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Gamma

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Notations

Traditional name

Gamma function

Traditional notation

$\Gamma(z)$

Mathematica StandardForm notation

Gamma[z]

Primary definition

06.05.02.0001.01

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} \, dt \;/;\; \operatorname{Re}(z) > 0$$

Specific values

Specialized values

06.05.03.0001.01

$$\Gamma(n) = (n-1)! \;/;\; n \in \mathbb{N}^+$$

06.05.03.0002.01

$$\Gamma\left(\frac{n}{2}\right) = \frac{2^{1-n} \sqrt{\pi}\, (n-1)!}{\frac{n-1}{2}!} \;/;\; n \in \mathbb{N}^+$$

06.05.03.0003.01

$$\Gamma(-n) = \tilde{\infty} \;/;\; n \in \mathbb{N}$$

06.05.03.0004.01

$$\Gamma\left(-\frac{n}{2}\right) = \frac{(-1)^{\frac{n+1}{2}}\, 2^n \sqrt{\pi}\, \frac{n-1}{2}!}{n!} \;/;\; n \in \mathbb{N}^+$$

06.05.03.0005.01

$$\Gamma\left(n+\frac{1}{4}\right) = \frac{1}{4^n}\, \Gamma\left(\frac{1}{4}\right) \prod_{k=1}^{n} (4k-3) \;/;\; n \in \mathbb{N}$$

06.05.03.0006.01
$$\Gamma\left(\frac{1}{4}-n\right) == \frac{(-1)^n\, 4^n}{\prod_{k=1}^{n}(4\,k-1)}\,\Gamma\left(\frac{1}{4}\right) /;\, n \in \mathbb{N}$$

06.05.03.0007.01
$$\Gamma\left(n+\frac{1}{3}\right) == \frac{1}{3^n}\,\Gamma\left(\frac{1}{3}\right)\prod_{k=1}^{n}(3\,k-2) /;\, n \in \mathbb{N}$$

06.05.03.0008.01
$$\Gamma\left(\frac{1}{3}-n\right) == \frac{(-1)^n\, 3^n}{\prod_{k=1}^{n}(3\,k-1)}\,\Gamma\left(\frac{1}{3}\right) /;\, n \in \mathbb{N}$$

06.05.03.0009.01
$$\Gamma\left(n+\frac{1}{2}\right) == \frac{\sqrt{\pi}}{2^n}\prod_{k=1}^{n}(2\,k-1) /;\, n \in \mathbb{N}$$

06.05.03.0010.01
$$\Gamma\left(\frac{1}{2}-n\right) == \frac{(-1)^n\sqrt{\pi}\; 2^n}{\prod_{k=1}^{n}(2\,k-1)} /;\, n \in \mathbb{N}$$

06.05.03.0011.01
$$\Gamma\left(n+\frac{2}{3}\right) == \frac{1}{3^n}\,\Gamma\left(\frac{2}{3}\right)\prod_{k=1}^{n}(3\,k-1) /;\, n \in \mathbb{N}$$

06.05.03.0012.01
$$\Gamma\left(\frac{2}{3}-n\right) == \frac{(-1)^n\, 3^n}{\prod_{k=1}^{n}(3\,k-2)}\,\Gamma\left(\frac{2}{3}\right) /;\, n \in \mathbb{N}$$

06.05.03.0013.01
$$\Gamma\left(n+\frac{3}{4}\right) == \frac{1}{4^n}\,\Gamma\left(\frac{3}{4}\right)\prod_{k=1}^{n}(4\,k-1) /;\, n \in \mathbb{N}$$

06.05.03.0014.01
$$\Gamma\left(\frac{3}{4}-n\right) == \frac{(-1)^n\, 4^n}{\prod_{k=1}^{n}(4\,k-3)}\,\Gamma\left(\frac{3}{4}\right) /;\, n \in \mathbb{N}$$

06.05.03.0015.01
$$\Gamma\left(n+\frac{p}{q}\right) == \frac{1}{q^n}\,\Gamma\left(\frac{p}{q}\right)\prod_{k=1}^{n}(p+k\,q-q) /;\, n \in \mathbb{N} \bigwedge p \in \mathbb{N}^+ \bigwedge q \in \mathbb{N}^+ \bigwedge p < q$$

06.05.03.0016.01
$$\Gamma\left(\frac{p}{q}-n\right) == \frac{(-1)^n\, q^n}{\prod_{k=1}^{n}(q\,k-p)}\,\Gamma\left(\frac{p}{q}\right) /;\, n \in \mathbb{N} \bigwedge p \in \mathbb{N}^+ \bigwedge q \in \mathbb{N}^+ \bigwedge p < q$$

Values at fixed points

06.05.03.0017.01
$$\Gamma(-3) == \tilde{\infty}$$

06.05.03.0018.01
$$\Gamma\left(-\frac{5}{2}\right) == -\frac{8}{15}\sqrt{\pi}$$

06.05.03.0019.01

$\Gamma(-2) == \tilde{\infty}$

06.05.03.0020.01

$\Gamma\left(-\frac{3}{2}\right) == \frac{4\sqrt{\pi}}{3}$

06.05.03.0021.01

$\Gamma(-1) == \tilde{\infty}$

06.05.03.0022.01

$\Gamma\left(-\frac{1}{2}\right) == -2\sqrt{\pi}$

06.05.03.0023.01

$\Gamma(0) == \tilde{\infty}$

06.05.03.0024.01

$\Gamma\left(\frac{1}{2}\right) == \sqrt{\pi}$

06.05.03.0025.01

$\Gamma(1) == 1$

06.05.03.0026.01

$\Gamma\left(\frac{3}{2}\right) == \frac{\sqrt{\pi}}{2}$

06.05.03.0027.01

$\Gamma(2) == 1$

06.05.03.0028.01

$\Gamma\left(\frac{5}{2}\right) == \frac{3\sqrt{\pi}}{4}$

06.05.03.0029.01

$\Gamma(3) == 2$

Values at infinities

06.05.03.0030.01

$\Gamma(\infty) == \infty$

06.05.03.0031.01

$\Gamma(-\infty) ==$ ¿

06.05.03.0032.01

$\Gamma(i\,\infty) == 0$

06.05.03.0033.01

$\Gamma(-i\,\infty) == 0$

06.05.03.0034.01

$\Gamma(\tilde{\infty}) ==$ ¿

General characteristics

Domain and analyticity

$\Gamma(z)$ is an analytical function of z which is defined over the whole complex z-plane with the exception of countably many points $z = -k$ /; $k \in \mathbb{N}$. $1/\Gamma(z)$ is an entire function.

06.05.04.0001.01

$$z \longrightarrow \Gamma(z) :: \mathbb{C} \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

06.05.04.0002.01

$$\Gamma(\bar{z}) == \overline{\Gamma(z)}$$

Periodicity

No periodicity

Poles and essential singularities

The function $\Gamma(z)$ has an infinite set of singular points:

a) $z == -k$ /; $k \in \mathbb{N}$ are the simple poles with residues $\frac{(-1)^k}{k!}$;

b) $z == \tilde{\infty}$ is the point of convergence of poles, which is an essential singular point.

06.05.04.0003.01

$$\mathcal{S}ing_z(\Gamma(z)) == \{\{\{-k, 1\} /; k \in \mathbb{N}\}, \{\tilde{\infty}, \infty\}\}$$

06.05.04.0004.01

$$\text{res}_z(\Gamma(z))(-k) == \frac{(-1)^k}{k!} /; k \in \mathbb{N}$$

06.05.04.0005.01

$$\text{res}_z(\Gamma(a+z)\, w^{-z})(-k-a) == \frac{(-1)^k}{k!} w^{a+k} /; k \in \mathbb{N}$$

06.05.04.0006.01

$$\text{res}_z(\Gamma(a-z)\, w^{-z})(k+a) == \frac{(-1)^{k-1}}{k!} w^{-a-k} /; k \in \mathbb{N}$$

Formulas for residues of ratios of products of gamma functions such as $\text{res}_s\left(\frac{\left(\prod_{k=1}^{A}\Gamma(s+a_k)\right)\prod_{k=1}^{B}\Gamma(b_k-s)}{\left(\prod_{k=1}^{C}\Gamma(s+c_k)\right)\prod_{k=1}^{D}\Gamma(d_k-s)} z^{-s}\right)\left(-a_j-l\right)$ in regular and logarithmic cases are represented in the subsection *Residues of ratios* below.

Branch points

The function $\Gamma(z)$ does not have branch points.

06.05.04.0007.01

$$\mathcal{BP}_z(\Gamma(z)) == \{\}$$

Branch cuts

The function $\Gamma(z)$ does not have branch cuts.

06.05.04.0008.01

$$\mathcal{BC}_z(\Gamma(z)) == \{\}$$

Series representations

Generalized power series

Expansions at $z == 0$

For the function itself

06.05.06.0001.02

$$\Gamma(z) \propto \frac{1}{z} - \gamma + \frac{1}{6}\left(3\gamma^2 + \frac{\pi^2}{2}\right) z + \left(-\frac{\zeta(3)}{3} - \frac{\gamma^3}{6} - \frac{\gamma\,\pi^2}{12}\right) z^2 + \left(\frac{\gamma\,\zeta(3)}{3} + \frac{\gamma^4}{24} + \frac{\pi^4}{160} + \frac{\gamma^2\,\pi^2}{24}\right) z^3 + \ldots \;/;\; (z \to 0)$$

06.05.06.0018.01

$$\Gamma(z) \propto \frac{1}{z} - \gamma + \frac{1}{6}\left(3\gamma^2 + \frac{\pi^2}{2}\right) z + \left(-\frac{\zeta(3)}{3} - \frac{\gamma^3}{6} - \frac{\gamma\,\pi^2}{12}\right) z^2 + \left(\frac{\gamma\,\zeta(3)}{3} + \frac{\gamma^4}{24} + \frac{\pi^4}{160} + \frac{\gamma^2\,\pi^2}{24}\right) z^3 + O\left(z^4\right)$$

06.05.06.0002.01

$$\Gamma(z) == \frac{1}{z} \sum_{k=0}^{\infty} \frac{\Gamma^{(k)}(1)\, z^k}{k!} \;/;\; |z| < 1$$

