COMPUTING HOMOTOPY TYPES USING CROSSED N-CUBES OF GROUPS*

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Dedicated to the memory of Frank Adams

Introduction

The aim of this paper is to explain how, through the work of a number of people, some algebraic structures related to groupoids have yielded algebraic descriptions of homotopy n-types. Further, these descriptions are explicit, and in some cases completely computable, in a way not possible with the traditional Postnikov systems, or with other models, such as simplicial groups.

These algebraic structures take into account the action of the fundamental group. It follows that the algebra has to be at least as complicated as that of groups, and the basic facts on the use of the fundamental group in 1-dimensional homotopy theory are recalled in Section 1. It is reasonable to suppose that it is these facts that a higher dimensional theory should imitate.

However, modern methods in homotopy theory have generally concentrated on methods as far away from those for the fundamental group as possible. Such a concentration has its limitations, since many problems in the applications of homotopy theory require a non-trivial fundamental group (low dimensional topology, homology of groups, algebraic K-theory, group actions, ...). We believe that the methods outlined here continue a classical tradition of algebraic topology. Certainly, in this theory non-Abelian groups have a clear role, and the structures which appear arise directly from the geometry, as algebraic structures on sets of homotopy classes[1].

It is interesting that this higher dimensional theory emerges not directly from groups, but from groupoids. In Sections 1 and 2 we state some of the main facts about the use of multiple groupoids in homotopy theory, including two notions of *higher homotopy groupoid*, and the related notions of *crossed complex* and of *crossed n-cube of groups*. Theorem 2.4 shows how to calculate standard homotopy invariants of 3-types for the classifying space of

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[1]This philosophy is developed in the paper [13]

a crossed square. We also show in Section 2 how crossed n-cubes of groups and the notion of n-cube of fibrations, with the use of the Generalized Seifert-Van Kampen Theorem due to Brown and Loday, 1987a, [27], cf. Theorem 3.2, can be used in some practical cases for the computation of homotopy types (Proposition 3.3).

An interesting methodological point is that the description of the whole n-type has, by these methods, better algebraic properties than do the individual invariants (homotopy groups, Whitehead products, etc.). As an application, we give some explicit results on 3-types, including computations of Whitehead products at this level. In Section 4 we prove a result from Section 1. In Section 5, we show that many simply connected 3-types arise from a crossed square of Abelian groups (Theorem 5.1).

Baues, in [2, 3] also considers algebraic models of homotopy types involving non-Abelian groups, and in the second reference considers *quadratic modules* and *quadratic chain complexes*. It seems that the sets of results of the two techniques have a non-trivial intersection, but neither is contained in the other. Thus a further comparison, and possibly integration, of the two types of methods would be of interest[2].

Joyal and Tierney have also announced a model of 3-types using *braided 2-groupoids*. These models are equivalent to the *braided crossed modules* of Brown and Gilbert, [16], which are there related to simplicial groups and used to discuss automorphism structures of crossed modules[3].

1 Groups and homotopy 1-types

The utility of groups in homotopy theory arises from the standard functors

$$\begin{aligned} \pi_1 &: (\text{spaces with base point}) \to (\text{groups}) \\ B &: (\text{groups}) \to (\text{spaces with base point}) \end{aligned}$$

known as the *fundamental group* and *classifying space* functors respectively. The classifying space functor is the composite of geometric realisation and the nerve functor N from groups to simplicial sets. These functors have the properties:

1.1 *There is a natural equivalence of functors* $\pi_1 \circ B \simeq 1$.

1.2 *If* G *is a group, then* $\pi_i BG$ *is 0 if* $i \neq 1$.

The fundamental group of many spaces may be calculated using the Seifert-Van Kampen theorem, or using fibrations of spaces. Further, if X is a connected CW-complex and G is a group, then there is a natural bijection

$$[X, BG] \cong Hom(\pi_1 X, G),$$

[2] In particular, quadratic modules do not satisfy the methods explained in [13, Sections 1,2].

[3] The relation of such ideas to potential developments in "higher order symmetry" is also postulated in [10].

where the square brackets denote pointed homotopy classes of maps. It follows that there is a map

$$X \to B\pi_1 X$$

inducing an isomorphism of fundamental groups. It is in this sense that groups are said to model homotopy 1-types, and a computation of a group G is also regarded as a computation of the 1-type of the classifying space BG.

A standard block against generalising this theory to higher dimensions has been that higher homotopy groups are Abelian. The algebraic reason for this is that group objects in the category of groups are Abelian groups. This seems to kill the case for a 'higher dimensional group theory', and in 1932 was the reason for an initial dissatisfaction with Čech's definition of higher homotopy groups, [30], Incidentally, Čech also suggested the idea of higher homotopy groups went back to Dehn, who never published it, [32] (Dieudonné, 1989). The difficulties of basic homotopy theory are shown by the fact that Hurewicz never published the proofs of the results announced in his four notes on homotopy groups, [46] (Hurewicz, 1935, 1936); that a proof of the Homotopy Addition Theorem did not appear in print till [43] Hu, 1953; and that even current proofs of this basic theorem are not easy (e.g. [55] G.W.Whitehead, 1978)[4].

It has for some time been established that most of the theory of the fundamental group is better expressed in terms of the fundamental groupoid (Brown, 1968, 1988) [9] in that theorems:

- have more natural and convenient expression;
- have more powerful statements;
- and have simpler proofs.

As an example, we mention the description in Brown, 1988, [9] of the fundamental groupoid of the orbit space of a discontinuous action. Thus it is natural to ask:
Can a 'better' higher homotopy theory be obtained using some notion of 'higher homotopy groupoid'?
Expectations in this direction were expressed in Brown, 1967, [7].

