$$X \bigvee (Y \bigwedge Z) = (X \bigvee Y) \bigwedge (X \bigvee Z). \tag{5.3}$$

(Note how the lattice operators are 'distributed' symmetrically around each other when they appear in front of a parenthesis.) A *non-distributive* lattice is not subject to either restriction (4.2) or (4.3). An example of a non-distributive lattice is (p. 135 of Pedicchio and Tholen 2004):

$$\begin{tikzcd} & 1 & \\ A \arrow[ur] & B \arrow[u] & C \arrow[ul] \\ & 0 \arrow[ul] \arrow[u] \arrow[ur] & \end{tikzcd} \tag{5.4}$$

A lattice will be called complete when it has all small limits and small colimits (e.g., small products and coproducts, respectively). It can be shown (p. 51 of MacLane and Moerdijk 1992) that any complete and infinitely distributive lattice is a Heyting algebra.

5.3 Łukasiewicz Quantum Logic (LQL)

With all assertions of the type *system A* is excitable to the *i*-th level and system B is excitable to the *j*-th level" one can form a distributive lattice, **L** (as defined above in Sect. 5.1). The composition laws for the lattice will be denoted by $\bigcup$ and $\bigcap$. The symbol $\bigcup$ will stand for the logical non-exclusive 'or', and $\bigcap$ will stand for the logical conjunction 'and'. Another symbol "$\preceq$ " allows for the ordering of the 'truth levels' and is defined as *the canonical ordering* of the lattice. Then, one is able to give a symbolic characterization of the system dynamics with respect to each 'energy', or 'truth', level *i*. This is achieved by means of the maps δt: $L \rightarrow L$ and N: $L \rightarrow L$, (with N being the negation). The necessary logical restrictions on the actions of these maps lead to an *n-valued Łukasiewicz Algebra*:

(I) There is a map $N : L \longrightarrow L$, so that

$$N(N(X)) = X, \tag{5.5}$$

$$N(X \bigcup Y) = N(X) \bigcap N(Y) \tag{5.6}$$

and

$$N(X \bigcap Y) = N(X) \bigcup N(Y), \tag{5.7}$$

for any $X, Y \in \mathbf{L}$.

(II) There are $(n-1)$ maps $\delta_i : L \longrightarrow L$ which have the following properties:
- (a) $\delta_i\,(0) = 0$, $\delta_i(1) = 1$, for any $1 \leq i \leq n-1$;
- (b) $\delta_i(X \bigcup Y) = \delta_i(X) \bigcup \delta_i(Y), \delta_i(X \bigcap Y) = \delta_i(X) \bigcap \delta_i(Y)$, for any $X, Y \in \mathbf{L}$, and $1 \leq i \leq n-1$;
- (c) $\delta_i(X) \bigcup N(\delta_i(X)) = 1, \delta_i(X) \bigcap N(\delta_i(X)) = 0$, for any $X \in \mathbf{L}$,
- (d) $\delta_i\,(X) \subset \delta_2\,(X) \subset \ldots \subset \delta_{(n-1)}\,(X)$, for any $X \in \mathbf{L}$,
- (e) $\delta_i * \delta_j = \delta_i$ for any $1 \leq i, j \leq n-1$;
- (f) If $\delta_i\,(X) = \delta_i\,(Y)$ for any $1 \leq i \leq n-1$, then $X = Y$;
- (g) $\delta_i\,(N(X)) = N(\delta_j\,(X))$, for $i + j = n$.

(Georgescu and Vraciu 1970).

The first axiom states that the double negation has no effect on any assertion concerning any level, and that a simple negation changes the disjunction into conjunction and conversely. The second axiom presents ten sub-cases that are summarized in equations (a)–(g). Sub-case (IIa) states that the dynamics of the system is such that it maintains the structural integrity of the system. It does not allow for structural changes that would alter the lowest and the highest energy levels of the system. Thus, maps $\delta : L \longrightarrow L$ are chosen to represent the dynamic behavior of the quantum or classical systems in the absence of structural changes. Equation (IIb) shows that the maps (d) maintain the type of conjunction and disjunction. Equations (IIc) are chosen to represent assertions of the following type: ⟨the sentence "a system component is excited to the i-th level or it is not excited to the same level" is true⟩, and ⟨ the sentence "a system component is excited to the i-th level and it is not excited to the same level, at the same time" is always false⟩.

Equation (IId) actually defines the actions of maps δ_t. Thus, Eq. (I) is chosen to represent a change from a certain level to another level as low as possible, just above the zero level of **L**. δ_2 carries a certain level x in assertion X just above the same level in δ_1 (X), δ_3 carries the level x-which is present in assertion X-just above the corresponding level in δ_2 (X), and so on. Equation (IIe) gives the rule of composition for the maps δ_t. Equation (IIf) states that any two assertions that have equal images under all maps δ_t, are equal. Equation (IIg) states that the application of δ to the negation of proposition X leads to the negation of proposition $\delta(X)$, if $i + j = n$.

In order to have the n-valued Łukasiewicz Logic Algebra represent correctly the basic behaviours of quantum systems (observed through measurements that involve a quantum system interactions with a measuring instrument—which is a macroscopic object), several of these axioms have to be significantly changed so that the resulting lattice becomes non-distributive and also, possibly, non-associative (Dalla Chiara 2004).

On the other hand, for classical systems, modeling with the unmodified Łukasiewicz Logic Algebra can include both stochastic and fuzzy behaviours. For an example of such models the reader is referred to a previous publication (Baianu 1977) modeling the activities of complex genetic networks from a classical standpoint. Further important results for the Category of Łukasiewicz Logic Algebras that may be also of importance for Quantum Logics are discussed in Brown et al. (2007) and Baianu et al. (2007).

Note also that the above Łukasiewicz Logic Algebra is *distributive* whereas the quantum logic requires a *non-distributive* lattice of quantum 'events'. Therefore, in order to generalize the standard Łukasiewicz Logic Algebra to the appropriate Łukasiewicz Logic Algebra, axiom I needs modifications, such as: $N(N(X)) = Y \neq X$ (instead of the restrictive identity $N(N(X)) = X$, and, in general, giving up its 'distributive' restrictions, such as

$$N(X \bigcup Y) = N(X) \bigcap N(Y) \text{ and } N(X \bigcap Y) = N(X) \bigcup N(Y), \tag{5.8}$$

for any X, Y in the Łukasiewicz Quantum Logic Algebra LQ whenever the context, 'reference frame for the measurements', or 'measurement preparation' interaction conditions for quantum systems are incompatible with the standard 'negation' operation N of the Łukasiewicz Logic Algebra that remains however valid for classical systems, such as various complex networks with n-states (see, for example, Sects. 7 and 8).

5.4 The Category of Łukasiewicz Logic Algebras vs. the Boolean Logic Category

Theorem 5.1 Adjointness Theorem (Georgescu and Vraciu 1970) *There exists an Adjointness between the Category of Centered Łukasiewicz-n Logic Algebras and the Category of Boolean Logic Algebras (Bl).*

(*Note*: this adjointness (in fact, actual equivalence) relation, and the adjointness between the Heyting Algebra Category and Bl have a logical basis: non (non(A)) = A in both Bl and Luk-*n*).

Conjecture 5.1 *There exist adjointness relationships amongst the Centered Heyting Logic Algebra, Bl, and the Centered Luk-n Categories.*

Remark 5.1 R1. Both a Boolean Logic Algebra and a Centered Łukasiewicz Logic Algebra are Heyting Logic algebras (the converse is, of course, generally false!).

R2. The natural equivalence logic classes defined by the adjointness relationships in the above Conjecture define a fundamental, '*logical groupoid*' structure.

5.5 Heyting–Brouwer Intuitionistic Foundations of Categories and Toposes

5.5.1 Subobject Classifier and the notion of a Topos

To an extent our interest concern the notion of *topos*, a special type of category for which several (equivalent) definitions can be found in the literature, the most prevailing being the category of (pre) sheaves on a set X. We will need an essential component of the topos concept called a *subobject classifier*. In order to motivate the discussion, suppose we take a set X and a subset $A \subseteq X$. A characteristic function $\chi_A : X \longrightarrow \{0, 1\}$ specifies 'truth values' in the sense that one defines

$$\chi_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \neq A. \end{cases} \tag{5.9}$$

A topos $\mathcal{C}$ is required to possess an analogue of the truth-value sets {0,1}. In order to specify this particular property, we consider a category $\mathcal{C}$ with a covariant functor $\mathcal{C} \longrightarrow Set$, called a *presheaf*. The collection of presheaves on $\mathcal{C}$ forms a category in its own right, once we have specified the arrows. If $\mathcal{E}$ and $\mathcal{F}$ are two presheaves, then an arrow is a natural transformation $N : \mathcal{C} \longrightarrow \mathcal{F}$, defined in the following way. Given $a \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(a, b)$, then there is a family of maps $N_a : \mathcal{E}(a) \longrightarrow \mathcal{F}(a)$, such that the diagram

$$\begin{array}{ccc} \mathcal{E}(a) & \xrightarrow{\mathcal{E}(f)} & \mathcal{E}(b) \\ {\scriptstyle N_a}\big\downarrow & & \big\downarrow{\scriptstyle N_b} \\ \mathcal{F}(a) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(b) \end{array} \tag{5.10}$$

commutes. Intuitively, an arrow between $\mathcal{E}$ and $\mathcal{F}$ serves to replicate $\mathcal{E}$ inside of $\mathcal{F}$.

Towards classifying subobjects we need the notion of a *sieve* on an object a of $\mathrm{Ob}(\mathcal{C})$. This is a collection S of arrows f in $\mathcal{C}$ such that if $f : a \longrightarrow b$ is in S and $g \in \mathrm{Hom}_{\mathcal{C}}(b, c)$ is any arrow, then the composition $f \circ g$ is in S.

We define a presheaf $\Omega : \mathcal{C} \longrightarrow Set$, as follows. Let $a \in \mathrm{Ob}(\mathcal{C})$, then $\Omega(a)$ is defined as the set of all sieves on a. Given an arrow $f : a \longrightarrow b$, then $\Omega(f) : \Omega(a) \longrightarrow \Omega(b)$, is defined as

$$\Omega(f)(S) := \{g : b \longrightarrow c : g \circ f \in S\}, \tag{5.11}$$

for all $S \in \Omega(a)$. Let $\uparrow b$ denote the set of all arrows having domain the object b. We say that $\uparrow b$ is the *principal sieve on* b, and from the above definition, if $f : a \longrightarrow b$ is in S, then

$$\Omega(f)(S) = \{g : b \longrightarrow c : g \circ f \in S\} = \{g : b \longrightarrow c\} = \uparrow b. \tag{5.12}$$

Let us return for the moment to our motivation for defining Ω. The set of truth values $\{0,1\}$ is itself a set and therefore an object in *Set*, furthermore, the set of subsets of a given set X corresponds to the set of characteristic functions χ_A as above. Likewise if $\mathcal{C}$ is a topos, Ω is an object of $\mathcal{C}$, and there exists a bijective correspondence between subobjects of an object a and arrows $a \longrightarrow \Omega$, leading to the nomenclature *subobject classifier*. In this respect, a typical element of Ω relays a string of answers about the status of a given object in the topos. Furthermore, for a given object a, the set $\Omega(a)$ enjoys the structure of a Heyting algebra (a distributive lattice with null and unit elements, that is relatively complemented).