06.05.06.0003.02

$$\Gamma(z) \propto \frac{1}{z} - \gamma + O(z)$$

Expansions at $z == z_0 \;/;\; z_0 \neq -n$

For the function itself

06.05.06.0004.02

$$\Gamma(z) \propto \Gamma(z_0)\left(1 + \psi(z_0)(z - z_0) + \frac{1}{2}\left(\psi(z_0)^2 + \psi^{(1)}(z_0)\right)(z - z_0)^2 + \frac{1}{6}\left(\psi(z_0)^3 + 3\,\psi^{(1)}(z_0)\,\psi(z_0) + \psi^{(2)}(z_0)\right)(z - z_0)^3 + \ldots\right) /;$$
$$(z \to z_0) \bigwedge \neg\,(z_0 \in \mathbb{Z} \bigwedge z_0 \leq 0)$$

06.05.06.0019.01

$$\Gamma(z) \propto \Gamma(z_0)\left(1 + \psi(z_0)(z - z_0) + \frac{1}{2}\left(\psi(z_0)^2 + \psi^{(1)}(z_0)\right)(z - z_0)^2 + \frac{1}{6}\left(\psi(z_0)^3 + 3\,\psi^{(1)}(z_0)\,\psi(z_0) + \psi^{(2)}(z_0)\right)(z - z_0)^3 + O\left((z - z_0)^4\right)\right) /;$$
$$\neg\,(z_0 \in \mathbb{Z} \bigwedge z_0 \leq 0)$$

06.05.06.0005.02

$$\Gamma(z) == \Gamma(z_0) \sum_{k=0}^{\infty} \frac{\Gamma^{(k)}(z_0)}{\Gamma(z_0)\, k!} (z - z_0)^k \;/;\; \neg\,(z_0 \in \mathbb{Z} \bigwedge z_0 \leq 0)$$

06.05.06.0006.02

$$\Gamma(z) \propto \Gamma(z_0)\,(1 + O(z - z_0)) \;/;\; \neg\,(z_0 \in \mathbb{Z} \bigwedge z_0 \leq 0)$$

Expansions at $z == -n$

For the function itself

06.05.06.0007.01

$$\Gamma(z) \propto \frac{(-1)^n}{n!\,(z+n)} + \frac{(-1)^n}{n!}\Bigg(\psi(n+1) + \frac{1}{6}\left(3\,\psi(n+1)^2 + \pi^2 - 3\,\psi^{(1)}(n+1)\right)(z+n) + \frac{1}{6}\left(\psi(n+1)^3 + \left(\pi^2 - 3\,\psi^{(1)}(n+1)\right)\psi(n+1) + \psi^{(2)}(n+1)\right)(z+n)^2 + \frac{1}{360}\left(15\left(\psi(n+1)^4 + 2\left(\pi^2 - 3\,\psi^{(1)}(n+1)\right)\psi(n+1)^2 + 4\,\psi^{(2)}(n+1)\,\psi(n+1) + \psi^{(1)}(n+1)\left(3\,\psi^{(1)}(n+1) - 2\,\pi^2\right)\right) + 7\,\pi^4 - 15\,\psi^{(3)}(n+1)\right)(z+n)^3\Bigg) + O\left((z+n)^4\right) /;\ (z \to -n) \land n \in \mathbb{N}$$

06.05.06.0008.01

$$\Gamma(z) \propto \frac{(-1)^n\,(1 + O(z+n))}{n!\,(z+n)} /;\ (z \to -n) \land n \in \mathbb{N}$$

06.05.06.0009.01

$$\Gamma(z) \propto \frac{(-1)^n}{n!\,(z+n)} + \frac{(-1)^n\,\psi(n+1)}{n!} + O(z+n) /;\ (z \to -n) \land n \in \mathbb{N}$$

Expansions of $\Gamma(z+\epsilon)$ at $\epsilon == 0$ /; $z \neq -n$

For the function itself

06.05.06.0020.01

$$\Gamma(z+\epsilon) \propto \Gamma(z)\,(1 + O(\epsilon)) /;\ \neg\,(z \in \mathbb{Z} \land z \leq 0)$$

06.05.06.0021.01

$$\Gamma(z+\epsilon) \propto \Gamma(z)\left(1 + \psi(z)\,\epsilon + O\left(\epsilon^2\right)\right) /;\ \neg\,(z \in \mathbb{Z} \land z \leq 0)$$

06.05.06.0022.01

$$\Gamma(z+\epsilon) \propto \Gamma(z)\left(1 + \psi(z)\,\epsilon + \frac{\psi(z)^2 + \psi^{(1)}(z)}{2}\,\epsilon^2 + O\left(\epsilon^3\right)\right) /;\ \neg\,(z \in \mathbb{Z} \land z \leq 0)$$

06.05.06.0023.01

$$\Gamma(z+\epsilon) \propto \Gamma(z)\left(1 + \psi(z)\,\epsilon + \frac{\psi(z)^2 + \psi^{(1)}(z)}{2}\,\epsilon^2 + \frac{\psi(z)^3 + 3\,\psi^{(1)}(z)\,\psi(z) + \psi^{(2)}(z)}{6}\,\epsilon^3 + O\left(\epsilon^4\right)\right) /;\ \neg\,(z \in \mathbb{Z} \land z \leq 0)$$

06.05.06.0024.01

$$\Gamma(z+\epsilon) \propto \Gamma(z)\Bigg(1 + \psi(z)\,\epsilon + \frac{\psi(z)^2 + \psi^{(1)}(z)}{2}\,\epsilon^2 + \frac{\psi(z)^3 + 3\,\psi^{(1)}(z)\,\psi(z) + \psi^{(2)}(z)}{6}\,\epsilon^3 + \frac{1}{24}\left(\psi(z)^4 + 6\,\psi^{(1)}(z)\,\psi(z)^2 + 4\,\psi^{(2)}(z)\,\psi(z) + 3\,\psi^{(1)}(z)^2 + \psi^{(3)}(z)\right)\epsilon^4 + O\left(\epsilon^5\right)\Bigg) /;\ \neg\,(z \in \mathbb{Z} \land z \leq 0)$$

06.05.06.0025.01

$$\Gamma(z+\epsilon) == \Gamma(z)\sum_{k=0}^{\infty} \frac{\Gamma^{(k)}(z)}{\Gamma(z)\,k!}\,\epsilon^k /;\ \neg\,(z \in \mathbb{Z} \land z \leq 0)$$

Expansions of $\Gamma(-n+\epsilon)$ at $\epsilon == 0$

For the function itself

06.05.06.0026.01

$$\Gamma(-n+\epsilon) \propto \frac{(-1)^n}{n!\,\epsilon}\,(1+O(\epsilon)) \;/;\; n \in \mathbb{N}$$

06.05.06.0027.01

$$\Gamma(-n+\epsilon) \propto \frac{(-1)^n}{n!\,\epsilon}\left(1+\psi(n+1)\,\epsilon + O\!\left(\epsilon^2\right)\right) /;\; n \in \mathbb{N}$$

06.05.06.0028.01

$$\Gamma(-n+\epsilon) \propto \frac{(-1)^n}{n!\,\epsilon}\left(1+\psi(n+1)\,\epsilon + \frac{3\,\psi(n+1)^2+\pi^2-3\,\psi^{(1)}(n+1)}{6}\,\epsilon^2 + O\!\left(\epsilon^3\right)\right) /;\; n \in \mathbb{N}$$

06.05.06.0029.01

$$\Gamma(-n+\epsilon) \propto \frac{(-1)^n}{n!\,\epsilon}\left(1+\psi(n+1)\,\epsilon + \frac{3\,\psi(n+1)^2+\pi^2-3\,\psi^{(1)}(n+1)}{6}\,\epsilon^2 + \frac{1}{6}\left(\psi(n+1)^3+\left(\pi^2-3\,\psi^{(1)}(n+1)\right)\psi(n+1)+\psi^{(2)}(n+1)\right)\epsilon^3 + O\!\left(\epsilon^4\right)\right) /;\; n \in \mathbb{N}$$

06.05.06.0030.01

$$\Gamma(-n+\epsilon) \propto \frac{(-1)^n}{n!\,\epsilon}\left(1+\psi(n+1)\,\epsilon + \frac{3\,\psi(n+1)^2+\pi^2-3\,\psi^{(1)}(n+1)}{6}\,\epsilon^2 + \frac{1}{6}\left(\psi(n+1)^3+\left(\pi^2-3\,\psi^{(1)}(n+1)\right)\psi(n+1)+\psi^{(2)}(n+1)\right)\epsilon^3 + \frac{1}{360}\left(15\left(\psi(n+1)^4+2\left(\pi^2-3\,\psi^{(1)}(n+1)\right)\psi(n+1)^2+4\,\psi^{(2)}(n+1)\,\psi(n+1)+\psi^{(1)}(n+1)\left(3\,\psi^{(1)}(n+1)-2\,\pi^2\right)\right)+7\,\pi^4-15\,\psi^{(3)}(n+1)\right)\epsilon^4 + O\!\left(\epsilon^5\right)\right) /;\; n \in \mathbb{N}$$

06.05.06.0031.01

$$\Gamma(\epsilon-n) \propto \frac{(-1)^n}{n!\,\epsilon}\left(1+\pi\,\epsilon\sum_{k=0}^{\infty}\sum_{j=0}^{k+1}(j+1)\sum_{r=0}^{j}\frac{(-1)^{j+r}\binom{j}{r}}{r+1}\,p_{r,j}\,c_{k-j}\,\epsilon^k\right) /;\; n \in \mathbb{N} \bigwedge (\epsilon \to 0) \bigwedge c_{2k} = 0 \bigwedge c_{2k+1} = \frac{(-1)^k\,2\left(2^{2k+1}-1\right)B_{2k+2}\,\pi^{2k+1}}{(2k+2)!} \bigwedge p_{j,0}=1 \bigwedge p_{j,k} = \frac{1}{k}\sum_{m=1}^{k}(j\,m+m-k)\,b_m\,p_{j,k-m} \bigwedge b_k = \frac{\Gamma^{(k)}(n+1)}{n!\,k!}$$