By now, some of the answers to this question seem to be of the 'best possible situation' kind, and suggest that homotopy theory is in principle coincident with a 'higher dimensional group(oid) theory'. Such a theory is a significant generalisation of group theory. In view of the many applications of group theory in mathematics and science, the wider uses of this generalisation, and the principles underlying it, need considerable further study. For example, some possibilities are sketched in [11, 12], Brown, Gilbert, Hoehnke, and Shrimpton, 1991, [17]. Also, the known applications in homotopy theory have so far used what seems only a small part of the algebra.

[4]This theorem is linked with the monoidal structure on crossed complexes in [24, Section 9.9].

2 Multiple groupoids

The simplest object to consider as a candidate for a 'higher dimensional groupoid' is an *n-fold groupoid*. This is defined inductively as a *groupoid object in the category of* $(n-1)$*-fold groupoids*, or alternatively, as a *set with n compatible groupoid structures*. The compatibility condition is that if $\circ_i$ and $\circ_j$ are two distinct groupoid structures, then the *interchange law* holds, namely that

$$(a \circ_i b) \circ_j (c \circ_i d) = (a \circ_j c) \circ_i (b \circ_i d)$$

whenever both sides are defined. This is often expressed in terms of the diagram

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{array}{l} \longrightarrow i \\ \downarrow j \end{array}$$

Note that Ehresmann, 1963, [35], defines double categories, and the above definition is a simple extension of that concept. The argument that a group object in the category of groups is an Abelian group now yields that a double groupoid contains a family of Abelian groups, one for each vertex of the double groupoid. More generally, a basic result is that a double groupoid contains also two *crossed modules over groupoids* (Brown and Spencer, 1976) [29]. For example, the *horizontal crossed module* is defined analogously to the second relative homotopy group. It consists in dimension 2 of the elements of the form

$$\begin{array}{ccc} & \partial m & \\ 1_V & m & 1_V \\ & 1_H & \end{array}$$

where 1_H and 1_V denote identities for the horizontal and vertical 'edge' groupoid structures respectively. In dimension 1 it consists of the horizontal 'edge' groupoid. The boundary ∂m of an element m is as shown, and the action is not hard to define, as suggested by the following diagram:

$$m^b = \begin{array}{ccccc} & b^{-1} & \partial m & b & \\ 1_V & \mid\ \mid & m & \mid\ \mid & 1_V \\ & b^{-1} & 1_H & b & \end{array}$$

where $\mid\ \mid$ denotes a vertical identity.

The existence of these crossed modules in any double groupoid, and the fact that a particular kind of double groupoid can be constructed from any given crossed module (Brown and Spencer, 1976), [29], together illustrate that double groupoids are in some

sense *more non-Abelian than groups*. This in principle makes them more satisfactory as models for two dimensional homotopy theory than are the second homotopy groups. In fact it is known that crossed modules over groupoids, and hence also certain double groupoids, model all homotopy 2-types [51] (see Mac Lane and Whitehead, 1950, [51], but note that they use the term "3-type" for what is now called "2-type").

One of the features of the use of multiple groupoids is that they are most naturally considered as cubical objects of some kind, since they have structure in different directions. Analogous simplicial objects may in some cases be defined, but their properties are often difficult to establish, and are sometimes obtained by referring to the cubical analogue. For a general background to problems on algebraic models of homotopy types, see Grothendieck, 1983, [41], although this work does not take into account the use of multiple groupoids[5].

The first example of which we know of a 'higher homotopy groupoid' was found in 1974 (see [18] Brown and Higgins, 1978), 42 years after Čech's definition of homotopy groups, namely the *fundamental double groupoid of a pair of pointed spaces*. It is conveniently expressed in the more general situation of filtered spaces as follows (Brown and Higgins, 1981b, [20], as modified in [23] Brown and Higgins, 1991, Section 8). Let

$$X_* : X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X$$

be a filtered space. Let RX_* be the *singular cubical complex* of X_*, consisting for all $n \geqslant 0$ of the filtered maps $I^n_* \to X_*$, where the standard n-cube I^n is filtered by its skeleta, and with the standard face and degeneracy maps. Let ρX_* consist in dimension n of the homotopy classes, through filtered maps and rel vertices, of such filtered maps. (The modification in the 1991 paper[6] is to take the homotopies rel vertices.) The standard gluing of cubes in each direction imposes an extra structure of n compositions on $(RX_*)_n$ for each $n \geqslant 1$.

It is a subtle fact [20] (Brown and Higgins, 1981b) that this structure is inherited by ρX_* to give the latter the structure of n-fold groupoid in each dimension n. There is also an extra, easily verified structure, on both RX_* and ρX_*, namely that of *connections*: these are extra degeneracy operations which arise in the cubical context from the monoid structure max on the unit interval I. The total structure on ρX_* is called that of *ω-groupoid* [19, 20] (Brown and Higgins, 1981a,b). This gives our first example of a *higher homotopy groupoid*.

[5] Since this article was first published, there has been a large amount of work on the Grothendieck programme, developing models of homotopy types in terms of forms of weak infinity categories; see for example https://www.ncatlab.org/nlab/show/homotopy+hypothesis. However it seems the weak infinity categories considered do not seem suitable for explicit calculations. The philosophy of our approach is made more explicit in [13].

[6] See also [24].