6 Ontological Aspects of Quantum Gravity, SuperSymmetry and TQFT

As the experimental findings in high-energy physics—coupled with theoretical studies—have revealed the presence of new fields and symmetries, there appeared the need in modern physics to develop systematic procedures for generalizing space–times and Quantum State Space (QSS) representations that reflect the existence of these new concepts.

6.1 Quantum Fields, General Relativity, Quantum Gravity and Symmetries

In the General Relativity (GR) formulation, the local structure of space–time, characterized by its various tensors (of energy–momentum, torsion, curvature, etc.), incorporates the gravitational fields surrounding various masses. In Einstein's own representation, the physical space–time of GR has the structure of a Riemannian R^4 space over large distances, although the detailed local structure of space–time—as Einstein perceived it—is likely to be significantly different.

On the other hand, there is a growing consensus in theoretical physics that a valid theory of Quantum Gravity requires a much deeper understanding of the small(est)-scale structure of Quantum Space–Time (QST) than currently developed. In Einstein's GR theory and his subsequent attempts at developing a unified field theory (as in the space concept advocated by Leibnitz), space–time does *not* have an *independent existence* from objects, matter or fields, but is instead an entity generated by the *continuous* transformations of fields. Hence, the continuous nature of space–time was adopted in GR and Einstein's subsequent field theoretical developments. Furthermore, the quantum, or 'quantized', versions of space–time, QST, are operationally defined through local quantum measurements in general reference frames that are prescribed by GR theory. Such a definition is therefore subject to the postulates of both GR theory and the axioms of Local Quantum Physics. We must emphasize, however, that this is *not* the usual definition of position and time observables in 'standard' QM. Therefore, the general reference frame positioning in QST is itself subject to the Heisenberg uncertainty principle, and therefore it acquires through quantum measurements, a certain 'fuzziness' at the Planck scale which is intrinsic to all microphysical quantum systems,

A notable feature of current 21-st century physical thought involves the questioning validity of the classical model of space–time as a 4-dimensional manifold equipped with a Lorentz metric. The expectation of the earlier approaches to quantum gravity (QG) was to cope with microscopic length scales where, as we have mentioned, a traditional manifold structure (in the conventional sense) needs to be forsaken (for instance, at the Planck length $L_p = (\frac{G\hbar}{c^3})^{\frac{1}{2}} \approx 10^{-35}m$). On the other hand, one needs to reconcile the discreteness versus continuum approach in view of space–time diffeomorphisms and that space–time may be suitably modeled as some type of 'combinatorial space' such as a simplicial complex, a poset, or a spin foam (a cluster of spin networks) The monumental difficulty is that to the present day, apart from a dire absence of experimental evidence, there is no consensus of agreement on the actual nature of the data necessary, neither upon on the actual conceptual framework to obtaining the data in the first place(!) This difficulty equates with how one can gear the approach to QG to run the gauntlet of conceptual problems in QFT and (General Relativity) GR.

Whereas Newton, Riemann, Einstein, Weyl, Hawking, Penrose, Weinberg and many other exceptional theoreticians regarded physical space as represented by a *continuum*, there is an increasing number of proponents for a *discrete, 'quantized'* structure of space–time, since space itself is considered as discrete on the Planck scale. Like most radical theories, the latter view carries its own set of problems. The biggest problem for any discrete, 'point-set' (or discrete topology), view of physical space–time is not only its immediate conflict with Einstein's General Relativity representation of space–time as a *continuous Riemann* space, but also the impossibility of carrying out quantum measurements to localize precisely either quantum events or masses at singular (in the sense of disconnected, or isolated), sharply defined, geometric points in space–time.

We mention some attempts at this problem. In Sorkin (1991) 'finitary topological spaces' were introduced to approximate or to reproduce in the limit, a topological space such as a manifold. The motivation reflects upon the patent inadequacies of the traditional differentiable manifold structure of space–time. Such a structure is perhaps too artificial for a 'laboratory' model. A main premise is that the smooth structure at small time scales breaks down to one that is more discrete- and 'quantum'-in form; there is an ideal character of the event as observed classically and this occurs within the presence of singularities. The continuum of events and their infinitesimal separation do not yield to the usual experimental analysis.

Differential structures in a non-commutative setting are replaced by such objects as quantized differential forms, Fredholm modules and quantum groups. Again, since GR breaks down at the Planck scale, space–time would no longer be describable by a smooth manifold structure. While not neglecting the large scale classical model, one may propose the structure of 'ideal observations' as manifest in a limit, in some sense, of discrete measurements, where such a limit accomodates the classical event. Then the latter is represented as a *'point'* which is not influenced by quantum interference; nevertheless, the idea is to admit *coherent* quantum superposition of events. Thus, at the quantum level, the events can *decohere* to the classical point in the limit, somewhat in accordance with the correspondence principle. Algebraic developments of the Sorkin model can be seen in Raptis and Zapatrin (2000) and quantum causal sets are considered in Raptis (2000). A main framework is Abstract Differential Geometry (ADG) which employs sheaf—theoretic methods enabling one to avoid point—based smooth manifolds, dodgy gyroscopic frames and the chimera of 'classical' singularities (see for instance Mallios and Raptis 2003).

Another proposed resolution of the problem is through non-commutative Geometry (NCG), or 'Quantum Geometry', where QST has 'no points', in the sense of visualizing a 'geometrical space' as some kind of a distributive and commutative lattice of space–time 'points'. The quantum 'metric' of QST in NCG would be related to a certain, fundamental quantum field operator, or a 'fundamental triplet (or quintet)' construction (Connes 2004). Although quantization is standard in Quantum Mechanics (QM) for most quantum observables, it does encounter major difficulties when applied to position and time. In standard QM, there are at least two implemented approaches to solve the problem, one of which was conceived by von Neumann (1932).

To quote an example, the space–time metric tensor $\gamma = (\gamma_{ab})$ is less engaging a fundamental field than perhaps once considered because it leads to describing an essentially classical gravitational field. A case study (Butterfield and Isham 2001) involves quantizing one side of Einstein's field equations by a quantum expectation value, so that a coupling of γ to quantized matter is given by an expression such as:

$$G_{\mu\nu}(\gamma) = \langle\psi|T_{\mu\nu}(g, \hat{\phi})|\psi\rangle, \tag{6.1}$$

where $|\psi\rangle$ denotes a state in the Hilbert space of quantized matter variables $\hat{\phi}$, and the subsequent source of the gravitational field is given by the expectation of the corresponding energy–momentum tensor $T_{\mu\nu}$. Unfortunately, this supposition leads to ontological problems which are serious enough to prevent the development of a complete QG theory that would include this expression. Three possible approaches were suggested by Butterfield and Isham (1998–2001) (see also the survey article by Rovelli 1998):

(1) to develop and test a quantized form of classical relativity theory;
(2) to recover GR as the low energy limit of a QFT approach which is not a quantization of a classical theory (e.g., *via quantum algebras/groups and their representations*);
(3) to develop a new theory, such as a 'quantization of topology' or 'causal' structures where, for instance, microphysical states provide amplitudes to the values of quantities whose norms squared define probabilities of occurrence for *physical*, quantum events.

We consider that the key ideas for such a program should be based in the development of a *non-Abelian algebraic topology* for which Sect. 7 to 9 provide a concise overview.

6.2 Supergravity Theories, The Metric Superfield and Supersymmetry Algebras

6.2.1 Supergravity Theories

Supergravity, in essence, is an extended supersymmetric theory of both matter and gravitation (Weinberg 2000). A first approach to supersymmetry relies on a curved 'superspace' (Wess and Bagger 2000) and is analogous to supersymmetric gauge theories (see, for example, Sects. 27.1 to 27.3 of Weinberg 2000). Unfortunately, a complete non-linear supergravity theory would be "forbiddingly complicated" and, furthermore, the constraints that need be made on the graviton superfield appear somewhat subjective (cf. Weinberg 2000). On the other hand, the second approach to supergravity is much more transparent than the first, albeit theoretically less elegant. The physical components of the gravitational superfield can be identified in this approach based on flat-space superfield methods (Chaps. 26 and 27 of Weinberg 2000). By employing the weak-field approximation one obtains several of the most important consequences of supergravity theory, including masses for the hypothetical gravitino and gaugino 'particles' whose existence is expected from supergravity theories. Furthermore, by adding on the higher order terms in G (the gravitational constant) to the supersymmetric transformation, the general coordinate transformations form a closed algebra and the Lagrangian that describes the interactions of the physical fields is invariant under such transformations. Quantization of such a flat-space superfield would obviously involve its 'deformation' as discussed in the following Sect. 8 above, and as a result its corresponding *supersymmetry algebra* would become *non-commutative*.

6.2.2 The Metric Superfield

Because in supergravity both spinor and tensor fields are being considered, the gravitational fields are represented in terms of *tetrads*, $e^{a}_{\mu}(x)$, rather than in terms of the general relativistic metric $g_{\mu\nu}(x)$. The connections between these two distinct representations are as follows:

$$g_{\mu\nu}(x) = \eta_{ab} e^a_\mu(x) e^b_\gamma(x), \tag{6.2}$$

with the general coordinates being labeled by μ,ν, etc., whereas local coordinates that are being defined in a locally inertial coordinate system are labeled with superscripts a, b, etc. :η^{ab} is the diagonal matrix with elements +1, +1, +1 and −1. The tetrads are invariant to two distinct types of symmetry transformations, the local Lorentz transformations:

$$e^a_\mu(x) \longrightarrow \Lambda^a_b(x) e^b_\mu(x), \tag{6.3}$$

(where Λ^a_b is an arbitrary real matrix), and the general coordinate transformations for which $x^\mu \longrightarrow x'^\mu$.

In a low intensity gravitational field the tetrad may be represented as:

$$e^a_\mu(x) = \delta^a_\mu(x) + 2\kappa \Phi^a_\mu(x), \tag{6.4}$$

where $\Phi^a_\mu(x)$ is small compared with $\delta^a_\mu(x)$ for all x values, and $\kappa = \sqrt{8\pi} G$, where G is Newton's gravitational constant.