06.05.06.0032.01

$$\Gamma(-n+\epsilon) \propto \frac{(-1)^n}{n!\,\epsilon}\sum_{q=0}^{\infty}\sum_{k=0}^{q}\frac{p_k\,\epsilon^q}{(q-k)!}\left(1+\sum_{i=1}^{\infty}\left(\sum_{j=1}^{n}\frac{\binom{n}{j}(-1)^{j-1}}{j^i}\right)\epsilon^i\right) /;\; n \in \mathbb{N} \land s_1=\gamma \land s_k=\zeta(k) \;/;\; k>1 \bigwedge a_0=-\gamma \bigwedge a_k = \frac{(-1)^{k+1}\,s_{k+1}}{k+1} \;/;\; k>0 \bigwedge p_0=(-1)^m\,\gamma^m \bigwedge p_k = \frac{1}{a_0\,k}\sum_{j=1}^{k}(m\,j+j-k)\,a_j\,p_{k-j} \;/;\; k>0 \land m=q-k$$

06.05.06.0033.01

$$\Gamma(\epsilon-n) = \frac{(-1)^n\,\pi\csc(\pi\,\epsilon)}{\Gamma(n-\epsilon+1)} \;/;\; n \in \mathbb{N}$$

Expansions of $1/\Gamma(z)$

For the function itself

06.05.06.0034.01

$$\frac{1}{\Gamma(z)} = \frac{1}{\Gamma(z_0)}\,(1 + O(z - z_0))$$

06.05.06.0035.01

$$\frac{1}{\Gamma(z)} = \frac{1}{\Gamma(z_0)}\left(1 - \psi(z_0)\,(z - z_0) + O\left((z - z_0)^2\right)\right)$$

06.05.06.0036.01

$$\frac{1}{\Gamma(z)} = \frac{1}{\Gamma(z_0)}\left(1 - \psi(z_0)\,(z - z_0) + \frac{1}{2}\left(\psi(z_0)^2 - \psi^{(1)}(z_0)\right)(z - z_0)^2 + O\left((z - z_0)^3\right)\right)$$

06.05.06.0037.01

$$\frac{1}{\Gamma(z)} = \frac{1}{\Gamma(z_0)}\left(1 - \psi(z_0)\,(z - z_0) + \frac{1}{2}\left(\psi(z_0)^2 - \psi^{(1)}(z_0)\right)(z - z_0)^2 - \frac{1}{6}\left(\psi(z_0)^3 - 3\,\psi^{(1)}(z_0)\,\psi(z_0) + \psi^{(2)}(z_0)\right)(z - z_0)^3 + O\left((z - z_0)^4\right)\right)$$

06.05.06.0038.01

$$\frac{1}{\Gamma(z)} = \frac{1}{\pi}\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\frac{\pi^{k-j}\,(-1)^j\,\Gamma^{(j)}(1 - z_0)}{j!\,(k - j)!}\sin\left(\pi\,z_0 + \frac{\pi\,(k - j)}{2}\right)\right)(z - z_0)^k$$

06.05.06.0039.01

$$\frac{1}{\Gamma(z + \epsilon)} \propto \frac{1}{\Gamma(z)}\sum_{k=0}^{\infty}(k + 1)\sum_{r=0}^{k}\frac{(-1)^r}{r + 1}\binom{k}{r}p_{r,k}\,\epsilon^k \;/;$$

$$(\epsilon \to 0) \bigwedge p_{j,0} = 1 \bigwedge p_{j,k} = \frac{1}{k}\sum_{m=1}^{k}(j\,m + m - k)\,b_m\,p_{j,k-m} \bigwedge b_k = \frac{\Gamma^{(k)}(z)}{\Gamma(z)\,k!} \bigwedge k \in \mathbb{N} \bigwedge n \in \mathbb{N}$$

06.05.06.0040.01

$$\frac{1}{\Gamma(-n + \epsilon)} \propto \epsilon\,(-1)^n\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}a_j\,b_{k-j}\right)\epsilon^k \;/;\; a_{2k} = \frac{(-1)^k\,\pi^{2k}}{(2k + 1)!} \bigwedge a_{2k+1} = 0 \bigwedge b_k = \frac{(-1)^k}{k!}\Gamma^{(k)}(n + 1) \bigwedge k \in \mathbb{N} \bigwedge n \in \mathbb{N}$$

Asymptotic series expansions

06.05.06.0010.01

$$\Gamma(x) \propto \sqrt{2\pi}\;x^{x-\frac{1}{2}}\,e^{-x} \;/;\; (x \to \infty)$$

06.05.06.0011.01

$$\Gamma(z) \propto \sqrt{2\pi}\;z^{z-\frac{1}{2}}\,e^{-z}\left(1 + \frac{1}{12\,z} + \frac{1}{288\,z^2} - \frac{139}{51\,840\,z^3} - \frac{571}{2\,488\,320\,z^4} + \frac{163\,879}{209\,018\,880\,z^5} + \frac{5\,246\,819}{75\,246\,796\,800\,z^6} - \frac{534\,703\,531}{902\,961\,561\,600\,z^7} - \frac{4\,483\,131\,259}{86\,684\,309\,913\,600\,z^8} + \frac{432\,261\,921\,612\,371}{514\,904\,800\,886\,784\,000\,z^9} + O\left(\frac{1}{z^{10}}\right)\right) \;/;\; |\arg(z)| < \pi \land (|z| \to \infty)$$

06.05.06.0041.01

$$\Gamma(z) \propto \sqrt{2\pi}\;z^{z-\frac{1}{2}}\,e^{-z}\,e^{\sum_{k=0}^{\infty}\frac{B_{2k+2}}{2\,(k+1)\,(2k+1)\,z^{2k+1}}} \;/;\; |\arg(z)| < \pi \land (|z| \to \infty)$$

06.05.06.0012.01

$$\Gamma(z) \propto \sqrt{2\pi}\, z^{z-\frac{1}{2}}\, e^{-z} \left(1 + \sum_{k=1}^{\infty} \sum_{j=1}^{2k} \frac{(-1)^j P(2(j+k), j)\, z^{-k}}{2^{j+k}\, (j+k)!}\right) /;$$
$$(|\arg(z)| < \pi \bigwedge (|z| \to \infty) \bigwedge P(n, j) == (n-1)((n-2) P(n-3, j-1) + P(n-1, j)) \bigwedge P(0, 0) == 1 \bigwedge P(n, 1) == (n-1)! \bigwedge P(n, j) == 0 /; n \leq 3j - 1)$$

06.05.06.0042.01

$$\Gamma(z) \propto \frac{\sqrt{\pi}\, \csc(z\pi)}{\sqrt{2}} (-z)^{z-\frac{1}{2}}\, e^{-z}\, e^{\sum_{k=0}^{\infty} \frac{B_{2k+2}}{2(k+1)(2k+1) z^{2k+1}}} /; \arg(z) = \pi \bigwedge z \notin \mathbb{Z} \bigwedge (|z| \to \infty)$$

06.05.06.0043.01

$$\Gamma(z) \propto \left(\frac{1}{2}\csc(z\pi)\right)^{\left\lfloor \frac{\arg(z)+\pi}{2\pi} \right\rfloor} \sqrt{2\pi} \left(e^{\pi i \left\lfloor \frac{\arg(z)+\pi}{2\pi} \right\rfloor} z\right)^{z-\frac{1}{2}} e^{-z}\, e^{\sum_{k=0}^{\infty} \frac{B_{2k+2}}{2(k+1)(2k+1) z^{2k+1}}} /; \neg (z \in \mathbb{Z} \land z < 1) \land (|z| \to \infty)$$

06.05.06.0013.01

$$\Gamma(z) \propto \sqrt{2\pi}\, z^{z-\frac{1}{2}}\, e^{-z} \left(1 + O\left(\frac{1}{z}\right)\right) /; |\arg(z)| < \pi \bigwedge (|z| \to \infty)$$

06.05.06.0014.01

$$\Gamma(z) \propto \sqrt{2\pi} \left(\frac{z - \frac{1}{2}}{e}\right)^{z-\frac{1}{2}} \left(1 + O\left(\frac{1}{z}\right)\right) /; |\arg(z)| < \pi \bigwedge (|z| \to \infty)$$

06.05.06.0015.01

$$|\Gamma(x + i\, y)| \propto \sqrt{2\pi}\, |y|^{x-\frac{1}{2}}\, e^{-\frac{\pi}{2}|y| - x} /; (|y| \to \infty) \bigwedge x \in \mathbb{R} \bigwedge y \in \mathbb{R}$$

06.05.06.0016.02

$$\frac{\Gamma(a+z)}{\Gamma(b+z)} \propto z^{a-b} \sum_{k=0}^{\infty} \frac{(-1)^k (b-a)_k}{k!} B_k^{(a-b+1)}(a)\, z^{-k} /; |\arg(a+z)| < \pi \bigwedge (|z| \to \infty)$$

06.05.06.0017.01

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} \propto z^{a-b} \left(1 + \frac{(a+b-1)(a-b)}{2z} + O\left(\frac{1}{z^2}\right)\right) /; |\arg(a+z)| < \pi \bigwedge (|z| \to \infty)$$

Integral representations

On the real axis

Of the direct function

06.05.07.0001.01

$$\Gamma(z) == \int_0^{\infty} t^{z-1}\, e^{-t}\, dt /; \mathrm{Re}(z) > 0$$

06.05.07.0002.01

$$\Gamma(z) == \int_1^{\infty} t^{z-1}\, e^{-t}\, dt + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k+z)}$$

06.05.07.0003.01

$$\Gamma(z) == \int_0^\infty \left(e^{-t} - \sum_{k=0}^{n} \frac{(-t)^k}{k!} \right) t^{z-1} \, dt \; /; \; n \in \mathbb{N} \land -n-1 < \text{Re}(z) < -n$$

(Cauchy-Saalschutz integral)

06.05.07.0004.01

$$\Gamma(z) == \int_0^1 \log^{z-1}\left(\frac{1}{t}\right) dt \; /; \; \text{Re}(z) > 0$$

06.05.07.0009.01

$$\Gamma(z) == e^{-\gamma z} \, e^{\int_0^1 \frac{x^z - \log(x^z) - 1}{(x-1)\log(x)} dx} \; /; \; \text{Re}\,(z) > 0$$

A. Radovi■

06.05.07.0010.01

$$\Gamma(z) == e^{\int_0^1 \frac{x^z - z\,(x-1) - 1}{(x-1)\log(x)} dx} \; /; \; \text{Re}(z) > 0$$

A. Radovi■

Contour integral representations

06.05.07.0005.01

$$\Gamma(z) == s^z \int_0^{e^{i\delta}\infty} t^{z-1} e^{-s\,t} \, dt \; /; \; \text{Re}(z) > 0 \bigwedge |\delta + \arg(s)| < \frac{\pi}{2} \bigvee 0 < \text{Re}(z) < 1 \bigwedge |\delta + \arg(s)| == \frac{\pi}{2}$$

06.05.07.0006.01

$$\frac{1}{\Gamma(z)} == \frac{i}{2\pi} \int_L (-t)^{-z} e^{-t} \, dt$$

(Hankel's contour integral.) The path of integration L starts at $\infty + i\,0$ on the real axis, goes to $\epsilon + i\,0$, circles the origin in the counterclockwise direction with radius ϵ to the point $\epsilon - i\,0$, and returns to the point $\infty - i\,0$.