The aim of the introduction of this functor

$$\rho : (\text{filtered spaces}) \to (\omega\text{-groupoids})$$

was that the proof of the usual Seifert-Van Kampen Theorem for the fundamental group generalised to a corresponding theorem for ρ [20] (Brown and Higgins, 1981b). One main feature of the proof is that ω-groupoids, being cubical objects, are appropriate for encoding subdivision methods, since they easily allow an 'algebraic inverse to subdivision'. It is not easy to formulate a corresponding simplicial method. (See Jones, 1984, [47], for a possible approach.) Another feature crucial in the proof is the use of the connections to express facts related to the Homotopy Addition Theorem, [43]. It seems that connections are an important new part of the cubical theory, since they allow for 'degenerate' elements in which adjacent faces are identical, as in the simplicial theory[7].

The *classifying space* BG of an ω-groupoid G is the geometric realisation of its underlying cubical set. These classifying spaces model only a restricted range of homotopy types, namely those which fibre over a $K(\pi, 1)$ with fibre a topological Abelian group [23] (Brown and Higgins, 1991). Nonetheless, these restricted models have useful applications. A principal reason for this is the equivalence proved in [19] Brown and Higgins, 1981a, between ω-groupoids and the classical tool in homotopy theory of crossed complex.

A *crossed complex* is a structure which encapsulates the properties of the relative homotopy groups $\pi_n(X_n, X_{n-1}, p)$, $p \in X_0$, $n \geqslant 2$, for a filtered space X_*, together with the boundary maps and the actions of the fundamental groupoid $\pi_1(X_1, X_0)$ on these relative homotopy groups. The notion was first considered in the reduced case (i.e. when X_0 is a singleton) by Blakers, 1948, [6], under the name *group system*. It was studied in the free case, and under the name *homotopy system*, by Whitehead, 1949 [56]. The term *crossed complex* is due to Huebschmann, 1980, [44] who used crossed n-fold extensions to represent the elements of the $(n+1)$-st cohomology group of a group (see also Holt, 1979,[42] Mac Lane, 1979, [50] Lue, 1981, [49]), and to determine differentials in the Lyndon-Hochschild-Serre spectral sequence (Huebschmann, 1981)[45]. Lue, 1981, [49] gives a good background to the general algebraic setting of crossed complexes. Crossed complexes have the advantage of being able to include information on chain complexes with a group (or groupoid) G of operators, [24], and on presentations of the group G. The category of crossed complexes also has a monoidal closed structure [21] (Brown and Higgins, 1987), which is convenient for expressing homotopies and higher homotopies[8].

The Generalized Seifert-Van Kampen Theorem for the fundamental ω-groupoid of a filtered space (Brown and Higgins, 1981b, [20]) implies immediately a similar theorem for the fundamental crossed complex, and this theorem has a number of applications, including the Relative Hurewicz Theorem. The latter theorem is thus seen in a wider

[7] This development in cubical theory is discussed further in [13], and is different from that developed in [52], which concentrates on connectivity results.

[8] These results are now covered in [24].

context, related to excision, and in a formulation dealing initially with the natural map $\pi_n(X, A) \to \pi_n(X \cup CA)$. This formulation was a model for the $(n+1)$-ad Hurewicz theorem [28] (Brown and Loday, 1987b). Other recent applications of crossed complexes are given in Baues, 1988, 1991, [2, 3] Brown and Higgins, 1987, 1989, 1991, [21, 22, 23] [4] Baues and Brown, 1990,[5] Baues and Conduché, 1991. See also [24].

More general algebraic models related to groupoids are associated not with filtered objects but with n-cubes of objects. Let $\langle n \rangle$ denote the set $\{1, 2, \ldots, n\}$. An *n-cube* C in a category $\mathcal{C}$ is a commutative diagram with vertices C_A for $A \subseteq \langle n \rangle$ and morphisms $C_A \to C_{A \cup \{i\}}$ for $A \subseteq \langle n \rangle$, $i \in \langle n \rangle$, and $i \notin A$. In particular, a 1-cube is a morphism, and a 2-cube is a commutative square.

Let X_* be an n-cube of pointed spaces. Loday, 1982, [48], defines the *fundamental cat^n-group* ΠX_*. (We are following the terminology and notation of [27] Brown and Loday, 1987a.) Here, a *cat^n-group* may be defined to be an n-fold groupoid in the category of groups. Alternatively, it is an $(n+1)$-fold groupoid in which one of the structures is a group. (This is one of several equivalent definitions considered in [**?**] Loday, 1982.)

For simplicity, we describe ΠX_* in a special case, namely when X_* arises from a pointed $(n+1)$-ad $\mathcal{X} = (X; X_1, \ldots, X_n)$ by the rule: $X_{\langle n \rangle} = X$ and for A properly contained in $\langle n \rangle$, $X_A = \bigcap_{i \notin A} X_i$, with maps the inclusions. Let Φ be the space of maps $I^n \to X$ which take the faces of I^n in the ith direction into X_i. Notice that Φ has the structure of n compositions derived from the gluing of cubes in each direction. Let $* \in \Phi$ be the constant map at the base point. Then $G = \pi_1(\Phi, *)$ is certainly a group. Gilbert, 1988, identifies G with Loday's ΠX_*, so that Loday's results, obtained by methods of simplicial spaces, show that G becomes a cat^n-group. It may also be shown that the extra groupoid structures are inherited from the compositions on Φ. It is this cat^n-group which is written $\Pi \mathcal{X}$ and is called the *fundamental cat^n-group of the $(n+1)$-ad $\mathcal{X}$*. This construction of Loday is our second example of a *higher homotopy groupoid*. We emphasise that the existence of this structure is itself a non-trivial fact, containing homotopy theoretic information. Also the results of Gilbert, 1988, [39], are for the case of n-cubes of spaces.