As it will be discussed next, the supersymmetry algebra (SA) implies that the graviton has a fermionic superpartner, the hypothetical *gravitino*, with helicities ± 3/2. Such a self-charge-conjugate massless particle as the gravitiono with helicities ± 3/2 can only have *low-energy* interactions if it is represented by a Majorana field $\psi_\mu(x)$ which is invariant under the gauge transformations

$$\psi_\mu(x) \longrightarrow \psi_\mu(x) + \delta_\mu \psi(x), \tag{6.5}$$

with $\psi(x)$ being an arbitrary Majorana field as defined by Grisaru and Pendleton in 1977. The tetrad field $\Phi_{\mu\nu}(x)$ and the graviton field $\psi_\mu(x)$ are then incorporated into a *vector superfield* $H_\mu(x,\theta)$ defined as the *metric superfield*. The relationships between $\Phi_{\mu\nu}(x)$ and $\psi_\mu(x)$, on the one hand, and the components of the metric superfield $H_\mu(x,\theta)$, on the other hand, can be derived from the transformations of the whole metric superfield:

$$H_\mu(x,\theta) \longrightarrow H_\mu(x,\theta) + \Delta_\mu(x,\theta), \tag{6.6}$$

by making the simplifying- and physically realistic-assumption of a weak gravitational field. Further details can be found, for example, in Ch.31 of vol.3. of Weinberg (2000). The interactions of the whole superfield H_μ (x) with matter would be then described by considering how a weak gravitational field, $h_{\mu\nu}$ interacts with an energy–momentum tensor $T^{\mu\nu}$ represented as a linear combination of components of a real vector superfield Θ^μ. Such interaction terms would, therefore, have the form:

$$I_{\text{matter}} = 2\kappa \int dx^4 [H_\mu \Theta^\mu]_D, \tag{6.7}$$

where the integration space is a four-dimensional ('Minkowski-like') space–time with the metric defined by the superfield H_μ (x,θ). The quantity H_1, as defined here, is physically a *supercurrent* and satisfies the conservation conditions:

$$\gamma^\mu \mathbf{D}\Theta_\mu = \mathbf{D}X, \tag{6.8}$$

where $\mathbf{D}$ is the four-component super-derivative and X is a real chiral scalar superfield. This leads immediately to the calculation of the interactions of matter with a weak gravitational field as:

$$I_{\text{matter}} = \kappa \int d^4xT^{\mu\nu}(x)h_{\mu\nu}(x). \tag{6.9}$$

It is quite interesting that the gravitational actions for the superfield that are invariant under the generalized gauge transformations $H_\mu \longrightarrow H_\mu + \Delta_\mu$ lead to solutions of the Einstein field equations for a homogeneous, non-zero vacuum energy density ρ_V that are either a deSitter space for $\rho_V > 0$, or an anti-deSitter space for $\rho_V < 0$. Such spaces can be represented then as surfaces:

$$x_5^2 \pm \eta_{\mu\nu}x^\mu x^\nu = R^2, \tag{6.10}$$

in a quasi-Euclidean five-dimensional space with the 'distance' (line element) specified as:

$$ds^2 = \eta_{\mu\nu}x^\mu x^\nu \pm dx_5^2, \tag{6.11}$$

with '+' for deSitter spaces and '$-$' for anti-deSitter space, respectively.

The space–time symmetry groups, or groupoids—as the case may be—are different from the 'classical' Poincaré symmetry group of translations and Lorentz transformations. Such space–time symmetry groups, in the simplest case, are therefore the orthogonal O(4,1) group for deSitter space and the O(3,2) group for anti-deSitter space. A detailed calculation indicates that the transition from ordinary flat space to a bubble of anti-deSitter space is *not* favored energetically and, therefore, the ordinary (deSitter) flat space is stable (cf. Coleman and deLuccia 1980), even though quantum fluctuations might occur to an anti-deSitter bubble within the limits permitted by the Heisenberg uncertainty principle.

7 Non-Abelian Algebraic Topology (NAAT): A new basis for SpaceTime Representation Strategies based on Algebraic Topology

7.1 Abelian vs. Non-Abelian Representations

The current state of play reveals several distinct pathways to the global structure of Quantum Space–Time from 'local space–time regions' or sectors, as in both Algebraic and Topological Quantum Field theories. However, we shall place here the emphasis on either Quantum Algebra (QA) or the 'Quantum Topology'(QT) of space–time. An alternative approach to this problem, and/or its construction, is that provided by Quantum Algebraic Topology (QAT) and involves considering *jointly* the *algebraic and topological* structures of QST, as well as defining and determining the fundamental *algebraic invariants* of possible QST topologies that might be relevant to corresponding *Quantum Gravity* theories. Although there is only a physically *unique* QST, already present is a rapid proliferation of proposed mathematical representations of physical space–time, ranging from partially ordered sets (i.e., with *discrete* topology) to *continuous* topological space representations such as various manifolds (with dimensions of 4, such as Riemannian R^4, or those of 10, 11, 26 as encountered in string theory), 'group manifolds', 'monoidal' categories, small 'intertwined' categories, 2-categories, 'tensor' 2-categories, and a 'quantum' topos. The systematic classification and rigorous characterization of such potential candidates for the mathematical representation of QST can also be considered as a significant task in QAT which is under development. Furthermore, a completely satisfactory resolution of the problem of QST structural representation will undoubtedly involve the consistent linking of Quantum Logics (QL) with QAT, thus referring back to the theoretical constructions of QST to quantum measurements and experimental data in terms of systematic Quantum Logic (QL) analysis of quantum events and their consequences for both QA (Alfsen and Schultz 2003), and QAT. Linking consistently QL with QAT for representing the structure of QST is an approach that was pursued in Baianu et al. (2006b). Algebraic developments related to quantum theories have a long and successful history. The more challenging aspects of such developments are recently based on Algebraic Topology, and also in algebraic treatments of 'Quantum Geometry'.

The consideration of possible candidates for representing the complete structure of our physical space–time thus runs into the basic problem of classifying such space–time candidates into *equivalence classes* determined by homeomorphisms of topological spaces. As the explicit mathematical construction of homemorphisms can be a very daunting problem for topological spaces in general, the computation of *algebraic invariants* of such spaces is the chosen, basic methodology of Algebraic Topology (AT). Thus, if one can assign the algebraic structure of a group to a topological space, then one can compare two homeomorphic, or equivalent, topological spaces and find that their corresponding groups are *isomorphic*. However, the converse does not necessarily hold: even though two arbitrary topological spaces may have assigned *isomorphic* groups of a sort (homotopy, homology, etc.), the two spaces are not necessarily *homeomorphic*, that is, they are not continuously deformable to each other. Therefore, one needs to consider first the simpler problem of finding a coarser equivalence of topological spaces in terms of *homotopy equivalence* and their associated homotopy groups by assembling *equivalence classes* of continuous path deformations in such topological spaces. Whereas many *homotopy groups* may be readily computed for standard spaces such as the n-dimensional spheres S^n, certain polyhedra-like spaces ('simplicial complexes' and their generalized forms—*CW-complexes*), their computation for arbitrary spaces with corresponding, 'dual' higher dimensional algebras is, more difficult. Therefore, other refined algebraic approaches to classifying topological spaces have been developed since the second half of the twentieth century. One such approach was conceived in terms of *map transformations* and exact sequences that involve both *singular* *homology and cohomology* constructions allowing the systematic computation of certain required (co)homology groups, or *groupoids*, especially for *CW-complexes*. The latter can be constructed either as equivalent 'cellular spaces' by attaching cells to spaces in a systematic, precisely-defined construction, or else they can be defined as a special type of Hausdorff space subject to several restrictions imposed by their equivalent cellular construction.

7.2 Clifford Algebra as an Example of a Non-Abelian Structure

The quaternion algebra was first formulated by W. R. Hamilton in his work on geometric representations and phase factors. Sometime after, H. Grassmann developed his theory of the exterior algebra calculus, originally as a framework for studying rotational mechanics, but eventually becoming a fundamental concept in differential geometric systems. Following Grassmann, W. Clifford combined these ideas with Hamilton's quaternion algebra to study rotational physical and planetary systems, thus anticipating the notion of *spinors* in an algebraic framework. Spinors, as fundamental objects in QT, have been attributed to E. Cartan, P. A. M. Dirac and H. Weyl, but the algebraic foundations of this work can be traced back in part to the Clifford algebra. We also provide here the essential definition of the (non-commutative) Clifford algebra.

7.2.1 Definition of a Clifford Algebra

Consider a pair (V, Q), where V denotes a real vector space and Q is a quadratic form on V. *The Clifford algebra associated to* V denoted $\mathrm{Cl}(V) = \mathrm{Cl}(V, Q)$, is the algebra over $\mathbb{R}$ generated by V, where for all $v, w \in V$, the relations

$$v \cdot w + w \cdot v = -2Q(v, w), \tag{7.1}$$

are satisfied; in particular, we have $v^2 = -2Q(v,v)$.

If W is an algebra and $c : V \longrightarrow W$ is a linear map satisfying

$$c(w)c(v) + c(v)c(w) = -2Q(v, w), \tag{7.2}$$

then there exists a unique algebra homomorphism $\phi : \mathrm{Cl}(V) \longrightarrow W$ such that the diagram

$$\begin{tikzcd} & \mathrm{Cl}(V) \arrow[dr, "\phi"] & \\ V \arrow[ur, "\mathrm{Cl}"] \arrow[rr, "c"'] & & W \end{tikzcd}$$

commutes. It is in this sense that $\mathrm{Cl}(V)$ is considered to be 'universal'.

For a given Hilbert space H, there is an associated *C*-Clifford algebra* $\mathrm{Cl}[H]$ which admits a canonical representation on $\mathcal{L}(\mathbb{F}(H))$ the bounded linear operators on the fermionic Fock space $\mathbb{F}(H)$ of H as in Plymen and Robinson (1994), and hence we a have a natural sequence of maps

$$H \longrightarrow \mathrm{Cl}[H] \longrightarrow \mathcal{L}(\mathbb{F}(H)). \tag{7.3}$$

7.3 Local-to-Global Constructions in a Generalized Topos with Quantum Logics

Another very interesting aspect of such algebraic constructions in the context of Quantum theory and Quantum Gravity leading from local to global structures of topological spaces is the representation of a topological space as the categorical *colimit* of a sequence of 'simpler' spaces, such as CW complexes, at least as an approximation. This also occurs in the generalized van Kampen theorem in terms of colimits of homotopy double groupoids. As an illustration, a specific example is presented in a following report for local subgroupoids that are defined in terms of *sheaves*, thus leading towards the concept of a *Generalized Topos with a Quantum Logic, subobject classifier* (Baianu et al. 2007) which links Quantum Multi-Valued Logics with generalized QAT structures in categories generated by sheaves, such as the Grothendieck categories. The relevance of such colimit constructions to the QAT representation of fundamental quantum space–time structure in our inflationary universe will be shown in the following sections. Therefore, instead of utilizing flat, or almost-flat, pieces of space–time as the local, *'linearized'* structure that approximates our inflationary universe only for small masses with *weak* gravitational fields (as in the 'standard' supergravity theory that was concisely reviewed in Sect. 6), one should also be able to employ categorical *colimits* to construct representations of quantized space–time that incorporate huge masses and correspondingly *intense* gravitational fields. Such generalized space–time representations—based on QAT constructions—will also be endowed with the prerequisite covariance, metric and broken supersymmetry properties, as well as will be able to avoid the severe problems associated with singularities and renormalization. It is conjectured at this point that such a physical representation of the emerging, *nonlinear* supergravity theory for *intense* gravitational (and other coupled) fields—which is obtained by including the appropriate QAT structure of space–time (both local and global)—will be at least *consistent* with the accepted results of the Standard Model in the limit of the currently attainable energies with the existing particle accelerators (i.e. $E < 0.2\ TeV$ in the laboratory reference frame).