06.05.07.0007.01

$$\Gamma(z) == \frac{1}{e^{2\pi i z} - 1} \int_L e^{-t} t^{z-1} \, dt$$

(Hankel's contour integral.) The path of integration L starts at $\infty + i\,0$ on the real axis, goes to $\epsilon + i\,0$, circles the origin in the counterclockwise direction with radius ϵ to the point $\epsilon - i\,0$, and returns to the point $\infty - i\,0$.

06.05.07.0008.01

$$\frac{1}{\Gamma(z)} == \frac{1}{2\pi i} \int_L t^{-z} e^t \, dt$$

(Hankel's contour integral.) The path of integration L starts at $-\infty - i\,0$ on the real axis, goes to $-\epsilon - i\,0$, circles the origin in the counterclockwise direction with radius ϵ to the point $-\epsilon + i\,0$, and returns to the point $-\infty + i\,0$.

Product representations

06.05.08.0001.01

$$\frac{1}{\Gamma(z)} == z\, e^{z\gamma} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}}$$

06.05.08.0002.01

$$\Gamma(z) == \frac{1}{z} \prod_{k=1}^{\infty} \frac{\left(1 + \frac{1}{k}\right)^z}{1 + \frac{z}{k}} /; -z \notin \mathbb{N}$$

06.05.08.0005.01

$$\Gamma(z) == \left(\frac{z-1}{e}\right)^{z-1} \left(1 + \frac{1}{z-1}\right)^{z-1} \prod_{k=1}^{\infty} \frac{\left(1 + \frac{1}{k+z-1}\right)^{k+z-1}}{\left(1 + \frac{1}{k}\right)^k} /; -z \notin \mathbb{N}$$

(van der Pol 1954)

06.05.08.0003.01

$$\Gamma(z+1) == \frac{1}{e^{z\gamma}} \sqrt{\frac{\pi z}{\sin(\pi z)}} \prod_{k=1}^{\infty} \exp\left(-\frac{\zeta(2k+1)\, z^{2k+1}}{2k+1}\right)$$

06.05.08.0004.01

$$\Gamma(z+2) == e^{z(1-\gamma)} \prod_{k=2}^{\infty} \exp\left(\frac{(-1)^k (\zeta(k)-1)\, z^k}{k}\right)$$

Limit representations

06.05.09.0001.01

$$\Gamma(z) == \lim_{n\to\infty} \frac{(n+1)^z\, n!}{(z)_{n+1}}$$

Euler—Gauss limit

06.05.09.0002.01

$$\Gamma(z) == \lim_{n\to\infty} \frac{n!\, n^z}{(z)_{n+1}}$$

06.05.09.0003.01

$$\Gamma(z) == \lim_{n\to\infty} \frac{(1)_n\, n^{z-1}}{(z)_n}$$

06.05.09.0007.01

$$\Gamma(z) == \left(\frac{z-1}{e}\right)^{z-1} \left(\lim_{n\to\infty} \sqrt{2\pi n} \prod_{k=0}^{n} \frac{\left(1 + \frac{1}{k+z-1}\right)^{k+z-1}}{e}\right) /; -z \notin \mathbb{N}$$

06.05.09.0004.01

$$\Gamma(z) == \lim_{n\to\infty} n^z\, \mathrm{B}(z, n)$$

06.05.09.0005.01

$$\Gamma(z) == \lim_{w\to\infty} \frac{w^z}{z}\, {}_1F_1(z; z+1; -w)$$

06.05.09.0006.01

$$\Gamma(z) = \lim_{n\to\infty} \int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}\, dt \;/;\; \mathrm{Re}(z) > 0$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

The gamma function does not satisfy any algebraic differential equation (O.Hölder, 1887). But it is the solution of the following nonalgebraic equation:

06.05.13.0001.01

$$\frac{\partial w(z)}{\partial z} = w(z)\,\psi(z) \;/;\; w(z) = \Gamma(z)$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

06.05.16.0001.01

$$\Gamma(-z) = -\frac{\pi \csc(\pi z)}{\Gamma(z+1)}$$

06.05.16.0002.01

$$\Gamma(z+1) = z\,\Gamma(z)$$

06.05.16.0003.01

$$\Gamma(z+n) = (z)_n\,\Gamma(z)$$

06.05.16.0004.01

$$\Gamma(z-1) = \frac{\Gamma(z)}{z-1}$$

06.05.16.0005.01

$$\Gamma(z-n) = \frac{(-1)^n\,\Gamma(z)}{(1-z)_n} \;/;\; n \in \mathbb{Z}$$

Multiple arguments

06.05.16.0006.01

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\,\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right)$$

06.05.16.0007.01

$$\Gamma(3z) = \frac{3^{3z-\frac{1}{2}}}{2\pi}\,\Gamma(z)\,\Gamma\left(z+\frac{1}{3}\right)\Gamma\left(z+\frac{2}{3}\right)$$

06.05.16.0008.01

$$\Gamma(n\,z) == n^{n\,z-\frac{1}{2}}\,(2\,\pi)^{\frac{1-n}{2}}\prod_{k=0}^{n-1}\Gamma\left(z+\frac{k}{n}\right)\,/;\,n\in\mathbb{N}^+$$

06.05.16.0009.01

$$\Gamma(n\,z+b) == n^{n\,z+b-\frac{1}{2}}\,(2\,\pi)^{\frac{1-n}{2}}\prod_{k=0}^{n-1}\Gamma\left(z+\frac{b+k}{n}\right)\,/;\,n\in\mathbb{N}^+$$

Products, sums, and powers of the direct function

Products of the direct function

06.05.16.0010.01

$$\Gamma(z)\,\Gamma(1-z) == \frac{\pi}{\sin(\pi\,z)}$$

(additional formula)

06.05.16.0012.01

$$\Gamma(z)\,\Gamma(-z) == -\frac{\pi}{z\sin(\pi\,z)}$$

06.05.16.0019.01

$$\Gamma(z)\,\Gamma(n-z) == \frac{\pi}{\sin(\pi\,z)}\,(1-z)_{n-1}\,/;\,n\in\mathbb{Z}$$

06.05.16.0011.01

$$\Gamma\left(z+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}-z\right) == \frac{\pi}{\cos(\pi\,z)}$$

06.05.16.0013.01

$$\Gamma(1+z)\,\Gamma(1-z) == \frac{z\,\pi}{\sin(\pi\,z)}$$

06.05.16.0014.01

$$\Gamma(z)\,\Gamma(w) == \frac{(w+z-2)!}{\binom{w+z-2}{z-1}}$$

06.05.16.0015.01

$$\frac{\Gamma(z)}{\Gamma(w)} == (w)_{z-w}$$

06.05.16.0016.01

$$\frac{\Gamma(z)}{\Gamma(w)} == (z-w)!\binom{z-1}{z-w}$$

06.05.16.0017.01

$$\frac{\Gamma(z+n)}{\Gamma(z)} == (z)_n$$

06.05.16.0020.01

$$\frac{\Gamma(z+n)}{\Gamma(z)} == \prod_{k=0}^{n-1}(z+k)\,/;\,n\in\mathbb{N}^+$$

06.05.16.0021.01

$$\frac{\Gamma(z-n)}{\Gamma(z)} = \frac{(-1)^n}{(1-z)_n} \;/;\; n \in \mathbb{N}$$

06.05.16.0022.01

$$\frac{\Gamma(z-n)}{\Gamma(z)} = \prod_{k=1}^{n} \frac{1}{z-k} \;/;\; n \in \mathbb{N}^+$$

06.05.16.0018.01

$$\frac{\Gamma(z)\,\Gamma(w)}{\Gamma(z+w)} = B(z, w)$$

Identities

Recurrence identities

Consecutive neighbors

06.05.17.0001.01

$$\Gamma(z) = (z-1)\,\Gamma(z-1)$$

06.05.17.0002.01

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}$$

Distant neighbors

06.05.17.0003.01

$$\Gamma(z) = \frac{\Gamma(z+n)}{(z)_n}$$

06.05.17.0004.01

$$\Gamma(z) = (-1)^n\,(1-z)_n\,\Gamma(z-n) \;/;\; n \in \mathbb{Z}$$

Functional identities

Relations of special kind

06.05.17.0005.01

$$f(z) = (z-1)\,f(z-1) \;/;\; f(z) = \Gamma(z)\,g(z) \land g(z+1) = g(z) \land f(1) = 1$$

$\Gamma(x)$ is the unique nonzero solution of the functional equation $f(x+1) = x\,f(x)$ which is logarithmically convex for all real $x > 0$, that is, for which $\log(f(x))$ is a convex function for $x > 0$.