The nerve NG mentioned in Section 1 may be defined, not only for a group but also for a groupoid G, to be in dimension i the set of groupoid maps $\pi_1(\Delta^i, \Delta^i_0) \to G$, where Δ^i_0 is the set of vertices of the i-simplex Δ^i . It follows by iteration that N defines also a functor

$$((n+1)\text{-fold groupoids}) \to ((n+1)\text{-simplicial sets}).$$

Hence there is a *classifying space functor*

$$B : (\text{cat}^n\text{-groups}) \to (\text{pointed spaces})$$

defined as the composite of geometric realisation and the nerve functor to $(n+1)$-simplicial sets. Loday, 1982, [48], proves that if G is a cat^n-group, then BG is *$(n+1)$-coconnected*, i.e. $\pi_i BG = 0$ for $i > n+1$. He also shows, with a correction due to Steiner, 1986,

[54], that if X is a connected, $(n+1)$-coconnected CW-complex, then there is a cat^n-group G such that X is of the homotopy type of BG. In fact, Loday gives an equivalence between a localisation of the category of cat^n-groups and the pointed homotopy category of connected, $(n+1)$-coconnected CW-complexes. This can be put provocatively as

$$(n+1)\text{-fold groupoids model all homotopy } (n+1)\text{-types.}$$

That is, the generalisation from groups or groupoids to $(n+1)$-fold groupoids is as good for modelling homotopy types as might be expected. This result also shows the surprising richness of the algebraic structure of $(n+1)$-fold groupoids.

There is an important structure equivalent to that of cat^n-groups, namely that of *crossed n-cubes of groups* ([38] Ellis and Steiner, 1987). The main intuitive idea is that a crossed n-cube of groups is a crossed module in the category of crossed $(n-1)$-cubes of groups. This leads to the following definition (*loc. cit.*).

Definition 2.1 Let $\langle n\rangle$ denote the set $1, 2, \ldots, n$. A *crossed n-cube of groups* is a family of groups, $M_A, A \subseteq \langle n\rangle$, together with morphisms $\mu_i : M_A \to M_{A\setminus\{i\}}$, $(i \in \langle n\rangle,\ A \subseteq \langle n\rangle)$, and functions $h : M_A \times M_B \to M_{A\cup B}, (A, B \subseteq \langle n\rangle)$, such that if ${}^a b$ denotes $h(a,b)b$ for $a \in M_A$ and $b \in M_B$ with $A \subseteq B$, then for $a, a' \in M_A,\ b, b' \in M_B, c \in M_C$ and $i, j \in \langle n\rangle$, the following hold:

1. $\mu_i a = a$ if $i \notin A$
2. $\mu_i\mu_j a = \mu_j\mu_i a$
3. $\mu_i h(a,b) = h(\mu_i a, \mu_i b)$
4. $h(a,b) = h(\mu_i a, b) = h(a, \mu_i b)$ if $i \in A$ and $i \in B$
5. $h(a,a') = [a,a']$
6. $h(a,b) = h(b,a)^{-1}$
7. $h(a,b) = 1$ if $a = 1$ or $b = 1$
8. $h(aa',b) = {}^a h(a',b)h(a,b)$
9. $h(a,bb') = h(a,b)\ {}^b h(a,b')$
10. ${}^a h(h(a^{-1},b),c)\ {}^c h(h(c^{-1},a),b)\ {}^b h(h(b^{-1},c),a) = 1$
11. ${}^a h(b,c) = h({}^a b, {}^a c)$ if $A \subseteq B \cap C$.

A *morphism* of crossed n-cubes $(M_A) \to (N_A)$ is a family of morphisms of groups $f_A : M_A \to N_A (A \subseteq \langle n\rangle)$ which commute with the maps μ_i and the functions h. This gives us a category $Crs^n gp$. Ellis and Steiner, 1987, [38], show that this category is equivalent to that of cat^n-groups, and this is the reason for the choice of structure and axioms in Definition 2.1. This equivalence shows that there is a *classifying space functor*

$$B : Crs^n gp \to Top.$$

This functor would be difficult to describe directly[9]. The results for cat^n-groups imply that a localisation of the category $Crs^n gp$ is equivalent to the homotopy category of pointed, connected, $(n+1)$-coconnected CW-complexes.

The *fundamental crossed n-cube of groups functor* Π' is defined from n-cubes of pointed spaces to crossed n-cubes of groups: $\Pi' X_*$ is simply the crossed n-cube of groups equivalent to the cat^n-group ΠX_*. It is easier to identify Π' in classical terms in the case X_* is the n-cube of spaces arising as above from a pointed $(n+1)$-ad $\mathcal{X} = (X; X_1, \ldots, X_n)$. That is, let $X_{\langle n \rangle} = X$ and for A properly contained in $\langle n \rangle$ let $X_A = \bigcap_{i \notin A} X_i$. Then $M = \Pi' \mathcal{X}$ is given as follows (Ellis and Steiner, 1987, [38],): $M_\emptyset = \pi_1(X)$; if $A = i_1, \ldots, i_r$, then M is the homotopy $(r+1)$-ad group $\pi_{r+1}(X; X_1 \cap X_i, \ldots, X_n \cap X_i)$; the maps μ are given by the usual boundary maps; the h-functions are given by generalised Whitehead products. Note that whereas these separate elements of structure had all been considered previously, the aim of this theory is to consider the whole structure, despite its apparent complications. This global approach is necessary for the Generalized Seifert-Van Kampen Theorem 3.2 stated below. That $\Pi' \mathcal{X}$ satisfies the laws for a crossed n-cube of groups follows immediately since $\Pi' \mathcal{X}$ is the crossed n-cube of groups derived from the cat -group ΠX_*. From now on, we abbreviate Π' to Π , the meaning being clear from the context.