8 Non-Commutative Representations of SpaceTime

8.1 The Underlying Idea of Quantization

Quantization may be broadly viewed as the geometric procedure for relating classical (deterministic) systems to ones that are quantum mechanical (indeterministic). The term reflects upon the fact that at the microscopic level certain physical quantities only assume discrete values (*quantum numbers*), as was verified for energy levels of bound states and electrical charge. In the Hilbert space setting, the discreteness arises as a question of whether a self-adjoint operator may or may not possess a discrete spectrum. More specifically, the aim of quantization is to relate classical phase spaces (Poisson, or in particular, symplectic manifolds) to quantum phase spaces (Hilbert spaces) and observables (bounded linear operators on Hilbert spaces). In the former case, one considers a non-associative (but possibly commutative) *Jordan—Lie algebra* of say, the differentiable functions on the phase space. A basic principle of quantum mechanics is that every bounded observable of some sub-theory corresponds to a bounded, self-adjoint operator on a Hilbert space, and conversely. Otherwise, the system may be tied to *superselection rules*. The observables, in particular, will most often form the self adjoint part of a C*-algebra which here is manifestly a *Jordan–Lie–Banach Algebra*. The state space of the C*-algebra consists of all positive and normalized linear functionals forming a compact convex set. Through a certain representation, *the Gelfand–Naimark–Segal* construction reveals a notable two-way connection between states on a C*-algebra to elements on a Hilbert space containing a unit cyclic vector. It is a fundamental concept used throughout the methods of quantization (Landsman 1998).

The *pure* states are those states that cannot be expressed in terms of linear combinations of other states. In a C*-algebra such states form a generalized Poisson manifold. An important feature of QM concerns the existence of (symmetric) *transition probabilities* between pure states, thus leading to the concept of a *Poisson space with a transition probability*. The original theoretical formulations of quantization were due to Bohr–Sommerfeld who postulated the existence of families of linear maps between the space of classical observables and the self-adjoint part of a C*-algebra. These ideas were later taken up by Berezin and Wigner–Weyl–Moyl in order to deal with more general phase spaces (such as the symplectic cotangent bundle of flat and non-flat spaces) and implement quite sophisticated mathematical techniques. Ideally, whatever the quantization procedure employed, it should be geared towards uncovering the classical limit of the QM system. Further technical details on how quantization procedures can be developed and how they can be categorically extended to more general spacetimes are given in two subsections in Brown et al. 2007 (in this issue).

8.2 The Basic Principle of Quantization

At the microscopic/indeterministic level certain physical quantities assume only discrete values. The means of quantization describe the passage from a classical to an associated quantum theory where, at the probabilistic level, Bayesian rules are replaced by theorems on the composition of amplitudes. The classical situation is considered as ‘commutative’: one considers a pair (A, Π) where typically A is a commutative algebra of a class of continuous functions on some topological space and Π is a state on A. Quantization involves the transference to a ‘non-commutative’ situation via an integral transform: $(\mathsf{A}, \Pi) \longrightarrow (\mathcal{A}^{\mathrm{ad}}, \psi)$ where $\mathcal{A}^{\mathrm{ad}}$

denotes the self-adjoint part of the non-commutative Banach algebra $\mathcal{A} = \mathcal{L}(H)$, of the bounded linear operators (observables) on a Hilbert space H. In this case, the state ψ can be specified as $\psi(T) = \mathrm{Tr}(\rho T)$, for T in $\mathcal{L}(H)$ and where ρ is a density operator. Alternative structures may involve a Poisson manifold (with Hamiltonian) and $(\mathcal{A}^{\mathrm{ad}}, \psi)$, possibly with time evolution. Such quantization procedures are realized by the transforms of Weyl-Heisenberg, Berezin, Wigner–Weyl–Moyal, along with certain variants of these. Problematic can be the requirements that the adopted quantum theory should converge to the classical limit, as $\hbar \longrightarrow 0$, meaning that in the Planck limit, $\hbar$ is small by comparison with other relevant quantities of the same dimension (Landsman 1998).

8.3 Wigner–Weyl–Moyal Quantization Procedures

We have mentioned that a governing principle of quantization involves 'deforming', in a certain way, an algebra of functions on a phase space to an algebra of operator kernels. The more general techniques revolve around using such kernels in representing asymptotic morphisms. A fundamental example is an asymptotic morphism $C_0(T^*\mathbb{R}^n) \longrightarrow \mathcal{K}(L^2(\mathbb{R}^n))$ as expressed by the *Moyal deformation*:

$$[T_\hbar(a)f](x) := \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} a(\frac{x+y}{2}, \xi) \exp\left[\frac{\imath}{\hbar}\right] f(y) dy\, d\xi, \tag{8.1}$$

where $a \in C_0(T^*\mathbb{R}^n)$ and the operators $T_\hbar(a)$ are of trace class. In Connes (1994), it is called the *Heisenberg deformation*. Such 'quantizing deformations' can be thought to generate *non-commutative 'spaces'*, or *non-commutative 'geometry'*, *loc. cit*.

An elegant way of generalizing this construction entails introducing *the tangent groupoid* $\mathcal{T}X$ of a suitable space X and using asymptotic morphisms. Putting aside a number of technical details which can be found in Connes (1994) or Landsman (1998), the tangent groupoid $\mathcal{T}X$ is defined as the normal groupoid of a pair Lie groupoid $X \times X \rightrightarrows X$ obtained by 'blowing up' the diagonal *diag*(X) in X. More specifically, if X is a (smooth) manifold let $G' = X \times \ X \times \ (0,1]$ and $G'' = TX$, from which it can be seen $diag(G') = \ X \times \ (0,1]$ and $diag(G'') = \ X$. Then in terms of disjoint unions we have

$$\begin{aligned} \mathcal{T}X &= G' \bigvee G'' \\ diag(\mathcal{T}X) &= diag(G') \bigvee diag(G''). \end{aligned} \tag{8.2}$$

In this way $\mathcal{T}X$ shapes up both as a smooth groupoid, as well as a manifold with boundary.

Quantization relative to $\mathcal{T}X$ is outlined by Várilly (1997) to which we refer for details. The procedure entails characterizing a function on $\mathcal{T}X$ in terms of a pair of functions on G' and G'' respectively, the first of which will be a kernel and the second will be the inverse Fourier transform of a function defined on T^*X. It will be instructive to consider the case $X = \mathbb{R}^n$ as a suitable example. So we take a function $a(x,\xi)$ on $T^*\mathbb{R}^n$ whose inverse Fourier transform

$$\mathcal{F}^{-1}(a(u,v)) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \exp[\imath\xi v]a(u,\xi)d\xi, \tag{8.3}$$

thus yields a function on $T\mathbb{R}^n$. Consider next the terms

$$x := \exp_u\left[\frac{1}{2}\hbar v\right] = u + \frac{1}{2}\hbar v, y := \exp_u\left[-\frac{1}{2}\hbar v\right] = u - \frac{1}{2}\hbar v, \tag{8.4}$$

which on solving leads to $u = \frac{1}{2}(x+y)$ and $v = \frac{1}{\hbar}(x-y)$. Then the following family of operator kernels

$$k_a(x,y,\hbar) := \hbar^{-n}\mathcal{F}^{-1}a(u,v) = \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n} a(\frac{x+y}{2},\xi)\exp\left[\frac{\imath}{\hbar}(x-y)\xi\right]a(u,\xi)d\xi, \tag{8.5}$$

realize the Moyal quantization.

8.4 Quantization on Categories

There remains the substantial question of how such approaches as those discussed above may reconcile or achieve the unification of quantum field theory with general relativity. The general categorical setting proposed in Isham (2003) starts by considering a system whose configuration (or ‘histories’) space is the set of $\mathrm{Ob}(\mathcal{C})$ of a category $\mathcal{C}$ whose momentum transformations are represented by arrows in $\mathrm{Hom}(\mathcal{C})$. There are several choices for $\mathcal{C}$; among these are:

(1) $\mathcal{C}$ is a category of finite causal sets interpreted as a history theory.
(2) $\mathcal{C}$ is a small category of posets interpreted as the structure of physical space at a given time. Here it may suffice to take a poset P as discussed above.
(3) $\mathcal{C}$ is a small category of topological spaces interpreted a history once objects represent spacetime.

Now assuming we work with small categories, then in each instance an arrow $f \in \mathrm{Hom}(\mathcal{C})$ is associated with an operator $\hat{d}(f)$ and the set $\mathrm{Hom}(\mathcal{C})$ is endowed with a semigroup structure $\mathrm{Sem}(\mathcal{C})$, a semigroup seen as generating gauge transformations on the set $\mathrm{Ob}(\mathcal{C})$, somewhat analogous to how a group G functions in a standard quantization procedure for a system whose configuration space is $Q = G/H$ (where H is some subgroup of G). More specifically, for $a \in \mathrm{Ob}(\mathcal{C})$, select an arrow in the semigroup $\mathrm{Hom}(\mathcal{C})$ whose domain is A ($\mathrm{id}_A : A \longrightarrow A$), and then let it act on A, so we may consider maps

$$\phi : \mathrm{Ob}(\mathcal{C}) \longrightarrow \mathrm{Hom}(\mathcal{C}), \tag{8.6}$$

such that for each $A \in \mathrm{Ob}(\mathcal{C})$, $Dom(\phi) = A$, we have $\phi(A) : A \longrightarrow B$, for some $B \in \mathrm{Ob}(\mathcal{C})$. Let us call such a map an *arrow field on* $\mathcal{C}$, and let $AF(\mathcal{C})$ denote the set of arrow fields acting on $\mathrm{Ob}(\mathcal{C})$. State vectors ψ can then be viewed as complex-valued functions on $\mathrm{Ob}(\mathcal{C})$ and $AF(\mathcal{C})$ inherits a semigroup structure, and thus play a role analogous to that of a diffeomorphism group of a space.

For $\sigma \in AF(\mathcal{C})$, we consider operators $\hat{a}(\sigma)$, and set $\ell_x A$ denote the action of $AF(\mathcal{C})$ on functions, so that we have

$$(\hat{a}(\sigma)\psi)(A) = \psi(\ell_x A) = \psi[Range\,\sigma(A)]. \tag{8.7}$$

Given a function $\beta : \mathrm{Ob}(\mathcal{C}) \longrightarrow \mathbb{R}$, we specify $\hat{\beta}$ by $(\hat{\beta}\psi)(A) = \beta(A)\psi(A)$. Furthermore, unitaries are defined by $\hat{U}(\beta) = \exp(-\imath\beta)$, satisfying $\hat{U}(\beta_1)\hat{U}(\beta_2) = \hat{U}(\beta_1 + \beta_2)$.

Provided $\mathrm{Ob}(\mathcal{C})$ is finite, a Hilbert space inner product is defined by

$$\langle\phi|\psi\rangle = \sum_{A\in\mathrm{Ob}(\mathcal{C})} \phi(A)^*\psi(A), \tag{8.8}$$

or, in the measurable sense

$$\langle\phi|\psi\rangle = \int_{\mathrm{Ob}(\mathcal{C})} \phi(A)^*\psi(A)d\mu(A). \tag{8.9}$$

For $\sigma \in AF(\mathcal{C})$ and an operator $\hat{a}(\sigma)$, this leads to a well-defined adjoint $\hat{a}(\sigma)^\dagger$. The operators $\hat{a}(\sigma), \hat{a}(\sigma)^\dagger$ could be conceivably seen as 'annihilation/creation' operators, and satisfy $[\hat{a}^\dagger\hat{a}, \hat{\beta}] = 0$, together with various other relationships involving the unitaries $\hat{U}$.