$\Gamma(x)$ is the unique non-zero continuously differentiable solution of the system of functional equations

$$f(2x) = \frac{2^{2x-1}}{\sqrt{\pi}}\,f(x)\,f\left(x+\tfrac{1}{2}\right) \text{ and } f(x+1) = x\,f(x).$$

06.05.17.0006.01

$$\Gamma(z) = \frac{2^{2z}}{z} \prod_{k=1}^{\infty} \frac{1}{\sqrt{\pi}}\,\Gamma\left(\frac{z}{2^k} + \frac{1}{2}\right)$$

Complex characteristics

Real part

06.05.19.0001.01

$$\operatorname{Re}(\Gamma(x+i\,y)) = \int_1^\infty e^{-t}\,t^{x-1}\cos(y\log(t))\,dt + \sum_{k=0}^\infty \frac{(-1)^k}{k!}\left(\frac{k}{(k+x)^2+y^2} + \frac{x}{(k+x)^2+y^2}\right)$$

Imaginary part

06.05.19.0002.01

$$\operatorname{Im}(\Gamma(x+i\,y)) = \int_1^\infty e^{-t}\,t^{x-1}\sin(y\log(t))\,dt - y\sum_{k=0}^\infty \frac{(-1)^k}{k!\left((k+x)^2+y^2\right)}$$

Absolute value

06.05.19.0003.01

$$|\Gamma(i\,y)| = \sqrt{\frac{\pi}{y\sinh(\pi\,y)}}\ /;\ y\in\mathbb{R}$$

06.05.19.0004.01

$$\left|\Gamma\left(\frac{1}{2}+i\,y\right)\right| = \sqrt{\pi\operatorname{sech}(\pi\,y)}\ /;\ y\in\mathbb{R}$$

06.05.19.0005.01

$$|\Gamma(1+i\,y)| = \sqrt{\pi\,y\operatorname{csch}(\pi\,y)}\ /;\ y\in\mathbb{R}$$

06.05.19.0006.01

$$\left|\frac{\Gamma(x+i\,y)}{\Gamma(x)}\right|^2 = \prod_{k=0}^\infty \frac{1}{\frac{y^2}{(k+x)^2}+1}$$

Argument

06.05.19.0007.01

$$\arg(\Gamma(x+i\,y)) = y\,\psi(x) + \sum_{k=0}^\infty\left(\frac{y}{k+x} - \tan^{-1}\left(\frac{y}{k+x}\right)\right)\ /;\ y\in\mathbb{R}\bigwedge x>0$$

Conjugate value

06.05.19.0008.01

$$\overline{\Gamma(z)} = \Gamma(\bar{z})$$

Differentiation

Low-order differentiation

06.05.20.0001.01

$$\frac{\partial\Gamma(z)}{\partial z} = \Gamma(z)\,\psi(z)$$

06.05.20.0002.01

$$\frac{\partial^2 \Gamma(z)}{\partial z^2} = \Gamma(z)\,\psi(z)^2 + \Gamma(z)\,\psi^{(1)}(z)$$

Symbolic differentiation

06.05.20.0003.02

$$\frac{\partial^n \Gamma(z)}{\partial z^n} = \Gamma(z)\,R(n, z) \,/;\, R(n, z) = \psi(z)\,R(n-1, z) + R^{(0,1)}(n-1, z) \bigwedge R(0, z) = 1 \bigwedge n \in \mathbb{N}$$

06.05.20.0004.02

$$\frac{\partial^n \Gamma(z)}{\partial z^n} = \int_1^\infty t^{z-1} \log^n(t)\, e^{-t}\, dt + \frac{(-1)^n\, n!}{z^{n+1}}\, {}_nF_n(z_1, z_2, \ldots, z_{n+1}; z_1+1, z_2+1, \ldots, z_{n+1}+1; -1) \,/;$$
$$z_1 = z_2 = \ldots = z_{n+1} = z \bigwedge n \in \mathbb{N}$$

Fractional integro-differentiation

06.05.20.0005.01

$$\frac{\partial^\alpha \Gamma(z)}{\partial z^\alpha} = \mathcal{FC}_{\exp}^{(\alpha)}(z, -1)\, z^{-\alpha-1} + \sum_{k=1}^\infty \frac{\Gamma^{(k)}(1)\, z^{k-\alpha-1}}{k\,\Gamma(k-\alpha)}$$

06.05.20.0006.01

$$\frac{\partial^\alpha \Gamma(z)}{\partial z^\alpha} = \mathcal{FC}_{\exp}^{(\alpha)}(z, -1)\, z^{-\alpha-1} + z^{-\alpha} \sum_{k=1}^\infty \frac{(-1)^k}{k\,k!}\, {}_2\tilde{F}_1\left(1, 1; 1-\alpha; -\frac{z}{k}\right) + z^{-\alpha} \int_1^\infty t^{z-1}\,(z\log(t))^\alpha\,(1 - Q(-\alpha, z\log(t)))\, e^{-t}\, dt$$

Integration

Definite integration

Involving the direct function

06.05.21.0001.01

$$\int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(a+t)\,\Gamma(b+t)\,\Gamma(c-t)\,\Gamma(d-t)\,dt = \frac{2\pi i\; \Gamma(a+c)\,\Gamma(a+d)\,\Gamma(b+c)\,\Gamma(b+d)}{\Gamma(a+b+c+d)} \,/;$$
$$-\min(\mathrm{Re}(a), \mathrm{Re}(b)) < \gamma < \min(\mathrm{Re}(c), \mathrm{Re}(d))$$

06.05.21.0002.01

$$\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(a+t)\,\Gamma(b+t)\,\Gamma(c-t)}{\Gamma(a-b-c+1+t)}\,dt = \frac{\pi i\,\Gamma\left(\frac{a+c}{2}\right)\Gamma(b+c)}{\Gamma\left(1-b+\frac{a-c}{2}\right)} \,/;\, -\min(\mathrm{Re}(a), \mathrm{Re}(b)) < \gamma < \mathrm{Re}(c)$$

06.05.21.0003.01

$$\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(a+t)\,\Gamma(b-t)}{\Gamma(c+t)\,\Gamma(d-t)}\,dt = \frac{2\pi i\; \Gamma(a+b)\,\Gamma(c+d-a-b-1)}{\Gamma(c+d-1)\,\Gamma(c-a)\,\Gamma(d-b)} \,/;\, -\mathrm{Re}(a) < \gamma < \mathrm{Re}(b) \bigwedge \mathrm{Re}(a+b-c-d) < -1$$

06.05.21.0004.01

$$\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(a+t)\,\Gamma(b+t)\,\Gamma(c+t)\,\Gamma(d-t)\,\Gamma(-t)}{\Gamma(a+b+c+d+t)}\,dt = \frac{2\pi i\; \Gamma(a)\,\Gamma(b)\,\Gamma(c)\,\Gamma(a+d)\,\Gamma(b+d)\,\Gamma(c+d)}{\Gamma(h-a)\,\Gamma(h-b)\,\Gamma(h-c)} \,/;$$
$$h = a+b+c+d \bigwedge -\min(\mathrm{Re}(a), \mathrm{Re}(b), \mathrm{Re}(c)) < \gamma < \min(\mathrm{Re}(d), 0)$$

06.05.21.0005.01

$$\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(a+t)\,\Gamma\left(\frac{a}{2}+1+t\right)\Gamma(b+t)\,\Gamma(c+t)\,\Gamma(d+t)\,\Gamma(-a+b-t)\,\Gamma(-t)}{\Gamma\left(\frac{a}{2}+t\right)\Gamma(a-c+1+t)\,\Gamma(a-d+1+t)}\,dt == \frac{\pi\, i\, \Gamma(b)\,\Gamma(c)\,\Gamma(d)\,\Gamma(-a+b+c)\,\Gamma(-a+b+d)}{\Gamma(a-c-d+1)\,\Gamma(-a+b+c+d)} \;/;$$
$$-\min\left(\mathrm{Re}(a),\, \frac{\mathrm{Re}(a)}{2}+1,\, \mathrm{Re}(b),\, \mathrm{Re}(c),\, \mathrm{Re}(d)\right) < \gamma < \min(\mathrm{Re}(b-a),\, 0)$$

06.05.21.0006.01

$$\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(a+t)\,\Gamma\left(\frac{a}{2}+1+t\right)\Gamma(b+t)\,\Gamma(c+t)\,\Gamma(d+t)\,\Gamma(e+t)\,\Gamma(f+t)\,\Gamma(b-a-t)\,\Gamma(-t)}{\Gamma\left(\frac{a}{2}+t\right)\Gamma(a-c+1+t)\,\Gamma(a-d+1+t)\,\Gamma(a-e+1+t)\,\Gamma(a-f+1+t)}\,dt ==$$
$$\pi\, i\, \Gamma(b)\,\Gamma(c)\,\Gamma(d)\,\Gamma(e)\,\Gamma(f)\,\Gamma(b+c-a)\,\Gamma(b+d-a)\,\Gamma(b+e-a)$$
$$\Gamma(b+f-a)\,/\,(\Gamma(a-c-e+1)\,\Gamma(a-d-e+1)\,\Gamma(a-c-d+1)\,\Gamma(a-c-f+1)\,\Gamma(a-d-f+1)\,\Gamma(a-e-f+1))\;/;$$
$$2a == b+c+d+e+f-1 \bigwedge -\min\left(\mathrm{Re}(a),\, \frac{\mathrm{Re}(a)}{2}+1,\, \mathrm{Re}(b),\, \mathrm{Re}(c),\, \mathrm{Re}(d),\, \mathrm{Re}(e),\, \mathrm{Re}(f)\right) < \gamma < \min(\mathrm{Re}(b-a),\, 0)$$

06.05.21.0007.01

$$\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\prod_{k=1}^{A}\Gamma(t+a_k)\prod_{k=1}^{B}\Gamma(b_k-t)}{\prod_{k=1}^{C}\Gamma(t+c_k)\prod_{k=1}^{D}\Gamma(d_k-t)}\, z^{-t}\,dt == 2\pi\, i\, G_{B+C,A+D}^{A,B}\left(t \,\middle|\, \begin{array}{l} 1-b_1,\,\ldots,\,1-b_B,\,c_1,\,\ldots,\,c_C \\ a_1,\,\ldots,\,a_A,\,1-d_1,\,\ldots,\,1-d_D \end{array}\right) /;\; \Delta == A-B-C+D \bigwedge$$
$$\mathrm{E} == A+B-C-D \bigwedge \nu == \sum_{k=1}^{A} a_k + \sum_{k=1}^{B} b_k - \sum_{k=1}^{C} c_k - \sum_{k=1}^{D} d_k \bigwedge -\min(\mathrm{Re}(a_1),\,\ldots,\,\mathrm{Re}(a_A)) < \gamma < \min(\mathrm{Re}(b_1),\,\ldots,\,\mathrm{Re}(b_B)) \bigwedge$$
$$\left(|\arg(z)| < \frac{\pi\,\mathrm{E}}{2} \bigwedge \mathrm{E} > 0 \bigvee |\arg(z)| == \frac{\pi\,\mathrm{E}}{2} \bigwedge \mathrm{E} > 0 \bigwedge \gamma\,\Delta + \mathrm{Re}(\nu) - \frac{\mathrm{E}}{2} < -1 \bigvee\right.$$
$$\left. z > 0 \bigwedge \mathrm{E} == 0 \bigwedge \Delta \neq 0 \bigwedge \gamma\,\Delta + \mathrm{Re}(\nu) < \frac{1}{2} \bigvee z > 0 \land \mathrm{E} == 0 \land \Delta == 0 \land (\mathrm{Re}(\nu) < 0 \land z \neq 1 \lor \mathrm{Re}(\nu) < -1 \land z == 1)\right)$$