A crossed n-cube of groups M gives rise to an n-cube of crossed n-cubes of groups $\boxplus M$ where

$$((\boxplus M)(A))_B = \begin{cases} M_B & \text{if } A' \subseteq B \\ 1 & \text{otherwise} \end{cases}$$

Then $B\boxplus M$ is an n-cube of spaces. The generalisation to this context of the result on the fundamental group of the classifying space of a group is that there is a natural isomorphism of crossed n-cubes of groups

$$\Pi B\boxplus M \cong M.$$

(See Loday, 1982, [48],for the cat^n-group case, and Brown and Higgins, 1981b, [20], [23] 1991, for the analogous crossed complex case.) This result confirms the appropriate nature of the axioms (1)-(11) of Definition 2.1.

A description of the homotopy groups of BG for a cat^n-group G has been given in Loday, 1982, [48], in terms of the homology groups of a non-Abelian chain complex. This, with some extra work, yields a result on the homotopy invariants of the classifying space of a crossed square (i.e. a crossed 2-cube of groups). It is useful first to give the axioms for this in a different notation.

A crossed square (Loday, 1982) [48], consists of a commutative square of morphisms

[9] (Porter, 1990, [53], gives a different account of such a functor, but that paper does not deal with the Generalised Seifert-van Kampen Theorem.)

of groups

$$\begin{array}{ccc} L & \xrightarrow{\lambda} & M \\ {\scriptstyle\lambda'}\downarrow & & \downarrow{\scriptstyle\mu} \\ N & \xrightarrow[\nu]{} & P \end{array} \tag{2.2}$$

together with actions of P on the groups L, M, N, and a function $h : M \times N \to L$. This structure shall satisfy the following axioms, in which we assume that M and N act on L, M, N via P:

(2.3)(i) the morphisms $\lambda, \lambda', \mu, \nu$ and $\mu\lambda = \nu\lambda'$ are crossed modules and λ and λ' are P-equivariant;

(2.3)(ii) $\lambda h(m,n) = m\,{}^n m^{-1}, \lambda' h(m,n) = {}^m n n^{-1}$;

(2.3)(iii) $h(\lambda l, n) = l\,{}^n l^{-1}, h(m, \lambda' l) = {}^m l l^{-1}$;

(2.3)(iv) $h(mm', n) = {}^m h(m', n) h(m, n), h(m, nn') = h(m, n)\,{}^n h(m, n')$;

(2.3)(v) $h({}^p m, {}^p n) = {}^p h(m, n)$,

for all $l \in L, m, m' \in M, n, n' \in N, p \in P$.

We now describe the homotopy groups of BG for a crossed square G as above. The first part of the following result is a special case of results in Loday, 1982, [48].

Theorem 2.4 *Let G be the crossed square (2.2). Then the homotopy groups of BG may be computed as the homology groups of the non-Abelian chain complex*

$$L \xrightarrow{(\lambda^{-1}, \lambda')} M \rtimes N \xrightarrow{\mu * \nu} P \tag{2.5}$$

where $\mu * \nu : (m, n) \mapsto (\mu m)(\nu n)$. This implies that

$$\pi_i BG \cong \begin{cases} P/(\mu M)(\nu N) & \text{if } i = 1 \\ (M \times_P N)/\{(\lambda l, \lambda' l) : l \in L\} & \text{if } i = 2 \\ (Ker\ \lambda) \cap (Ker\ \lambda') & \text{if } i = 3 \\ 0 & \text{if } i \geqslant 4. \end{cases} \tag{2.6}$$

Further, under these isomorphisms, the composition $\eta^* : \pi_2 BG \to \pi_3 BG$ with the Hopf map $\eta : S^3 \to S^2$ is induced by the function $M \times_P N \to L$, $(m, n) \mapsto h(m, n)$, and the Whitehead product $\pi_2 \times \pi_2 \to \pi_3$ on BG is induced by the function $((m, n), (m', n')) \mapsto h(m', n) h(m, n')$. The first Postnikov invariant of BG is the cohomology class determined by the crossed module

$$(M \rtimes N)/Im(\lambda^{-1}, \lambda') \xrightarrow{\mu * \nu} P.$$

We will explain the proof of this result in Section 4.

3 n-cubes of fibrations

As in Brown and Loday, 1987a, [27], an n-cube of maps X_* yields an n-cube of fibrations $\overline{X}_*$. (See [34] Edwards and Hastings, 1976, [31] Cordier and Porter, 1990.) Following Steiner, 1986, [54] we parametrize this as a commutative diagram consisting of spaces $X_{A,B}$ (A, B disjoint subsets of $\langle n \rangle$) and fibration sequences

$$\overline{X}_{A\cup\{i\},B} \to \overline{X}_{A,B} \to \overline{X}_{A,B\cup\{i\}}, A \cap B = \emptyset, i \in \langle n \rangle \setminus (A \cup B) \tag{3.1}$$

The n-cube of fibrations $(\overline{X}_{A,B})$ contains an n-cube of spaces $\overline{X}_{\emptyset,*}$ homotopy equivalent to X_* (i.e. there is a morphism $X_* \to \overline{X}_{\emptyset,*}$ consisting of homotopy equivalences $X_B \to \overline{X}_{\emptyset,B}$). The n-cube of maps X_* is called *connected* if all the spaces $\overline{X}_{A,B}$ are path-connected.