8.5 Topological and Homotopy Quantum Field Theories (TQFT and HQFT)

TQFT and HQFT are concerned mostly with *topological invariants* in *lower* dimensional spaces (i.e., $n < 4$) and partition functions or 'state sums'. HQFT can be defined as a *'TQFT with background'*, but it also utilizes Homotopy concepts and other tools from Algebraic Topology to investigate—in characteristic TFT style—the invariants of lower dimensional ($n \leq 3$) manifolds and their associated vector spaces. HQFT has considerably accelerated progress with identifying QSS invariants through 'standard' algebraic topology procedures even though its extensions to higher dimensions have not yet appeared. Its more interesting applications are for 3-manifolds and also those related to *Spin Networks and Quantum Spin Foams*. Other potential applications are related to the Ocneanu theory of type II_1 subfactors of von Neumann algebras, topological/state sum invariant determination for 3-manifolds and extensions of HQFT *via* cross modules. A recent development of TFTs to higher dimensions proposed by H. Quinn involves group-categories, as well as homology/cohomology theories of 3-and n-dimensional manifolds in TFT (H. Quinn 1999, arXiv ref.)

8.6 Categorical Basics of TQFT

Baez (2001) points out that a topological quantum field theory (TQFT) is representable in terms of a functor. This is pertinent to explain in the context of quantum gravity. Let M_1 and M_2 be $(n-1)$-dimensional manifolds, and consider an n-dimensional manifold M with boundary $\partial M = M_1 \cup M_2$, in which case we say that M_1, M_2 are *n-cobordant*. Loosely speaking, the manifolds M_1 and M_2 coalesce in order to form the manifold M. As far as TQFT is concerned, M_1, M_2 are ‘physical’ from the general relativity viewpoint, and the merging of M_1, M_2 into M is a ‘topology change’. Thus a cobordism, in a sense, represents the time span in passing from M_1 to M_2. A basic principle of quantum gravity requires that Hilbert spaces (as state spaces), H_1, H_2 say, are assigned to M_1 and M_2, respectively. Now any bounded linear operator $T : H_1 \longrightarrow H_2$ transforms the states of one into those of the other, and very often the unitary condition $T^*T = TT^* = \mathbf{1}$, is required. Thus, via the cobordism of M_1 to M_2, there corresponds a unitary operator $T : H_1 \longrightarrow H_2$.

8.6.1 Possible Decompositions of SpaceTimes. Spin Networks and Spin Foams

Example 8.6.1 A graph is a one-dimensional CW complex. Spin networks are one-dimensional CW complexes, whereas ‘spin foams’ are two-dimensional CW complexes representing two local spin networks with quantum transitions between them, sometimes represented also as functors (Baez 2001).

We are proposing here a new conjecture for *extensions* of TQFT to time-dependent QSS (ETQFTs) that also include Quantum Foams of Spin Networks as lower-dimensional ($n = 2$), specific examples.

Conjecture 8.1. (Baianu et al. 2006a). *The Quantum Fundamental Groupoid, $\Pi_1(D_{QS})$, of any time-dependent ETQFT State Space, D_{QS}, can be computed—at least in principle—via the Generalized van Kampen Theorem (see Brown et al. 2007) as the colimit of the sequence of fundamental groupoids $\{\pi_1^i\ (CW_i)\}$ of the sequence of CW complex subspaces, $\{CW_i\ \}$, forming the CW-approximation (colimit) sequence of the time-dependent ETQFT. In categorical form, this is concisely stated as*:

$$\Pi_1(D_{QS}) \approx \mathsf{colim}_{i=1,\ldots n}\{\Pi_1^i(CW_i)\}. \tag{8.10}$$

Note: In the simpler cases of one- and two-dimensional CW complexes (simplices), such as, respectively, the Quantum Spin Networks (QSN) and the time-dependent QSNs (or Spin Foams), this general conjecture can be proven directly through a step-by-step graph decomposition procedure for QSNs.

UV/IR (ultraviolet/infrared) mixing is a significant aspect of non-commutative QFT and in the context of open strings can be interepreted as a ‘double twist’ of the strings. Including a non-commutative time direction in Minkowski ST, acausal effects may be realized, such as, for instance, an event which precedes its cause, and objects that increase in size rather than Lorentz contract when they are boosted. Interestingly, however, in *non-commutative* open string theory with a background electric field, the string-like effects may team-up to cancel out such an effect (Szabo 2003).

8.7 Quantum Groups, Hopf Algebras and Quantum Groupoids

Groups when viewed as structures defined by symmetries have by definition the invertibility (or reversibility) property. *Quantum groups* (Majid 1995, 2002) originated out of quantum statistical physics in relationship to e.g. the Ising lattice and braid group representations, as well as from conformal field theories. The major conceptual carrier for a quantum groups is that of a much richer structure, namely a *Hopf algebra*; these two terms are often identified in practice. Because of their richer structure the algebras are particularly instrumental at the Planck-scale for several reasons. Hopf algebras enjoy the role of a generalized symmetry: the dual linear space of a Hopf algebra is an algebra specified by a *comultiplication*. This duality represents a desired symmetry between quantum states and observables. Another important feature is that of the *antipode* which provides a *local linearized inverse*. In this way, each quantum group can be represented in terms of its antipode thus leading to further useful algebraic machinery in terms of tensor products. Moreover, since quantum groups involve curvature and gauge theory, they provide a conceptual basis for extending Riemannian geometry to the *non-commutative* setting with the aim of theorizing a quantum gravitational field at the Planck scale. Whereas in Riemannian geometry the pertinent algebra to consider is the space of continuous functions (on a manifold), a quantum group fulfills the required mechanism for quantum geometry in terms of a *C*-Hopf algebra*. Indeed, the structure of a Hopf algebra can be weakened, and a *weak Hopf C*-algebra* inherently embraces the notion of a *quantum groupoid*, and thus a gateway into the applications of *Higher Dimensional Algebra* (HDA).

9 Towards a Higher Dimensional Algebra Approach to SpaceTime Structures

We shall begin this section with the background to the van Kampen theorem and its generalizations to groupoids and higher homotopy. The required concepts of groupoid, topological groupoid, and groupoid atlas leading to higher dimensional structures—such as the double groupoid and the 2-groupoid—are defined in Sect. 9.1. Further details are provided in the following paper by Brown et al. (2007).

9.1 The van Kampen Theorems as Local-to-Global Problems. Generalization of the van Kampen Theorem to Groupoids (GvKT). Higher Homotopy van Kampen theorems

Recall that a groupoid $\mathbb{G}$ is a small category with inverses over its set of objects $X = \mathrm{Ob}(\mathbb{G})$. One often writes $\mathbb{G}_x^y$ for the set of morphisms in $\mathbb{G}$ from x to y. Furthermore, we shall be concerned here with topological groupoids.

Brown (1968) noted that to compute the fundamental group of the circle one had to develop something of covering space theory. Then, Brown found the work of Higgins (1966) on groupoids, which defined free products with amalgamation of groupoids, and this led to a more general formulation of the van Kampen theorem presented for groupoids in the next Sect. 9.2. From this theorem, one can compute a particular fundamental group $\pi_1(X,x_0)$ using combinatorial information on the graph of intersections of path components of U,V,W, but for this it is useful to develop the algebra of groupoids. Notice two special features of this result:

(i) The computation of the *invariant* one wants to obtain, *the fundamental group*, is obtained from the computation of a larger structure, and so part of the work is to give methods for computing the smaller structure from the larger one. This usually involves non canonical choices, such as that of a maximal tree in a connected graph. The work on applying groupoids to groups gives many examples of such methods (Higgins 1966, 2005; Brown 2005).

(ii) The fact that the computation can be done is surprising in two ways: (a) The fundamental group is computed *precisely*, even though the information for it uses input in two dimensions, namely 0 and 1. This is contrary to the experience in homological algebra and algebraic topology, where the interaction of several dimensions involves exact sequences or spectral sequences, which give information only up to extension, and (b) the result is a *non-commutative invariant*, which is usually even more difficult to compute precisely.

The reason for this success seems to be that the fundamental groupoid $\pi_1(X,X_0)$ contains information in *dimensions 0 and 1*, and therefore it can adequately reflect the geometry of the intersections of the path components of U,V,W and the morphisms induced by the inclusions of W in U and V. This fact also suggested the question of whether such methods could be extended successfully to *higher dimensions*.

In order to see how this version of the van Kampen Theorem gives an interesting analogy between the geometry and the algebra we use the language of groupoids. Here $\pi_1(X,X_0)$ is the fundamental *groupoid* of X on a set X_0 of base points: so it consists of homotopy classes rel end points of paths in X joining points of $X_0 \cap X$.

In the case X is the circle S^1, one chooses U,V to be slightly extended semicircles including $X_0 = \{+1,-1\}$. The point is that in this case $W = U \cap V$ is not path connected, and so it is not clear where to choose a single base point. The day is saved by hedging one's bets, and using two base points. The proof of this theorem uses the same 'tricks'/procedures as those used to prove Theorem 1, but in a broader context. In order to compute fundamental groups from this theorem, one can set up some general combinatorial groupoid theory (see Brown et al. 2008). A key feature of this theory is the groupoid $\Im$, the indiscrete groupoid on two objects 0,1, which acts as a unit interval object in the category of groupoids. It also plays a rôle analogous to that of the infinite cyclic group Ç in the category of groups. One then compares the pushout diagrams, the first in spaces, the second in groupoids.

9.2 Generalization of the van Kampen Theorem to the Higher Homotopy, General Van Kampen Theorem: A Powerful NAAT Tool with potential QT and QFT Applications

The general setting of the van Kampen theorem is that of a local-to-global problem which can be explained as follows:

Given an open covering $\mathcal{U}$ of X and knowledge of each hdgb(U) *for U in $\mathcal{U}$, give a determination of* hdgb(X), where hdgb is a functor from Hausdorff spaces to double groupoids as defined in the following Eq. (9.1).

Of course we need also to know the values of the functor hdgb on intersections $U \cap V$ and on the inclusions from $U \cap V$ to U and V.

We first note that that the functor hdgb on the category Top preserves coproducts $\bigsqcup$, since these are just disjoint union in topological spaces and in double groupoids. It is an advantage of the groupoid approach that the coproduct of such objects is so simple to describe.

Suppose we are given a cover $\mathcal{U}$ of X. Then the homotopy double groupoids in the following *ρ-sequence of the cover* are well-defined:

$$\bigsqcup_{(U,V)\in\mathcal{U}^2} \rho^{\square}(U \cap V) \underset{b}{\overset{a}{\rightrightarrows}} \bigsqcup_{U\in\mathcal{U}} \rho^{\square}(U) \xrightarrow{c} \rho^{\square} X. \tag{9.1}$$

The morphisms a,b are determined by the inclusions

$$a_{UV} : U \cap V \longrightarrow U, b_{UV} : U \cap V \longrightarrow V$$

for each $(U, V) \in \mathcal{U}^2$ and c is determined by the inclusion $c_U : U \longrightarrow X$ for each $U \in \mathcal{U}$.