Integral transforms

Inverse Mellin transforms

06.05.22.0001.01

$$\mathcal{M}_s^{-1}[\Gamma(s)]\,(t) == e^{-t} \;/;\; \mathrm{Re}(s) > 0$$

06.05.22.0002.01

$$\mathcal{M}_s^{-1}[\Gamma(s)\,\Gamma(a-s)]\,(t) == (t+1)^{-a}\,\Gamma(a) \;/;\; 0 < \mathrm{Re}(s) < \mathrm{Re}(a)$$

06.05.22.0003.01

$$\mathcal{M}_s^{-1}\left[\frac{\Gamma(s)}{\Gamma(a-s)}\right](t) == t^{\frac{1-a}{2}}\, J_{a-1}\left(2\sqrt{t}\right) /;\; 0 < \mathrm{Re}(s) < \frac{2\,\mathrm{Re}(a)+1}{4}$$

06.05.22.0004.01

$$\mathcal{M}_s^{-1}\left[\frac{\Gamma(s)}{\Gamma(a+s)}\right](t) == \frac{(1-t)^{a-1}\,\theta(1-t)}{\Gamma(a)} \;/;\; \mathrm{Re}(a) > 0 \land \mathrm{Re}(s) > 0$$

06.05.22.0005.01

$$\mathcal{M}_s^{-1}\left[\frac{\prod_{k=1}^{A}\Gamma(s+a_k)\prod_{k=1}^{B}\Gamma(b_k-s)}{\prod_{k=1}^{C}\Gamma(s+c_k)\prod_{k=1}^{D}\Gamma(d_k-s)}\right](t) = G_{B+C,A+D}^{A,B}\left(t \left| \begin{array}{l} 1-b_1, \ldots, 1-b_B, c_1, \ldots, c_C \\ a_1, \ldots, a_A, 1-d_1, \ldots, 1-d_D \end{array}\right.\right) /;$$

$$\Delta = A - B - C + D \bigwedge \mathrm{E} = A + B - C - D \bigwedge \nu = \sum_{k=1}^{A} a_k + \sum_{k=1}^{B} b_k - \sum_{k=1}^{C} c_k - \sum_{k=1}^{D} d_k \bigwedge$$
$$-\min(\mathrm{Re}(a_1), \ldots, \mathrm{Re}(a_A)) < \mathrm{Re}(s) < \min(\mathrm{Re}(b_1), \ldots, \mathrm{Re}(b_B)) \bigwedge$$
$$\left(|\arg(t)| < \frac{\pi \mathrm{E}}{2} \bigwedge \mathrm{E} > 0 \bigvee |\arg(t)| = \frac{\pi \mathrm{E}}{2} \bigwedge \mathrm{E} > 0 \bigwedge -\frac{\mathrm{E}}{2} + \Delta \mathrm{Re}(s) + \mathrm{Re}(\nu) < -1 \bigvee\right.$$
$$\left. t > 0 \bigwedge \mathrm{E} = 0 \bigwedge \Delta \neq 0 \bigwedge \Delta \mathrm{Re}(s) + \mathrm{Re}(\nu) < \frac{1}{2} \bigvee t > 0 \wedge \mathrm{E} = 0 \wedge \Delta = 0 \wedge (\mathrm{Re}(\nu) < 0 \wedge t \neq 1 \vee \mathrm{Re}(\nu) < -1 \wedge t = 1)\right)$$

Summation

Infinite summation

06.05.23.0001.01

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \Gamma(k+z)\, \zeta(k+z) = -\Gamma(z) /; \mathrm{Re}(z) < 1$$

Products

Finite products

06.05.24.0001.01

$$\prod_{k=0}^{n-1} \Gamma\left(\frac{k+z}{n}\right) = n^{\frac{1}{2}-z} (2\pi)^{\frac{n-1}{2}} \Gamma(z) /; n \in \mathbb{N}^+$$

Operations

Limit operation

06.05.25.0001.01

$$\lim_{y\to\infty} |\Gamma(x+i y)|\, e^{\frac{\pi y}{2}}\, y^{\frac{1}{2}-x} = \sqrt{2\pi} /; x \in \mathbb{R} \bigwedge y \in \mathbb{R}$$

06.05.25.0002.01

$$\lim_{y\to\infty} |\Gamma(x+i y)|\, e^{-\frac{\pi}{2} y}\, (-y)^{\frac{1}{2}-x} = \sqrt{2\pi} /; x \in \mathbb{R} \bigwedge y \in \mathbb{R}$$

06.05.25.0003.01

$$\lim_{n\to\infty} \frac{\Gamma(a+n)\, n^{b-a}}{\Gamma(b+n)} = 1$$

Residues of ratios of gamma functions

Case of simple poles

06.05.25.0004.01

$$\operatorname{res}_s\left(\frac{\left(\prod_{k=1}^{\mathcal{A}}\Gamma(s+a_k)\right)\left(\prod_{k=1}^{\mathcal{B}}\Gamma(b_k-s)\right)}{\left(\prod_{k=1}^{\mathcal{C}}\Gamma(s+c_k)\right)\prod_{k=1}^{\mathcal{D}}\Gamma(d_k-s)}z^{-s}\right)(-a_1-l)=\Gamma\text{Res}\left(\begin{matrix}a_1,\ldots,a_{\mathcal{A}}; & b_1,\ldots,b_{\mathcal{B}};\\ c_1,\ldots,c_{\mathcal{C}}; & d_1,\ldots,d_{\mathcal{D}};\end{matrix}\,a_1,1,l;z\right)/;$$
$$l\in\mathbb{N}\land a_j-a_1\notin\mathbb{Z}\land 2\le j\le\mathcal{A}\land -b_j-a_1\notin\mathbb{N}\land 1\le j\le\mathcal{B}\land c_j-a_2-l\notin\mathbb{N}\land 1\le j\le\mathcal{C}\land -d_j-a_2-l\notin\mathbb{N}\land 1\le j\le\mathcal{D}$$

06.05.25.0005.01

$$\operatorname{res}_s\left(\frac{\left(\prod_{k=1}^{\mathcal{A}}\Gamma(s+a_k)\right)\left(\prod_{k=1}^{\mathcal{B}}\Gamma(b_k-s)\right)}{\left(\prod_{k=1}^{\mathcal{C}}\Gamma(s+c_k)\right)\prod_{k=1}^{\mathcal{D}}\Gamma(d_k-s)}z^{-s}\right)(b_1+l)=\Gamma\text{Res}\left(\begin{matrix}a_1,\ldots,a_{\mathcal{A}}; & b_1,\ldots,b_{\mathcal{B}};\\ c_1,\ldots,c_{\mathcal{C}}; & d_1,\ldots,d_{\mathcal{D}};\end{matrix}\,b_1,1,l;z\right)/;$$
$$l\in\mathbb{N}\land b_k-b_1\notin\mathbb{Z}\land 2\le j\le\mathcal{B}\land -a_j-b_1\notin\mathbb{N}\land 1\le j\le\mathcal{A}\land d_j-b_1-l\notin\mathbb{N}\land 1\le j\le\mathcal{D}\land -c_j-b_1-l\notin\mathbb{N}\land 1\le j\le\mathcal{C}$$

06.05.25.0006.01

$$\operatorname{res}_s\left(\frac{\left(\prod_{k=1}^{\mathcal{A}}\Gamma(\mathfrak{A}_k s+a_k)\right)\left(\prod_{k=1}^{\mathcal{B}}\Gamma(b_k-\mathfrak{B}_k s)\right)}{\left(\prod_{k=1}^{\mathcal{C}}\Gamma(\mathfrak{C}_k s+c_k)\right)\left(\prod_{k=1}^{\mathcal{D}}\Gamma(d_k-\mathfrak{D}_k s)\right)}z^{-s}\right)\left(-\frac{a_1+l}{\mathfrak{A}_1}\right)=\frac{(-1)^l\left(\prod_{k=2}^{\mathcal{A}}\Gamma\left(a_k-\mathfrak{A}_k\frac{a_1+l}{\mathfrak{A}_1}\right)\right)\left(\prod_{k=1}^{\mathcal{B}}\Gamma\left(b_k+\mathfrak{B}_k\frac{a_1+l}{\mathfrak{A}_1}\right)\right)}{\mathfrak{A}_1\,l!\left(\prod_{k=1}^{\mathcal{C}}\Gamma\left(c_k-\mathfrak{C}_k\frac{a_1+l}{\mathfrak{A}_1}\right)\right)\left(\prod_{k=1}^{\mathcal{D}}\Gamma\left(d_k+\mathfrak{D}_k\frac{a_1+l}{\mathfrak{A}_1}\right)\right)}z^{\frac{a_1+l}{\mathfrak{A}_1}}\;/;$$
$$l\in\mathbb{N}\bigwedge a_j-\mathfrak{A}_j\frac{a_1+l}{\mathfrak{A}_1}\notin\mathbb{Z}\bigwedge 2\le j\le\mathcal{A}\bigwedge -b_j-\mathfrak{B}_j\frac{a_1+l}{\mathfrak{A}_1}\notin\mathbb{N}\bigwedge$$
$$1\le j\le\mathcal{B}\bigwedge c_j-\mathfrak{C}_j\frac{a_1+l}{\mathfrak{A}_1}\notin\mathbb{N}\bigwedge 1\le j\le\mathcal{C}\bigwedge -d_j-\mathfrak{D}_j\frac{a_1+l}{\mathfrak{A}_1}\notin\mathbb{N}\bigwedge 1\le j\le\mathcal{D}$$