Just as the traditional Seifert-Van Kampen Theorem enables one to compute the fundamental group of a union of connected spaces, so the Generalised Seifert-Van Kampen Theorem (GSVKT) enables one to compute the fundamental crossed n-cube of a union of connected n-cubes. This result is Theorem 5.4 of Brown and Loday, 1987a, [27], where it is proved by induction on n starting with the usual SVKT. It may be restated in terms of crossed n-cubes of groups, rather than cat^n-groups, as follows:

Theorem 3.2 *Let X_* be an n-cube of spaces, and suppose that $\mathcal{U} = \{U^\lambda\}$ is an open cover of the space $X_{<n>}$, such that $\mathcal{U}$ is closed under finite intersections. Let $\mathcal{U}^\lambda$ be the n-cube of spaces obtained from X_* by inverse images of the U^λ. Suppose that each $\mathcal{U}^\lambda$ is a connected n-cube of spaces. Then:*
(C)*: the n-cube X_* is connected, and*
(I)*: the natural morphism of crossed n-cubes of groups*

$$colim^\lambda \ \Pi\mathcal{U}^\lambda \to \Pi X_*$$

is an isomorphism.

The colimit in this theorem is taken in the category of crossed n-cubes of groups, and so the validity of (I) confirms again that the axioms for crossed n-cubes of groups are well chosen.

The connectivity statement (C) of this theorem generalises the famous $(n + 1)$-ad connectivity theorem, which is usually regarded as a difficult result (at the time of initial writing, no recent proof was in print except that referred to here[10]). Of course, the connectivity result is related to the fact that a colimit of zero objects is zero.

The isomorphism statement (I) implies the characterisation by a universal property of the critical group of certain $(n+1)$-ads. (See Brown and Loday, 1987b, [28] for the general procedure and explicit results on the triad case, using a non-Abelian tensor product, and Ellis and Steiner, 1987, [38], for the general case.) The earlier result in this area is in

[10] More recent proofs of these connectivity results are given in the book [52].

Barratt and Whitehead, 1952, [1] but there the assumption is made of simply connected intersection, and the result is proved by homological methods, so that it has no possibility for dealing with the occurrence of a non-Abelian $(r+1)$-ad homotopy group. It is clearly advantageous to see the Barratt and Whitehead result, including the $(n+1)$-ad connectivity theorem, as a special case of a theorem which has other consequences, for example an $(n+1)$-ad Hurewicz theorem (Brown and Loday, 1987b) [28, 14]

These results, with Theorem 2.4, illustrate how situations in homotopy theory may require constructions on non-Abelian groups for the convenient statement of a theorem, let alone its proof. The methods of crossed n-cubes of groups give a (largely unstudied[11]) range of new constructions in group theory.

Theorem 3.2 allows in some cases for the computation of the fundamental crossed n-cube of groups ΠX_* of an n-cube of spaces X_*. We now consider to what extent it also allows computation of the $(n+1)$-type of the space $X_{<n>}$.

Let X_* be a connected n-cube of spaces, and let $X = X_{<n>}$. It is proved in Loday, 1982, that there is an n-cube of fibrations Z_* and maps of n-cubes of fibrations

$$\overline{X} \xleftarrow{f} Z_* \xrightarrow{g} \overline{B\square(\Pi X_*)}$$

such that f is a level weak homotopy equivalence and g induces an isomorphism of π_1 at each level. Assume now that X is of the homotopy type of a CW-complex. Then from f and g we obtain a map

$$\phi : X \to B\Pi X_*$$

inducing an isomorphism of π_1, namely the composite, *at this level*, of g with a homotopy inverse of f, and with the map $X_{<n>} \to \overline{X}_{<n>}$. We do not expect ϕ to be a homotopy equivalence in general, since the n-cube of fibrations $\overline{B\square(\Pi X_*)}$ has special properties not necessarily satisfied by $\overline{X}_*$.

We say an n-cube of spaces X_* is an *Eilenberg-Mac Lane* n-cube of spaces if it is connected and all the spaces $\overline{X}_{A,}$ are spaces of type $K(\pi, 1)$. A chief example of this is the n-cube of spaces $B\square M$ derived from a crossed n-cube of groups. In fact, $(B\square M)_{A,B}$ is not only path-connected but also $(|B|+1)$-coconnected. This n-cube of fibrations may also be constructed directly in terms of the structure of M, using the techniques of Loday, 1982, [48]. We have the following result.

Proposition 3.3 *Let X_* be a connected n-cube of spaces such that $X_{<n>}$ is of the homotopy type of a CW-complex. Suppose that for $A, B \subseteq \langle n\rangle$, such that $A \cap B = \emptyset$, $i \in \langle n\rangle \setminus (A \cup B)$, and $r = |B|$, the induced morphism $\pi_{r+2}\overline{X}_{A,B} \to \pi_{r+2}\overline{X}_{A,B\cup\{i\}}$ is zero. Then the canonical (up to homotopy) map $\phi : X_{<n>} \to B\Pi X_*$ is an $(n+1)$-equivalence.*

[11] The nonabelian tensor product of groups introduced in [26, 27] has by 2017 a bibliography of 160 items, starting in 1952, and mainly by group theorists: see http://groupoids.org.uk/nonabtens.html. There have been a lot of developments of the paper [15]. .