Theorem 9.1 (Brown et al. 2002) ***The General, Higher Homotopy van Kampen Theorem***. *If the interiors of the sets of $\mathcal{U}$ cover X, then in the above ρ-sequence, or diagram of the cover, c is the coequaliser of a,b in the category of double groupoids with connections.*

A special case of this result is when $\mathcal{U}$ has two elements. In this case the coequaliser reduces to a pushout.

9.3 Potential Applications of the Van Kampen Theorem to Crossed Complex Representations of Quantum Space–Time over a Quantum Groupoid

There are several possible applications of the generalized van Kampen theorem in the development of physical representations of a quantized space–time 'geometry'. For example, a possible application of the generalized van Kampen theorem is the construction of the initial, quantized space–time as the *unique colimit* of *quantum causal sets (posets)* which was precisely described in Sect. 4.5.1 in terms of *the nerve of an open covering $N\mathcal{U}$* of the topological space X that would be isomorphic to a *k*-simplex K underlying *X*. The corresponding, *non-commutative* algebra Ω associated with the finitary T_0-poset $P(S)$ is *the Rota algebra* Ω discussed above, and the *quantum topology* T_0 is defined by the partial ordering arrows for regions that can overlap, or superpose, coherently (in the quantum sense) with each other. When the poset $P(S)$ contains $2N$ points we write this as $P_{2N}(S)$. The *unique* (up to an isomorphism) $P(S)$ in the projective limit (*colimit*), $\underleftarrow{\lim}\, P_N X$, recovers a space homeomorphic to X (Sorkin 1991). Other non-abelian results derived from the generalized van Kampen theorem are discussed by Brown (2004).

The generalized van Kampen theorem is a local-to-global theorem–it allows for the computation of certain invariants of a space X which is built of simpler spaces in terms of the invariants of the simpler spaces. On the other hand, the general, direct computation of even the fundamental groupoid of an arbitrary topological space is a difficult, and generally unsolved, problem. In general, *homology* and *cohomology* groups are, more readily computed than homotopy groups for topological spaces of somewhat arbitrary complexity, but by no means an easy task. Cohomology does provide, however, a *more sensitive algebraic invariant* of topological spaces than homology by virtue of being able to introduce a *ring* structure through the definition of a product which is not possible for homology. Thus, *cohomology* can distinguish between topological spaces that have *isomorphic homology* groups.

Crossed complexes have several *advantages* in Algebraic Topology such as:

- They are good for *modeling CW-complexes*. Free crossed resolutions enable calculations with *small CW-models* of $K(G,1)$s *and their maps*.
- They have an interesting relation with the Moore complex of simplicial groups and of *simplicial groupoids*.
- They *generalise groupoids and crossed modules to all dimensions*. Moreover, the natural context for the second relative homotopy groups is crossed modules of groupoids, rather than groups.
- They are convenient for *calculation*, and the functor Π is classical, involving *relative homotopy groups*.
- They provide a kind of *'linear model'* for homotopy types which includes all 2-types. Thus, although they are not the most general model by any means (they do not contain quadratic information such as Whitehead products), this simplicity makes them easier to handle and to relate to classical tools. The new methods and results obtained for crossed complexes can be used as a model for more complicated situations. For example, this is how a general n-adic Hurewicz Theorem was found.
- Crossed complexes have a *good homotopy theory*, with a *cylinder object, and homotopy colimits*. (A *homotopy classification* result generalises a classical theorem of Eilenberg and Mac Lane).
- They are close to chain complexes with a group(oid) of operators, and related to some classical homological algebra (e.g. *chains of syzygies*). In fact if SX is the simplicial singular complex of a space, with its skeletal filtration, then the crossed complex $\Pi(SX)$ can be considered as a slightly *non-commutative version of the singular chains of a space*.

Also note that a replacement for the excision theorem in homology is obtained by using cubical methods to prove *a colimit theorem for the fundamental crossed complex functor on filtered spaces*. This *colimit theorem* is a higher dimensional version of a classical example of a *non-commutative local-to-global theorem*, which itself was the initial motivation for the work by Brown on generalizations of the Van Kampen Theorem. This Seifert–Van Kampen Theorem (SVKT) determines completely the fundamental group $\pi_1(X,x)$ of a space X with base point which is the union of open sets U,V whose intersection is path connected and contains the base point x; the *'local information'* is on the morphisms of fundamental groups induced by the inclusions $U \cap V \to U$, $U \cap V \to V$. The importance of this result reflects the importance of the *fundamental group* in algebraic topology, *algebraic geometry*, *complex analysis*, and many other, mathematical subjects. Indeed, the origin of the fundamental group was in Poincaré's work on *monodromy for complex variable theory*. Essential to this use of crossed complexes and the colimit theorem, is a construction of *higher homotopy groupoids*, with properties described by *an algebra of cubes*. Such a construction is particularly important for conjecturing and proving *local-to-global* theorems since homotopical methods play a key role in many areas. There are applications to *local-to-global problems in homotopy theory* which are more powerful than purely classical tools, while shedding light on those tools.

Furthermore, with the advent of Quantum Groups, Quantum Groupoids, Quantum Algebra and Quantum Algebraic Topology such fundamental theorems in Algebraic Topology also acquire an enhanced importance through their applications to current problems in Theoretical Physics, such as those described in an available preprint (Baianu et al. 2007).

Thus, the Van Kampen Theorem was generalized by formulating it for the *fundamental groupoid* $\pi_1(X,X_0)$ *on a set* X_0 *of base points*, therefore enabling computations in the *non-connected* case, including those in Van Kampen's original paper (Van Kampen 1993). This use of groupoids in dimension 1 suggested the possibility of utilising groupoids in *higher homotopy theory*, and especially the question of the existence of *higher homotopy groupoids*. It will be useful to consider briefly the statement and special features of this generalised Van Kampen Theorem for the fundamental groupoid. First, if X_0 is a set, and X is a space, then $\pi_1(X,X_0)$ denotes the fundamental groupoid on the set $X \cap X_0$ of base points. This allows the set X_0 to be chosen in a way which is appropriate to the geometry. Consider the simple example of the circle S^1 written as the union of two semicircles $E_+ \cup E_-$, then the intersection $\{-1,1\}$ of the semicircles is not connected, so it is not clear where to take the base point. Instead one takes $X_0 = \{-1,1\}$, and so has two base points. This flexibility is very important in computations, and this simple example of S^1 was a motivating example for this development.

We see here how this version of the van Kampen Theorem gives an analogy between the geometry and the algebra provided by the notion of groupoid.

The *fundamental group* is a kind of anomaly in algebraic topology because of its *non-Abelian* nature. Topologists in the early part of the 20th century were aware that:

- the *non-commutativity* of the fundamental group was useful in applications;
- for path connected X there was an isomorphism

$$H_1(X) \cong \pi_1(X, x)^{ab};$$

- the Abelian homology groups existed in all dimensions.

Consequently there was a desire to generalise the nonabelian fundamental group to all dimensions.

In 1932 Čech submitted a paper on higher homotopy groups $\pi_n(X,x)$ to the ICM at Zurich, but it was quickly proved that these groups were abelian for $n \geq 2$, and on these grounds Čech was persuaded to withdraw his paper, so that only a small paragraph appeared in the Proceedings.

We now see the reason for the commutativity as the result (Eckmann–Hilton) that a group internal to the category of groups is just an abelian group. Thus the vision of a *non-commutative* higher dimensional version of the fundamental group has since 1932 been generally considered to be a mirage.

Theorem 2 is also anomalous: it is a colimit type theorem, and so, even in the non-connected case, yields *complete information* on the fundamental groups which are contained in it; whereas the usual method in algebraic topology is to relate different dimensions by exact sequences or even spectral sequences, which usually yield information only up to extension. Thus exact sequences by themselves cannot show that a group is given as an HNN-extension: however such a description may be obtained from a pushout of groupoids, generalizing the pushout of groupoids in Brown et al. (2007), (see also Chap. 9 in Brown 2005).

It was then found that the theory of covering spaces could be given a nice exposition using the notion of covering morphism of groupoids (Chap. 10 in Brown 2005). It was also found by Higgins and Taylor (1968) that there was a nice theory of orbit groupoids which gave models for the fundamental groupoids of orbit spaces (Chap. 11 in Brown 2005).

The objects of a groupoid add to group theory a 'spatial component', which is essential in many applications. This is evident in many parts of Ehresmann's work. Another view of this anomalous success of groupoids is that they have structure in two dimensions, 0 with the objects and 1 with the arrows. We have a colimit type theorem for this larger structure, and so a good model of the geometry. Useful information on fundamental groups is carried by the fundamental groupoid. It is therefore natural to seek for higher homotopy theory algebraic models which:

- have structure in a range of dimensions;
- contain useful information on classical invariants, and
- satisfy van Kampen type theorems.

That is, we seek *non-Abelian methods for higher dimensional local-to-global problems in homotopy theory*.

9.4 Local-to-Global Construction Principles Consistent with Quantum 'Axiomatics'

An alternative approach involves generalizing fundamental theorems of algebraic topology from specialized, 'globally well-behaved' topological spaces, to arbitrary ones. In this category are both the GvKT and the generalized Hurewicz theorem of AT. Several fundamental theorems of Algebraic Topology, such as the Hurewicz (1955), the Whitehead (1965) and the van Kampen (1933) theorems were first proven for CW complexes and subsequently extended to a broader category of topological spaces. Such theorems greatly aid the calculation of homology, cohomology and homotopy groups of topological spaces. In the case of the Hurewicz theorem, this was generalized to arbitrary topological spaces (Spanier 1966), and establishes that certain homology groups are isomorphic to 'corresponding' homotopy groups of an arbitrary topological space. Brown and coworkers (1999, 2004a,b,c) went further and generalized the van Kampen theorem, at first to homotopy groupoids (Brown 1967), and then, to higher dimensional algebras involving, for example, homotopy double groupoids and 2-categories (Brown 2004a). The more sensitive *algebraic invariant* of topological spaces seems to be, however, captured only by *cohomology* theory through an algebraic *ring* structure that is not accessible either in homology theory, or in the existing homotopy theory. Thus, two arbitrary topological spaces that have isomorphic homology groups may not have isomorphic cohomological ring structures, and may also not be homeomorphic, even if they are of the same homotopy type. The corollary of this statement may lead to an interesting cohomology-based classification in a category of certain Coh topological spaces that have isomorphic ring structures and are also homeomorphic. Furthermore, several *nonabelian* results in algebraic topology could only be derived from the generalized van Kampen theorem (cf. Brown 2004a), so that one may find links of such results to the expected '*non-commutative* geometrical' structure of quantized space–time (Connes 1994). In this context, the important algebraic–topological concept of a *Fundamental Homotopy Groupoid (FHG) is applied to a Quantum Topological Space (QTS)* as a "partial classifier" of the *invariant* topological properties of quantum spaces of *any* dimension; quantum topological spaces are then linked together in a *crossed complex over a quantum groupoid* (Sect. 8.7), thus suggesting the construction of global topological structures from local ones with well-defined quantum homotopy groupoids. The latter theme is then further pursued through defining locally topological groupoids that can be globally characterized by applying the Globalization Theorem, which involves the *unique* construction of the Holonomy Groupoid. In a real quantum system, a unique quantum holonomy groupoid may represent parallel transport processes and the 'phase-memorizing' properties of such remarkable quantum systems. This theme can be similarly pursued in the continuous case through locally Lie groupoids and their corresponding Globalization theorem. The converse approach may involve the use of fundamental theorems of Algebraic Topology such as the generalized van Kampen theorem for characterizing the topological invariants of a higher- dimensional, or 'composite', topological space in terms of the (known) invariants of its 'simpler' subspaces (such as CW complexes in the case of Whitehead's theorem and the original version of the van Kampen theorem).