06.05.25.0007.01

$$\operatorname{res}_s\left(\frac{\left(\prod_{k=1}^{\mathcal{A}}\Gamma(a_k+s\,\mathfrak{A}_k)\right)\left(\prod_{k=1}^{\mathcal{B}}\Gamma(b_k-\mathfrak{B}_k s)\right)}{\left(\prod_{k=1}^{\mathcal{C}}\Gamma(c_k+s\,\mathfrak{C}_k)\right)\prod_{k=1}^{\mathcal{D}}\Gamma(d_k-\mathfrak{D}_k s)}z^{-s}\right)\left(\frac{b_1+l}{\mathfrak{B}_1}\right)=\frac{(-1)^{l-1}\prod_{k=1}^{\mathcal{A}}\Gamma\left(a_k+\mathfrak{A}_k\frac{b_1+l}{\mathfrak{B}_1}\right)\prod_{k=2}^{\mathcal{B}}\Gamma\left(b_k-\mathfrak{B}_k\frac{b_1+l}{\mathfrak{B}_1}\right)}{\mathfrak{B}_1\,l!\prod_{k=1}^{\mathcal{C}}\Gamma\left(c_k+\mathfrak{C}_k\frac{b_1+l}{\mathfrak{B}_1}\right)\prod_{k=1}^{\mathcal{D}}\Gamma\left(d_k-\mathfrak{D}_k\frac{b_1+l}{\mathfrak{B}_1}\right)}z^{-\frac{l+b_1}{\mathfrak{B}_1}}\;/;$$
$$l\in\mathbb{N}\bigwedge b_j-\mathfrak{B}_j\frac{b_1+l}{\mathfrak{B}_1}\notin\mathbb{Z}\bigwedge 2\le j\le\mathcal{B}\bigwedge -a_j-\mathfrak{A}_j\frac{b_1+l}{\mathfrak{B}_1}\notin\mathbb{N}\bigwedge$$
$$1\le j\le\mathcal{A}\bigwedge d_j-\mathfrak{D}_j\frac{b_1+l}{\mathfrak{B}_1}\notin\mathbb{N}\bigwedge 1\le j\le\mathcal{D}\bigwedge -c_j-\mathfrak{C}_j\frac{b_1+l}{\mathfrak{B}_1}\notin\mathbb{N}\bigwedge 1\le j\le\mathcal{C}$$

Case of double poles

06.05.25.0008.01

$$\operatorname{res}_s\left(\frac{\left(\prod_{k=1}^{\mathcal{A}}\Gamma(s+a_k)\right)\left(\prod_{k=1}^{\mathcal{B}}\Gamma(b_k-s)\right)}{\left(\prod_{k=1}^{\mathcal{C}}\Gamma(s+c_k)\right)\prod_{k=1}^{\mathcal{D}}\Gamma(d_k-s)}z^{-s}\right)(-a_2-l)=\Gamma\text{Res}\left(\begin{matrix}a_1,\ldots,a_{\mathcal{A}}; & b_1,\ldots,b_{\mathcal{B}};\\ c_1,\ldots,c_{\mathcal{C}}; & d_1,\ldots,d_{\mathcal{D}};\end{matrix}\,a_2,2,l;z\right)/;a_2-a_1\in\mathbb{N}\land l\in\mathbb{N}\land$$
$$a_j-a_1\notin\mathbb{Z}\land 3\le j\le\mathcal{A}\land -b_j-a_1\notin\mathbb{N}\land 1\le j\le\mathcal{B}\land c_j-a_2-l\notin\mathbb{N}\land 1\le j\le\mathcal{C}\land -d_j-a_2-l\notin\mathbb{N}\land 1\le j\le\mathcal{D}$$

06.05.25.0009.01

$$\operatorname{res}_s\left(\frac{\left(\prod_{k=1}^{\mathcal{A}}\Gamma(s+a_k)\right)\left(\prod_{k=1}^{\mathcal{B}}\Gamma(b_k-s)\right)}{\left(\prod_{k=1}^{\mathcal{C}}\Gamma(s+c_k)\right)\prod_{k=1}^{\mathcal{D}}\Gamma(d_k-s)}z^{-s}\right)(b_2+l)=\Gamma\text{Res}\left(\begin{matrix}a_1,\ldots,a_{\mathcal{A}}; & b_1,\ldots,b_{\mathcal{B}};\\ c_1,\ldots,c_{\mathcal{C}}; & d_1,\ldots,d_{\mathcal{D}};\end{matrix}\,b_2,2,l;z\right)/;b_2-b_1\in\mathbb{N}\land l\in\mathbb{N}\land$$
$$b_k-b_1\notin\mathbb{Z}\land 3\le j\le\mathcal{B}\land -a_j-b_1\notin\mathbb{N}\land 1\le j\le\mathcal{A}\land d_j-b_2-l\notin\mathbb{N}\land 1\le j\le\mathcal{D}\land -c_j-b_2-l\notin\mathbb{N}\land 1\le j\le\mathcal{C}$$

Case of triple poles

06.05.25.0010.01

$$\operatorname{res}_s\left(\frac{\left(\prod_{k=1}^{\mathcal{A}}\Gamma(s+a_k)\right)\left(\prod_{k=1}^{\mathcal{B}}\Gamma(b_k-s)\right)}{\left(\prod_{k=1}^{C}\Gamma(s+c_k)\right)\prod_{k=1}^{\mathcal{D}}\Gamma(d_k-s)}z^{-s}\right)(-a_3-l)=\Gamma\mathrm{Res}\left(\begin{matrix}a_1,\,\ldots,\,a_{\mathcal{A}}; & b_1,\,\ldots,\,b_{\mathcal{B}};\\ c_:,\,\ldots,\,c_C; & d_1,\,\ldots,\,d_{\mathcal{D}};\end{matrix}\;a_3,\,3,\,l;\,z\right)/;\;a_2-a_1\in\mathbb{N}\bigwedge a_3-a_2\in\mathbb{N}\bigwedge l\in\mathbb{N}\bigwedge a_j-a_1\notin\mathbb{Z}\wedge 4\le j\le\mathcal{A}\bigwedge -b_j-a_1\notin\mathbb{N}\bigwedge 1\le j\le\mathcal{B}\bigwedge c_j-a_3-l\notin\mathbb{N}\bigwedge 1\le j\le C\bigwedge -d_j-a_3-l\notin\mathbb{N}\bigwedge 1\le j\le\mathcal{D}$$

06.05.25.0011.01

$$\operatorname{res}_s\left(\frac{\left(\prod_{k=1}^{\mathcal{A}}\Gamma(s+a_k)\right)\left(\prod_{k=1}^{\mathcal{B}}\Gamma(b_k-s)\right)}{\left(\prod_{k=1}^{C}\Gamma(s+c_k)\right)\prod_{k=1}^{\mathcal{D}}\Gamma(d_k-s)}z^{-s}\right)(b_3+l)=\Gamma\mathrm{Res}\left(\begin{matrix}a_1,\,\ldots,\,a_{\mathcal{A}}; & b_1,\,\ldots,\,b_{\mathcal{B}};\\ c_:,\,\ldots,\,c_C; & d_1,\,\ldots,\,d_{\mathcal{D}};\end{matrix}\;b_3,\,3,\,l;\,z\right)/;\;b_2-b_1\in\mathbb{N}\bigwedge b_3-b_2\in\mathbb{N}\bigwedge l\in\mathbb{N}\bigwedge b_k-b_1\notin\mathbb{Z}\bigwedge 4\le j\le\mathcal{B}\bigwedge -a_j-b_1\notin\mathbb{N}\bigwedge 1\le j\le\mathcal{A}\bigwedge d_j-b_3-l\notin\mathbb{N}\bigwedge 1\le j\le\mathcal{D}\bigwedge -c_j-b_3-l\notin\mathbb{N}\bigwedge 1\le j\le C$$

Case of quartic poles

06.05.25.0012.01

$$\operatorname{res}_s\left(\frac{\left(\prod_{k=1}^{\mathcal{A}}\Gamma(s+a_k)\right)\left(\prod_{k=1}^{\mathcal{B}}\Gamma(b_k-s)\right)}{\left(\prod_{k=1}^{C}\Gamma(s+c_k)\right)\prod_{k=1}^{\mathcal{D}}\Gamma(d_k-s)}z^{-s}\right)(-a_4-l)=\Gamma\mathrm{Res}\left(\begin{matrix}a_1,\,\ldots,\,a_{\mathcal{A}}; & b_1,\,\ldots,\,b_{\mathcal{B}};\\ c_:,\,\ldots,\,c_C; & d_1,\,\ldots,\,d_{\mathcal{D}};\end{matrix}\;a_4,\,4,\,l;\,z\right)/;$$

$$a_2-a_1\in\mathbb{N}\bigwedge a_3-a_2\in\mathbb{N}\bigwedge a_4-a_3\in\mathbb{N}\bigwedge l\in\mathbb{N}\bigwedge a_j-a_1\notin\mathbb{Z}\wedge 5\le j\le\mathcal{A}\bigwedge -b_j-a_1\notin\mathbb{N}\bigwedge 1\le j\le\mathcal{B}\bigwedge c_j-a_4-l\notin\mathbb{N}\bigwedge 1\le j\le C\bigwedge -d_j-a_4-l\notin\mathbb{N}\bigwedge 1\le j\le\mathcal{D}$$

06.05.25.0013.01

$$\operatorname{res}_s\left(\frac{\left(\prod_{k=1}^{\mathcal{A}}\Gamma(s+a_k)\right)\left(\prod_{k=1}^{\mathcal{B}}\Gamma(b_k-s)\right)}{\left(\prod_{k=1}^{C}\Gamma(s+c_k)\right)\prod_{k=1}^{\mathcal{D}}\Gamma(d_k-s)}z^{-s}\right)(b_4+l)=\Gamma\mathrm{Res}\left(\begin{matrix}a_1,\,\ldots,\,a_{\mathcal{A}}; & b_1,\,\ldots,\,b_{\mathcal{B}};\\ c_:,\,\ldots,\,c_C; & d_1,\,\ldots,\,d_{\mathcal{D}};\end{matrix}\;b_4,\,4,\,l;\,z\right)/;$$

$$b_2-b_1\in\mathbb{N}\bigwedge b_3-b_2\in\mathbb{N}\bigwedge b_4-b_3\in\mathbb{N}\bigwedge l\in\mathbb{N}\bigwedge b_k-b_1\notin\mathbb{Z}\bigwedge 5\le j\le\mathcal{B}\bigwedge -a_j-b_1\notin\mathbb{N}\bigwedge 1\le j\le\mathcal{A}\bigwedge d_j-b_4-l\notin\mathbb{N}\bigwedge 1\le j\le\mathcal{D}\bigwedge -c_j-b_4-l\notin\mathbb{N}\bigwedge 1\le j\le C$$