Proof This is a simple consequence of the five lemma applied by induction on $|B|$ to the maps of homotopy exact sequences of the fibration sequences (3.1) of the n-cubes of fibrations $\overline{X}_*$ and $\overline{B\square(\Pi\mathcal{X})}$. □

Example 3.4 Let M and N be normal subgroups of a group P, and let the space X be given as the homotopy pushout

$$\begin{array}{ccc} K(P,1) & \longrightarrow & K(P/M,1) \\ \downarrow & & \downarrow \\ K(P/N,1) & \longrightarrow & X \end{array}$$

Brown and Loday, 1987a, [27], apply the case $n = 2$ of Theorem 3.2 to show that the above square of spaces has fundamental crossed square given by the 'universal' crossed square

$$\begin{array}{ccc} M \otimes N & \to & M \\ \downarrow & & \downarrow \\ N & \longrightarrow & P \end{array} \tag{3.5}$$

where $M \otimes N$ is the non-Abelian tensor product (*loc. cit.*), with generators $m \otimes n$ for $m \in M$ and $n \in N$ and relations

$$mm' \otimes n = ({}^m m' \otimes {}^m n)(m \otimes n),$$
$$m \otimes nn' = (m \otimes n)({}^n m \otimes {}^n n')$$

for all $m, m' \in M, n, n' \in N$. The h-map of this crossed square is $(m, n) \mapsto m \otimes n$. It follows from Proposition 3.3 that the 3-type of X is also given by this crossed square. This result has been stated in Brown, 1989b, 1990, [11], and we have now given the proof. Note that Theorem 2.4 allows one to compute $\eta : \pi_2 \to \pi_3$ and the Whitehead product map $\pi_2 \times \pi_2 \to \pi_3$.

By contrast, the Postnikov description of the 3-type of X requires the description of the first k-invariant

$$k^{(3)} \in H^3(P/MN, (M \cap N)/[M, N]),$$

which in this case is represented by the crossed module $M \circ N \to P$, where $M \circ N$ is the coproduct of the crossed P-modules M and N (see Brown, 1984, [8], and also Gilbert and Higgins, 1989, [40]). This k-invariant determines (up to homotopy) a space $X^{(2)}$, which could be taken to be the classifying space of the above crossed module, constructed either by regarding the crossed module as a crossed 1-cube of groups, or as in Brown and Higgins, 1991, [23]. One then needs a second Postnikov invariant

$$k^{(4)} \in H^4(X^{(2)}, Ker(M \otimes N \to P)).$$

This description of the 3-type of X is less explicit than that given by the crossed square (2.2), from which we obtained the homotopy groups and the action of π_1 in the first place.

Note also that if M, N, P are finite, then so also is $M \otimes N$ (Ellis, 1987, [37]), so that in this case the crossed square (2.2) is finite.

As an example, in this way one finds that if $P = M = N$ is the dihedral group D_n of order $2n$, with generators x and y and relations $x^2 = y^n = xyxy = 1$, where n is even, then the suspension $SK(D_n, 1)$ of $K(D_n, 1)$ has π_3 isomorphic to $(\mathbb{Z}_2)^4$ generated by the elements of $D_n \otimes D_n$:

$$x \otimes x, (x \otimes y)^{n/2}, y \otimes y, (x \otimes y)(y \otimes x).$$

Further, $\eta^*(\overline{x}) = x \otimes x, \eta^*(\overline{y}) = y \otimes y$, where $\overline{x}$ and $\overline{y}$ denote the corresponding generators of $\pi_2 SK(D_n, 1) = (D_n)^{\mathrm{ab}}$ (if n is odd, only the $x \otimes x$ term appears in π_3). The element $(x \otimes y)(y \otimes x)$ is the Whitehead product $[\overline{x}, \overline{y}]$. Other computations of η^* and of Whitehead products at this level in spaces $SK(P, 1)$ may be deduced from the calculations of non-Abelian tensor products given in Brown, Johnson and Robertson, 1987, [25]. (This paper covers the case of dihedral, quaternionic, metacyclic and symmetric groups, and all groups of order $\leqslant 31$.) Problems in this area are given in Brown, 1990, [12][12].

4 Proof of Theorem 2.4

We now explain the results on η^* and Whitehead products in the second part of Theorem 2.4. Let G be the crossed square (2.2). Then the square of crossed squares $\square G$ may be written in abbreviated form as follows:

$$\begin{array}{ccc} \begin{pmatrix} 1 & 1 \\ 1 & P \end{pmatrix} & \longrightarrow & \begin{pmatrix} 1 & 1 \\ N & P \end{pmatrix} \\ \downarrow & & \downarrow \\ \begin{pmatrix} 1 & M \\ 1 & P \end{pmatrix} & \longrightarrow & \begin{pmatrix} L & M \\ N & P \end{pmatrix} \end{array} \tag{4.1}$$

Let us write Y_* for the square of spaces $B\square G$. Then ΠY_* is isomorphic to the original crossed square G . Further the 2-cube of fibrations $\overline{Y}_*$ associated to Y_* is homotopy equivalent to the following diagram:

$$\begin{array}{ccccc} BL & \longrightarrow & BM & \longrightarrow & B(L \to M) \\ \downarrow & & \downarrow & & \downarrow \\ BN & \longrightarrow & BP & \longrightarrow & B(N \to P) \\ \downarrow & & \downarrow & & \downarrow \\ B(L \to N) & \to & B(M \to P) & \longrightarrow & B(G) \end{array} \tag{4.3}$$

[12]See also the nonabelian tensor product bibliography cited in footnote 11.

For a general square of spaces X_* as follows

$$\begin{array}{ccc} C & \xrightarrow{f} & A \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle a} \\ B & \xrightarrow[b]{} & X \end{array} \tag{4.4}$$

the associated 2-cube of fibrations is equivalent to the following diagram

$$\begin{array}{ccccc} F(X_*) & \to & F(g) & \to & F(a) \\ \downarrow & & \downarrow & & \downarrow \\ F(f) & \longrightarrow & C & \longrightarrow & A \\ \downarrow & & \downarrow & & \downarrow \\ F(b) & \longrightarrow & B & \longrightarrow & X \end{array} \tag{4.5}$$

where each row and column is a fibration sequence. So we deduce the second part of Theorem 2.4 from the following more general result.