9.5 Outline of A Higher Dimensional Algebra Approach to Non-Abelian Algebraic Topology and Quantum Gravity

In higher dimensional algebra the concept of a category generalizes to that of an n-category.

We list here a short (but tentative) dictionary of analogies between general relativity theory (GR) and quantum theory (QT), (Baez and Dolan 1995; Baez 2001):

(1) (GR) pairs of spatial $(n-1)$-manifolds (M_1, M_2)—(QT) assigned Hilbert spaces H_1, H_2, respectively
(2) (GR) cobordism leading to a spacetime n-manifold M—(QT) (unitary) operator $T : H_1 \longrightarrow H_2$
(3) (GR) composition of cobordisms—(QT) composition of operators
(4) (GR) identity cobordism—(QT) identity operator.

The next step is to re-phrase this interplay of ideas categorically. So let Hilb denote the category whose objects are Hilbert spaces H with arrows the bounded linear operators on H. Let nCob denote the category whose objects are $(n-1)$-dimensional manifolds as above, and whose arrows are cobordisms between objects. Next we define a functor

$$Z : \mathsf{nCob} \longrightarrow \mathsf{Hilb}, \tag{9.2}$$

which assigns to any $(n-1)$-manifold M_1, a Hilbert space of states $Z(H_1)$, and to any n-dimensional cobordism $M : M_1 \longrightarrow M_2$, a (bounded) linear operator $Z(M) : Z(M_1) \longrightarrow Z(M_2)$, satisfying:

i) given n-cobordisms $M : M_1 \longrightarrow M_2$ and $\check{M} : \check{M}_1 \longrightarrow \check{M}_2$, we have $Z(M\check{M}) = Z(\check{M})Z(M)$.
ii) $Z(\mathrm{id}_{M1}) = \mathrm{id}_{z(M_1)}$.

Observe that (i) means the duration of time corresponding to the cobordism M followed by that of the cobordism $\check{M}$, is the same as the combined duration for that of $M, \check{M}$ Part (ii) is the standard functorial mapping condition for identities. Since a TQFT omits local degrees of freedom, a topology change reflects a change in the physical universe. Such a theory necessitates further development; on the one hand, the relationship between nCob and n-categories (cf Baez and Dolan 1995), and on the other, that of a (non-commutative) theory of presheaves of Hilbert spaces/C*-algebras which can be fitted into some Quantum theory. Furthermore, there is a necessity to realize the Grothendieck (1971) idea of *fibrations of n-categories over n-categories* as a possible unifying model for all such TQF theories.

10 Conclusions and Discussion

Current developments in SpaceTime Ontology were here discussed with a view to bridging the gap between Quantum Field Theories and General Relativity, a long standing problem in the foundation of Mathematical and Theoretical Physics which is of considerable conceptual importance. Mathematical generalizations from quantum groups to quantum groupoids, and then further to quantum topological groupoids and double groupoids, as well as higher dimensional algebra are concluded to be logical requirements of the unification between quantum and relativity theories that would be leading towards a deeper understanding of quantum gravity and quantum space–time geometry through QAT. In a subsequent paper (Baianu et al. 2007), we shall further consider quantum algebraic topology from the standpoints of the theory of categories, functors, quantum logics, higher dimensional algebra, as well as the integrated viewpoint of the Quantum Logics in a Generalized 'Topos'—a new concept that links quantum logics with category theory. Other potential applications of quantum algebraic topology to operational quantum nano-automata were also recently suggested (Baianu 2004). Algebraically simpler representations of quantum space–time than QAT have also been proposed in terms of causal sets and quantized causal sets (see for example, Raptis 2003; Raptis and Zapatrin 2000) that might also prove to be useful in emerging quantum gravity theories and that may have a topology compatible with the QAT approach summarized in this paper.

The algebraic structure of lattices, the algebraic–topological structures of *quantum groupoids*—including quantum groups, compact groupoids, quantum 2-groupoids and certain categories of sheaves—are suggested as being especially important for further developments of unified quantum field theories. Such concepts could also link quantum field theories with general relativity, thus leading towards relativistic quantum gravity.

The existence of spacetime higher dimensions in Quantum Gravity is a moot, fundamental point for any theory of levels in ontology as it affects its entire conceptual and formal structure. The formal ability of handling higher dimensional spacetime structures through non-Abelian Algebraic Topology constructions and results—such as the Higher Homotopy van Kampen Theorem—paves the way towards developing non-Abelian Quantum Gravity theories founded upon non-commutative Quantum Logics that represent universal reality at both microscopic and astrophysical scales.

Acknowledgements The authors gratefully acknowledge fruitful suggestions from Dr. Roberto Poli. I.C.B gratefully acknowledges the support in part by a grant from Renessen Biotechnology Co.

References

Alfsen EM, Schultz FW (2003) Geometry of state spaces of operator algebras. Birkäuser, Boston–Basel–Berlin

Andersen PB, Emmeche C, Finnemann NO, Christiansen PV (2000) Downward causation. minds, bodies and matter. Aarhus University Press, Aarhus DK

Baez J (2001) Higher dimensional algebra and Planck scale physics. In: Callender C, Hugget N (eds) Physics Meets Philosophy at the Planck scale. Cambridge University Press, pp 177–195

Baez J, Dolan J (1995) Higher dimensional algebra and topological quantum field theory. J Math Phys 36:6073–6105

Baianu IC, Marinescu M (1968) Organismic supercategories: towards a unified theory of systems. Bull Math Biophys 30:148

Baianu IC (1970) Organismic supercategories: II. On multistable systems. Bull Math Biophys 32:539–561

Baianu IC (1971a) Organismic supercategories and qualitative dynamics of systems. Bull Math Biophys 33(3):339–354

Baianu IC (1971b) Categories, functors and quantum algebraic computations. In: Suppes P (ed) Proceed Fourth Intl Congress Logic-Mathematics-Philosophy of Science. September 1–4, 1971, Buch

Baianu IC, Scripcariu D (1973) On adjoint dynamical systems. Bull Math Biophys 35(4):475–486

Baianu IC (1973) Some algebraic properties of (M,R) – systems. Bull Math Biophys 35:213–217

Baianu IC, Marinescu M (1974) A Functorial Construction of (M,R) – Systems. Revue Roumaine de Mathematiques Pures et Appliquees 19:388–391

Baianu IC (1977) A logical model of genetic activities in Łukasiewicz algebras: the non-linear theory. Bull Math Biol 39:249–258

Baianu IC (1980) Natural transformations of organismic structures. Bull Math Biol 42:431–446

Baianu IC (1983) Natural transformation models in molecular biology. In: Proceedings of the SIAM Natl. Meet. Denver, CO. http://cogprints.org/3675/; http://cogprints.org/3675/0l/Naturaltransfmolbionu6.pdf

Baianu IC (1984) A molecular-set-variable model of structural and regulatory activities in metabolic and genetic networks. FASEB Proc 43:917

Baianu IC (1987a) Computer models and automata theory in biology and medicine. In: Witten M (ed) Mathematical models in medicine, vol. 7. Pergamon Press, New York, 1513–1577; CERN Preprint No. EXT-2004-072: http://doe.cern.ch//archive/electronic/other/ext/ext-2004-072.pdf

Baianu IC (1987b) Molecular Models of Genetic and Organismic Structures. In: Proceedings of relational biology symposium Argentina; CERN Preprint No. EXT-2004-067: http://doc.cern.ch/archive/electronic/other/ext/ext2004067/Molecular Models ICB3.doc

Baianu IC (2004a) Quantum Nano-Automata (QNA): Microphysical Measurements with Microphysical QNA Instruments. CERN Preprint EXT–2004–125

Baianu IC (2004b) Quantum Interactomics and Cancer Mechanisms, Preprint No. 00001978: http://bioline.utsc.utoronto.ca/archive/00001978/01/QuantumInteractomics In Cancer–Sept13k4E–cuteprt.pdf. http://bioline.utsc.utoronto.ca/archive/00001978/

Baianu IC, Glazebrook JF, Georgescu G (2004) Categories of Quantum Automata and N-Valued Łukasiewicz Algebras in Relation to Dynamic Bionetworks, (M,R)–Systems and Their Higher Dimensional Algebra, Abstract and Preprint of Report: http://www.ag.uiuc.edu/fs401/QAuto.pdf and http://www.medicalupapers.com/quantum+automata+math+categories+baianu/

Baianu IC (2006) Robert Rosen's Work and Complex Systems Biology. Axiomathes 16(1–2):25–34

Baianu IC, Brown R, Glazebrook JF (2006a) Quantum Algebraic Topology and Field Theories, (manuscript in preparation) http://www.ag.uiuc.edu/fs40l/QAT.pdf

Baianu IC, Brown R, Georgescu G, Glazebrook JF (2006b) Complex nonlinear biodynamics in categories, higher dimensional algebra and Łukasiewicz–Moisil topos: transformations of neuronal, genetic and neoplastic networks. Axiomathes 16(1–2):65–122

Baianu IC, Glazebrook JF, Georgescu G, Brown R (2007) Non-abelian Algebraic Topology Representations of Quantum Space–Time in a Generalized 'Topos' with a Quantum N–Valued Logic Classifier. (in submission)

Baianu IC, Poli R (2008) From simple to super- and ultra- complex systems: a paradigm shift towards a Non-Abelian dynamics. In: Poli R et al (eds) Theory and applications of ontology, vol 1. Springer, Berlin, p 30

Bennett M, Hacker P (2003) Philosophical foundations of neuroscience. Blackwell Publishing, London

Birkhoff G, von Neumann J (1936) The logic of quantum mechanics. Ann Math 37:823–843

Brown R, Glazebrook JF, Baianu IC (2007) A conceptual construction for complexity levels theory in spacetime categorical ontology: Non-Abelian algebraic topology, many-valued logics and dynamic systems. Axiomathes 17:(in this issue)

Brown R (2004) Crossed complexes and homotopy groupoids as non commutative tools for higher dimensional local-to-global problems. In: Proceedings of the Fields Institute Workshop on

Categorical Structures for Descent and Galois Theory, Hopf Algebras and Semiabelian Categories (September 23–28, 2002), Fields Institute Communications 43, 101–130

Brown R, Sivera R (2004) Non-Abelian Algebraic Topology: Part I, http://www.bangor.ac.uk/mas010/nonab-a-t.html; http://www.bangor.ac.uk/mas010/nonab-t/partI010604.pdf

Brown R, Janelidze (2004) Galois theory and a new homotopy double groupoid of a map of spaces. Appl Categ Struct 12:63–80

Brown R, Higgins PJ, Sivera R (2008) Noncommutative algebraic topology, to appear

Brown R, Paton R, Porter T (2004) Categorical language and hierarchical models for cell systems. In: Paton R, Bolouri H, Holcombe M, Parish JH, Tateson R (eds) Computation in Cells and Tissues - Perspectives and Tools of Thought. Natural Computing Series, Springer Verlag, Berlin, pp. 289–303

Brown R, Porter T (2003) Category theory and higher dimensional algebra: potential descriptive tools in neuroscience. Proceedings of the International Conference on Theoretical Neurobiology, Delhi, February 2003, edited by Nandini Singh, National Brain Research Centre, Conference Proceedings 1:80–92

Brown R, Hardie K, Kamps H, Porter T (2002) The homotopy double groupoid of a Hausdorff space. Theor. Appl. Categories 10:71–93.