Representations through more general functions

Through other functions

Involving some hypergeometric-type functions

06.05.26.0001.01

$$\Gamma(z)=\Gamma(z,\,0)\;/;\;\mathrm{Re}(z)>0$$

Representations through equivalent functions

With related functions

06.05.27.0001.01

$$\Gamma(z)=(z-1)!$$

06.05.27.0002.01

$$\Gamma(z)=2^{\frac{1}{4}(3-4z+\cos(2\pi z))}\,\pi^{\frac{1}{2}\sin^2(\pi z)}\,(2z-2)!!$$

Inequalities

06.05.29.0001.01

$$|\Gamma(z)|\le|\Gamma(\mathrm{Re}(z))|$$

06.05.29.0002.01

$$\Gamma(x) \leq x^x e^{1-x} /; x \in \mathbb{R} \wedge x \geq 1$$

06.05.29.0006.01

$$\left(\frac{x}{e}\right)^{x-1} \leq \Gamma(x) \leq \ \leq \left(\frac{x}{2}\right)^{x-1} /; x \in \mathbb{R} \wedge x \geq 2$$

06.05.29.0007.01

$$n^{1-x} < \frac{\Gamma(n+1)}{\Gamma(n+x)} > e^{(1-x)\psi(n+1)} /; n \in \mathbb{N}^+ \bigwedge x \in \mathbb{R} \bigwedge 0 < x < 1$$

06.05.29.0008.01

$$\sqrt{x+\frac{1}{2}} > \frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} > \sqrt{x+\frac{1}{4}} /; x \in \mathbb{R} \bigwedge x > -\frac{1}{4}$$

06.05.29.0009.01

$$\left(x+\frac{y}{2}\right)^{1-y} < \frac{\Gamma(x+1)}{\Gamma(x+y)} > e^{(1-y)\psi\left(\frac{y+1}{2}+x\right)} /; x \in \mathbb{R} \bigwedge x > 0 \bigwedge y \in \mathbb{R} \bigwedge 0 < y < 1$$

06.05.29.0010.01

$$(-1)^n \frac{\partial^n \left(\frac{\Gamma(x+y)}{\Gamma(x+1)} e^{(1-y)\psi\left(\frac{y+1}{2}+x\right)}\right)}{\partial x^n} > 0 /; n \in \mathbb{N} \bigwedge x \in \mathbb{R}^+ \bigwedge y \in \mathbb{R} \bigwedge 0 < y < 1$$

06.05.29.0011.01

$$(-1)^n \frac{\partial^n \frac{\Gamma(x+1)}{\Gamma(x+y)} \left(x+\frac{y}{2}\right)^{y-1}}{\partial x^n} > 0 /; n \in \mathbb{Z} \bigwedge n \geq 0 \bigwedge x \in \mathbb{R} \bigwedge x > 0 \bigwedge y \in \mathbb{R} \bigwedge 0 < y < 1$$

06.05.29.0003.01

$$\Gamma\left(\sum_{k=1}^n p_k x_k\right) \leq \sum_{k=1}^n p_k \Gamma(x_k) /; p_k > 0 \bigwedge x_k > 0 \bigwedge 1 \leq k \leq n \bigwedge \sum_{k=1}^n p_k == 1$$

06.05.29.0004.01

$$\Gamma\left(\prod_{k=1}^n x_k^{p_k}\right) \leq \prod_{k=1}^n \Gamma(x_k)^{p_k} /; p_k > 0 \bigwedge x_k \geq a > 0 \bigwedge 1 \leq k \leq n \bigwedge \psi(a) + a\,\psi^{(1)}(a) == 0 \bigwedge$$

$$a == 0.21609874531333414051081581677981987758705649255064454597496063524339035\backslash 5016505581272221260167439143622018214555137493697\ldots$$

06.05.29.0005.01

$$\Gamma\left(\prod_{k=1}^n x_k^{p_k}\right) \geq \prod_{k=1}^n \Gamma(x_k)^{p_k} /; p_k > 0 \bigwedge 0 < x_k \leq a \bigwedge 1 \leq k \leq n \bigwedge \psi(a) + a\,\psi^{(1)}(a) == 0 \bigwedge$$

$$a == 0.21609874531333414051081581677981987758705649255064454597496063524339035\backslash 5016505581272221260167439143622018214555137493697\ldots$$

Zeros

06.05.30.0001.01

$$\Gamma(z) \neq 0 /; \forall z$$

Theorems

Riemann-Liouville fractional integration

$$\hat{f}_\alpha(y) == \int_c^y f(x)\, \frac{(y-x)^{\alpha-1}}{\Gamma(\alpha)}\, dx \Leftrightarrow f(x) == \frac{1}{\Gamma(n-\alpha)}\, \frac{\partial^n}{\partial x^n} \int_c^x \hat{f}_\alpha(y)\, (x-y)^{n-\alpha-1}\, dy \,/;\, n-1 \leq \alpha < n \bigwedge n \in \mathbb{N}^+$$

The value of the Barnes integral

$$\int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(a+s)\,\Gamma(b+s)\,\Gamma(c-s)\,\Gamma(d-s)\, ds == \frac{\Gamma(a+c)\,\Gamma(a+d)\,\Gamma(b+c)\,\Gamma(b+d)}{\Gamma(a+b+c+d)} \,/;\, -\mathrm{Re}(a),$$
$-\mathrm{Re}(b) < \gamma < \mathrm{Re}(c), \mathrm{Re}(d)$.

The scattering amplitude in the outgoing wave function

The scattering amplitude $f(k; \boldsymbol{n}', \boldsymbol{n})$ in the outgoing wave function

$$\psi(\boldsymbol{r}) \underset{x\to\infty}{\sim} e^{i\, \boldsymbol{k}\,\boldsymbol{r} + i/k \log(\boldsymbol{k}\,\boldsymbol{r} - k\, r\, \boldsymbol{n}'.\boldsymbol{n})} + \frac{1}{r}\, e^{i\, k\, r - i/k \log(2\, k\, r)}\, f(k; \boldsymbol{n}', \boldsymbol{n})$$

for the Coulomb potential $V(r) = \frac{\alpha}{r}$ after renormalization is given by

$$f(k; \boldsymbol{n}', \boldsymbol{n}) == \frac{1}{2\, i\, k} \sum_{l=0}^{\infty} (2\, l+1)\, \frac{\Gamma(l+1+i\,/k)}{\Gamma(l+1-i\,/k)}\, P_l(\boldsymbol{n}'.\boldsymbol{n}) == \frac{i}{2\, k^2} \left(4\, |\, \boldsymbol{n} - \boldsymbol{n}'\, |^{-2}\right)^{1+i/k} \frac{\Gamma(1+i\,/k)}{\Gamma(1-i\,/k)}$$

where $\eta == \eta(\mathrm{k})$.

Selberg's integral

$$\int_0^1 \int_0^1 \ldots \int_0^1 \Bigg| \prod_{\substack{j,l=1 \\ j<l}}^{n} \left(x_j - x_l\right) |^{2\gamma} \prod_{k=1}^{n} \left(x_k^{\alpha-1}\, (1-x_k)\right)^{\beta-1} dx_1\, dx_2 \ldots dx_n ==$$

$$\prod_{k=0}^{n} \frac{\Gamma(1+\gamma+k\,\gamma)\,\Gamma(\alpha+k\,\gamma)\,\Gamma(\beta+k\,\gamma)}{\Gamma(1+\gamma)\,\Gamma(\alpha+\beta+(n+k-1)\,\gamma)} \,/;$$

$$\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0, \mathrm{Re}(\gamma) > -\min\left(\frac{1}{n}, \frac{\mathrm{Re}(\alpha)}{n-1}, \frac{\mathrm{Re}(b)}{n-1}\right).$$

Dirichlet integral

$$\underset{x_1+x_2+\ldots+x_n\leq 1}{\int_0^1 \int_0^1 \ldots \int_0^1} f\left(\sum_{k=1}^{n} x_k\right) \prod_{k=1}^{n} x_k^{\alpha_k-1}\, dx_1\, dx_2 \ldots dx_n == \prod_{k=1}^{n} \Gamma(\alpha_k) \Big/ \Gamma\left(\sum_{k=1}^{n} \alpha_k\right) \int_0^1 f(x)\, x^{\sum_{k=1}^{n} \alpha_k - 1}\, dx \,/;\, \mathrm{Re}(\alpha_k) > 0,$$
$k == 1, 2, \ldots, n$

Another Dirichlet-type integral

$$\underset{\substack{x_1, x_2, \ldots, x_{n>0} \\ x_1+x_2+\ldots+x_n \leq 1}}{\int \ldots \int} x_1^{\alpha_1-1} x_2^{\alpha_2-1} \ldots x_n^{\alpha_n-1} \, dx_1 \, dx_2 \ldots dx_n = \frac{\Gamma(\alpha_1)\,\Gamma(\alpha_2)\ldots\Gamma(\alpha_n)}{\Gamma(\alpha_1+\ldots+\alpha_n+1)}.$$

The micro-canonical single particle distribution function for a multi-dimensional ideal gas

The micro-canonical single particle distribution function $p(z, v)$ for an f-dimensional ideal gas in a gravitational field with acceleration g is given by

$$p(z, v) = \frac{2^{(2-f)/2}\,\Gamma((f+2)N/2)}{\Gamma(f/2)\,\Gamma((f+2)(N-1)/2)} \left(g\, m^{(2-f)/2}\, v^{f-1}\, \varepsilon^{(2-f)/2}\right) \left(-\frac{m v^2}{2\varepsilon} - \frac{m g z}{\varepsilon} + 1\right)^{((f+2)N-(f+4))/2},$$

where m is the particle mass and ε its energy.

History

– L. Euler (1729) derived integral representation for $n!$, leading to gamma function
– J. Stirling (1730) found main asymptotical term for $n!$
– A. M. Legendre (1808, 1814) suggested the notation Γ and discovered known duplication formula
– C. F. Gauss (1812) derived the multiplication formula
– F.W. Newman (1848) found product representation for the reciprocal of the gamma function that are valid for whole complex plane
– B. Riemann (1856) proved important relation between gamma and zeta functions
- K. Weierstrass (1856) widely used gamma function in his inverrstigations
– H. Hankel (1864, 1880) derived a contour integral representation for complex arguments
– O. Hölder (1887) proved that gamma function does not satisfy any algebraic differential equation
– H. Bohr and J. Mollerup (1922) proved that the gamma function is the only function that satisfies the recurrence relationship and is logarithmically convex

Encountered often in mathematics and the natural sciences.

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