Proposition 4.6 *Let X_* be the square of pointed spaces as in (4.4). Suppose that the induced morphism $\pi_2 C \to \pi_2 X$ is zero. Then there is a commutative diagram*

$$\begin{array}{ccccc} \pi_2 X & & \xleftarrow{\quad\delta'\quad} & & \pi_1 F(f) \times_{\pi_1 C} \pi_1 F(g) \\ {\scriptstyle \eta^*}\downarrow & & & & \downarrow{\scriptstyle h'} \\ \pi_3 X & \xrightarrow[\partial]{} & \pi_2 F(a) & \xrightarrow[\partial']{} & \pi_1 F(\mathbf{X}) \end{array} \tag{4.7}$$

in which δ' is defined by a difference construction, ∂, ∂' are boundaries in homotopy exact sequences of fibrations, η^* is induced by composition with the Hopf map η, and h' is the restriction of the h-map of the crossed square ΠX_*.

Proof This result is a refinement of Lemma 4.2 of Brown and Loday, 1987a, [27]. It is proved by similar methods. One first considers the suspension square of S^1:

$$\begin{array}{ccc} S^1 & \longrightarrow & E^2_+ \\ \downarrow & & \downarrow \\ E^2_- & \longrightarrow & S^2 \end{array}$$

The fundamental crossed square of this suspension square is determined by Theorem 3.2, compare Example 3.4, as in Brown and Loday, 1987a, [27] and is

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} \\ {\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle 1} \\ \mathbb{Z} & \xrightarrow[1]{} & \mathbb{Z} \end{array} \tag{4.7}$$

with h-map $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ given by $(m, n) \mapsto mn$, so that $h(1, 1)$ represents the Hopf map η. But the diagram (4.7) interpreted for the suspension square of S^1 may now be completely determined, and is the universal example for Proposition 4.6. This completes the proof of the Proposition. □

For the proof of the final part of Theorem 2.4 we have to explain how the particular crossed module given in the theorem determines the homotopy 2-type. This is proved by considering the Moore complex of the diagonal simplicial group of the bisimplicial group arising as the nerve of the associated cat^2-group (see also [53].)

5 Simply connected 3-types and crossed squares of Abelian groups

It is known that the 3-type of a simply connected space X is determined by the homotopy groups $\pi_2 X$, $\pi_3 X$ and the quadratic function $\eta^* : \pi_2 X \to \pi_3 X$ induced by composition with the Hopf map $\eta : S^3 \to S^2$. This essentially results from the fact that for abelian groups A and B the cohomology group $H^4(K(A, 2), B)$ is isomorphic to the group of quadratic functions $A \to B$ (Eilenberg and Mac Lane, 1954, [36]).

The aim of this section is to show that all simply connected 3-types can be modelled by a crossed square of Abelian groups. It is not known if simply connected $(n + 1)$-types can be modelled by crossed n-cubes of Abelian groups.

Theorem 5.1 *Let C and D be Abelian groups such that C is finitely generated, and let $t : C \to D$ be a quadratic function. Then there is a crossed square*

$$G: \qquad \begin{array}{ccc} L & \xrightarrow{\lambda} & M \\ {\scriptstyle \lambda'}\downarrow & & \downarrow{\scriptstyle 1} \\ M & \xrightarrow[-1]{} & M \end{array}$$

of abelian groups whose classifying space $X = BG$ satisfies $\pi_2 X \cong C$, $\pi_3 X \cong D$ and such that these isomorphisms map η^* to the quadratic map t.

Proof The quadratic function t has first to be extended to a biadditive map. We use a slight modification of a definition of [36] Eilenberg and Mac Lane, 1954, § 18.

Let $t : C \to D$ be a quadratic function on Abelian groups C, D. A *biadditive extension* of t is an abelian group M and an epimorphism $\alpha : M \to C$ of Abelian groups together with a biadditive map $\phi : M \times M \to D$ such that for all $m, m' \in M$

(5.1.1) $\phi(m, m) = t\alpha m$;

(5.1.2) $\phi(m, m') = 0$ if $\alpha m = \alpha m' = 0$;

(5.1.3) $\phi(m, m') = \phi(m', m)$.

It is shown in *loc. cit.* that such a biadditive extension exists assuming C is finitely generated. (In fact they do not assume the symmetry condition (5.1.3), but their proof of existence yields such a ϕ.)

Let $K = \mathrm{Ker}\ \alpha$ and let L be the product group $D \times K$. Let M act on L on the left by

$$^m(d,k) = (d + \phi(m,k), k),$$

for $m \in M,\ d \in D,\ k \in K$. Define $\lambda, \lambda' : L \to M$ by $\lambda(d,k) = -k, \lambda'(d,k) = k$, for $(d,k) \in L$, and let M act trivially on itself. Then λ and λ' are M-morphisms, and (5.1.2) shows that they are also crossed modules. Define $h : M \times M \to L$ by

$$h(m,m') = (\phi(m,m'), 0)$$

for $m, m' \in M$. A straightforward check shows that we have defined a crossed square G say. The symmetry condition, or even the weaker condition that $\phi(m,m') = \phi(m',m)$ if m or m' lies in K, is used to verify that

$$h(\lambda(d,k), m) = (d,k) -^m (d,k).$$

The homotopy groups of BG are computed as the homology groups of the chain complex

$$L \xrightarrow{(-\lambda,\lambda')} M \times M \xrightarrow{\psi} M$$

where $\psi(m,m') = m - m'$. Thus $\pi_2 BG \cong M/K \cong C$, $\pi_3 BG \cong D$. Further $h(m,m) = (\phi(m,m),0) = (t\alpha m, 0)$. This proves the final assertion of the theorem. □

Note that in the proof of this theorem, while the groups are Abelian, the actions are in general non-trivial. So the associated cat^2-group in general has non-Abelian big group.

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