Brown R (2006) Topology and groupoids. BookSurge, LLC.

Butterfield J, Isham CJ (2001) Spacetime and the philosophical challenges of quantum gravity. In: Callender C, Hugget N (eds) Physics meets philosophy at the Planck scale. Cambridge University Press, pp 33–89

Butterfield J, Isham CJ 1998, 1999, 2000–2002, A topos perspective on the Kochen–Specker theorem I–IV, Int J Theor Phys, 37, No 11.: 2669–2733 38, No 3.: 827–859, 39, No 6.: 1413–1436, 41, No 4.: 613–639

Chalmers DJ (1996) The Conscious Mind–In search of a fundamental theory. Oxford University Press, Oxford, UK

Chevalley C (1946) The theory of Lie groups. Princeton Univ Press, Princeton, NJ

Chodos A, Detweiler S (1980) Where has the fifth diemsion gone? Phys Rev D 21:2167–2170

Connes A (1994) Noncommutative geometry. Academic Press, New York and London

Cramer JG (1980) The transactional interpretation of quantum mechanics. Phys Rev D 22:362

Dalla Chiara M, Giuntini R, Greechie R (2004) Reasoning in quantum theory, trends in logic–studia logica library, vol 22. Kluwer, Dordrecht

Ehresmann C (1965) Catégories et structures. Dunod, Paris

Ehresmann C (1966) Trends toward unity in mathematics. Cahiers de Topologie et Geometrie Differentielle 8:1–7

Ehresmann AC, Vanbremeersch J-P (1987) Hierarchical evolutive systems: a mathematical model for complex systems. Bull of Math Biol 49(1):13–50

Ehresmann AC, Vanbremeersch J-P (2006) The memory evolutive systems as a model of Rosen's organisms. Axiomathes 16(1–2):13–50

Eilenberg S, Mac Lane S (1945) The general theory of natural equivalences. Trans Am Math Soc 58:231–294

Elsasser MW (1981) A form of logic suited for biology. In: Robert R (ed) Progress in theoretical biology, vol 6. Academic Press, New York and London, pp 23–62

Gabor D (1946) Theory of communication. J IEE (London) 93(III):429–457

Georgescu G, Popescu D (1968) On algebraic categories. Revue Roumaine de Mathematiques Pures et Appliquées 13:337–342

Georgescu G, Vraciu C. (1970) On the characterization of Łukasiewicz-Moisil algebras. J. Algebra 16(4):486–495.

Georgescu G (2006) N-valued logics and Łukasiewicz–Moisil algebras. Axiomathes 16(1–2):123–136

Grothendieck A (1971) Revêtements Étales et Groupe Fondamental (SGA1), chapter VI: Catégories fibrées et descente. Lecture Notes in Math. 224, Springer–Verlag, Berlin

Grothendieck A (1957) Sur quelque point d-algébre homologique. Tohoku Math J 9:119–121

Grothendieck A, Dieudonné J (1960–1967) Éléments de géométrie algébrique. Publ Inst des Hautes Etudes de Science Publ Math. : 4 (1960), 8(1961), 11 (1961), 17 (1963), 20 (1964), 24 (1965),28 (1966) and 32(1967)

Hawking SW, Ellis GFR (1973) The Large Scale Structure of Space–Time. Cambridge University Press

Heller A (1958) Homological algebra in Abelian categories. Ann of Math 68:484–525

Heller A, Rowe KA (1962) On the category of sheaves. Am J Math 84:205–216

Higgins PJ, Mackenzie KCH (1990) Fibrations and quotients of differentiable groupoids. J London Math Soc 42(1):101–110

Hurewicz E (1955) On the concept of fiber spaces. Proc Nat Acad Sci USA 41:956–961

van Kampen EH (1933) On the connection between the fundamental groups of some related spaces. Am J Math 55:261–267

Kleisli H (1962) Homotopy theory in Abelian categories. Can J Math 14:139–169

Knight JT (1970) On epimorphisms of non-commutative rings. Proc Cambridge Phil Soc 68:589–601

Krips H (1999) Measurement in quantum theory. In: Zalta EN (ed) The Stanford Encyclopedia of Philosophy (on line) (Winter 1999 Edition)

Landsman NP (1998) Mathematical topics between classical and quantum mechanics. Springer Verlag, New York

Lawvere FW (1966) The category of categories as a foundation for mathematics. In: Eilenberg S et al (eds) Proc conf categorical algebra- La Jolla. Springer–Verlag, Berlin, Heidelberg and New York, pp 1–20

Lawvere FW (1963) Functorial semantics of algebraic theories. Proc Natl Acad Sci USA Math 50:869–872

Lawvere FW (1969) Closed Cartesian Categories. Lecture held as a guest of the Romanian Academy of Sciences, Bucharest

Lôfgren L (1968) An axiomatic explanation of complete self-reproduction. Bull Math Biophys 30:317–348

Lubkin S (1960) Imbedding of Abelian categories. Trans Am Math Soc 97:410–417

Mac Lane S (1963) Homology. Springer, Berlin

Mac Lane S (2000) Categories for the working mathematician. Springer, New York–Berlin–Heidelberg

Mac Lane S, Moerdijk I (1992) Sheaves in Geometry and Logic. A first introduction in topos theory. Springer-Verlag, New York

Majid S (1995) Foundations of quantum group theory. Cambridge Univ Press

Majid S (2002) A quantum groups primer. Cambridge Univ Press, Cambridge, UK

Mallios A, Raptis I (2003) Finitary, causal and quantal vacuum Einstein gravity. Int J Theor Phys 42:1479

Manders KL (1982) On the space-time ontology of physical theories. Phil Sci 49(4):575–590

McGinn C (1995) Consciousness and space. In: Metzinger T (ed) Conscious experience. Imprint Academic, Thorverton

Mitchell B (1965) Theory of categories. Academic Press, London

Mitchell B (1964) The full imbedding theorem. Am J Math 86:619–637

Oberst U (1969) Duality theory for Grothendieck categories. Bull Am Math Soc 75:1401–1408

Oort F (1970) On the definition of an Abelian category. Proc Roy Neth Acad Sci 70:13–02

Ore O (1931) Linear equations in non-commutative fields. Ann Math 32:463–477

Penrose R (1994) Shadows of the mind. Oxford University Press, Oxford, UK

Plymen RJ, Robinson PL (1994) Spinors in Hilbert Space, Cambridge Tracts in Math. 114, Cambridge Univ Press, Cambridge, UK

Poli, Roberto (2006) Steps towards a synthetic methodology. In: Proc conf continuity and change: perspectives on science and religion. Metanexus Institute, June 2006, Philadelphia, PA. http://www.metanexus.net/conferences/pdf/conference2006/Poli.pdf

Poli R (2008) Ontology: the categorical stance. In: Poli R et al (eds) Theory and applications of ontology, vol 1. Springer, Berlin (in press)

Popescu N (1973) Abelian categories with applications to rings and modules, 2nd edn, 1975. Academic Press, New York 1975 (English translation by I.C. Baianu)

Popescu N (1966 and 1967) Elements of Sheaf theory. St. Cerc. Mat. V/VI, 18–19:205–240; 945–991

Popescu N (1967) La théorie générale de la décomposition. Rev Roum Math Pures et Appl 7:1365–1371

Pribram KH (1991) Brain and Perception: Holonomy and Structure in Figural processing. Lawrence Erlbaum Assoc, Hillsdale

Pribram KH (2000) Proposal for a quantum physical basis for selective learning. In: Farre (ed) Proceedings ECHO IV, pp 1–4

Raptis I, Zapatrin RR (2000) Quantisation of discretized spacetimes and the correspondence principle. Intl Jour Theor Phys 39:1

Raptis I (2003) Algebraic quantisation of causal sets. Int Jour Theor Phys 39:1233

Rashevsky N (1965) The representation of organisms in terms of predicates. Bull Math Biophys 27: 477–491

Rashevsky N (1969) Outline of a unified approach to physics, biology and sociology. Bull Math Biophys 31:159–198
Roos JE (1964) Sur la condition Ab6 et ses variantes dans les catégories abéliennes. CRAS Paris 257:2368–2371
Roux A (1964) Sur une équivalence de catégories abéliennes. CRAS Paris 258:5566–5569
Roberts JE (2004) More lectures on algebraic quantum field theory. In: Connes A et al Noncommutative Geometry. Springer, Berlin
Rosen R (1985) Anticipatory systems. Pergamon Press, New York
Rosen R (1958a) A relational theory of biological systems. Bull Math Biophys 20:245–260
Rosen R (1958b) The representation of biological systems from the standpoint of the theory of categories. Bull Math Biophys 20:317–341
Rovelli C (1998) Loop Quantum Gravity. In: Dadhich N et al (eds) Living reviews in relativity (refereed electronic journal) http://www.livingreviews.org/Articles/Volume1/1998 1 rovelli
Russell B, Whitehead AN (1925) Principia mathematica. Cambridge Univ Press, Cambridge
Ryle G (1949) The concept of mind. Hutchinson, London
Silver L (1967) Non-commutative localization and applications. J Algebra 7:44–76
Sorkin RD (1991) Finitary substitute for continuous topology. Int J Theor Phys 30(7):923–947
Smolin L (2001) Three roads to quantum gravity. Basic Books, New York
Spanier EH (1966) Algebraic Topology. McGraw Hill, New York
Spencer-Brown G (1969) Laws of form. George Allen and Unwin, London
Stenström B (1968) Direct sum decomposition in Grothendieck categories. Arkiv Math 7:427–432
Szabo RJ (2003) Quantum field theory on non-commutative spaces. Phys Rep 378:207–209
Takahashi H (1963) Adjoint pair of functors on Abelian categories. J Fac Soc Univ Tokyo 13:175–181
Unruh WG (2001) Black holes, dumb holes, and entropy. In: Callender C, Hugget N (eds) Physics meets philosophy at the Planck scale. Cambridge University Press, Cambridge, UK, pp 152–173
Várilly JC (1997) An introduction to noncommutative geometry arXiv:physics/9709045
Velmans M (2000) Understanding consciousness. Routledge, London
von Neumann J (1932) Mathematische Grundlagen der Quantenmechanik. Springer, Berlin
Wess J, Bagger J (1983) Supersymmetry and supergravity. Princeton University Press, Princeton, NJ
Wheeler J, Zurek W (1983) Quantum theory and measurement. Princeton University Press, Princeton NJ
Weinberg S (1995–2000) The quantum theory of fields, vols I–III. Cambridge Univ Press, Cambridge, UK
Weinstein A (1996) Groupoids: unifying internal and external symmetry. Notices Am Math Soc 43: 744–752
Whitehead JHC (1941) On adding relations to homotopy groups. Ann Math 42(2):409–428