The Identity of the Abelian Sandpile Group

Noah Doman
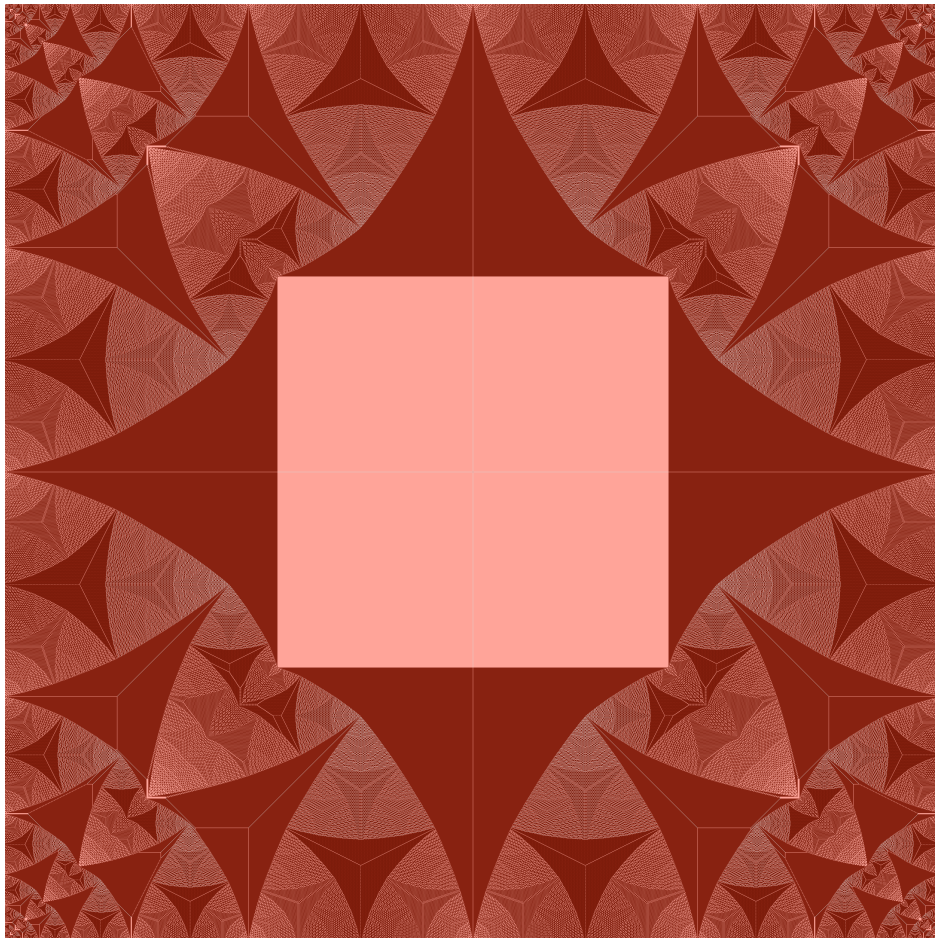
Supervisor: Jaap Top
Second Assessor: Steffen Müller

Abstract

Sandpiles, or configurations, are non-negative integer vectors indexed by the vertices of a finite connected graph. Under toppling, the set of stable configurations forms a finite commutative monoid. One possible definition of the abelian sandpile group is that it is the minimal ideal of this monoid. The identity element of the group is a particular configuration; in the case of a graph with symmetries (for example, a square grid), it possesses a fractal-like structure. In this bachelor's project, our main focus is on the abelian sandpile group of undirected rectangular grid graphs, and we present two different methods for calculating its identity. We also discuss the symmetric aspects of the identity element, and the existence of the symmetric abelian sandpile subgroup.

Contents

1 Introduction

Sand has been an interest to mathematics for a long time. Some might argue that the first research-expository paper is about sand. Addressed to Syracusan king Gelo II, Archemides wrote *The Sand-Reckoner* as an attempt to calculate an upper bound on the number of sand grains that could fit in the entire universe [1]. Archimedes did an adequate job for the methods available at that time. Yet, despite such an ancient history, the interest of sand in this paper comes from more recent work. Instead of analyzing sand as a physical object, we study the underlying algebra of its mechanical process in nature.

In 1987, physicists, Per Bak, Chao Tang and Kurt Wiesenfeld, published the first paper on the sandpile model. They approached the subject from a dynamical system perspective - discovering the first instance of self-organized criticality [2]. Essentially, when a sandpile cannot hold its weight and collapses, it re-organizes itself to a stable state without external tweaking of its parameters. Like most natural phenomenons, self-organized criticality has appeared in other fields like geophysics, ecology and economics. Then, in 1994, the physicists, Deepack Dhar, Phillippe Ruelle, S. Sen and D. Verma, wrote the paper *Algebraic Aspects of Abelian Sandpile Models* [3]. As the title suggests, they decided to study the sandpile model algebraically; and in doing so, they discovered an abelian group. By abstracting the physical process, their algebraic approach explained self-organized criticality by means of an abelian operator. The main goal of this paper is to present methods for calculating the identity of the abelian sandpile group. However, to do this, preliminaries must be established; sandpiles are mathematically defined and rules for stability are formed.

Sandpiles are non-negative integer vectors indexed by the vertices of a finite connected graph; undirected graphs are the main focus of this project. Furthermore, the process of stabilization (or collapse) of an unstable sandpile is described via a toppling transformation. When studied algebraically, the toppling transformation's abelian properties explains why sandpiles display self-organized critical behavior. Unstable sandpiles stabilize through a finite amount of steps to a unique stable state. Once these preliminaries are established, only then can an abelian group be discussed.

Most academic literature on sandpiles defines the abelian sandpile group as the set of recurrent sandpiles - the sandpiles that occur with positive probability. However, this paper takes a different approach. The set of stable sandpiles is a finite abelian monoid, an algebraic structure with interesting properties. One of the most important aspects of this paper is that the minimal ideal of a finite abelian monoid is a group, and that the set of recurrent sandpiles is the minimal ideal of the set of stable sandpiles. As it turns out, the identity of the abelian sandpile group is rarely the identity of the monoid that contains the group. So, finding the abelian sandpile group's identity can be a difficult task.

Two different methods are presented for calculating the identity of the group. Both methods come from an explicit isomorphism, which will be discussed later. Also, code for these methods and other important aspects of the group can be found in the appendix. Throughout the paper examples and figures are provided, all of which are created for the purpose of this project. Lastly, an example of the identity of the abelian sandpile group can be found on the title page. As the reader can see, the identity possesses a fractal-like structure and symmetries, which is the motivation for the last section of this project. Therein, we prove that the abelian sandpile group contains a subgroup of symmetric sandpiles.

2 Preliminaries

To understand the mathematical approach to sandpiles, we begin by imagining a sandpile on a table. One at a time, we randomly drop sand grains onto the pile, leaving it more or less unchanged. However, there comes a point when the pile accumulates too much sand for it to hold. This results in an 'avalanche'. The grains on these unstable sites displaces itself to nearby sites until the pile becomes stable again. If we continue the process of randomly dispensing sand grains, the pile will eventually reach the boundary of the table. Any further 'expansion' of the sandpile will result in grains falling off the boundary of the table; never to return. With this scenario in mind, we define the mathematical notion of a sandpile.

2.1 Graphs and Configurations

To begin, we place a finite connected *graph G* on our imaginary table. We define the *graph G* to be the pair (V, E), where V is a set of *n vertices* and E is a set of two-sets of vertices whose elements are called *edges*. An edge represents the connection of two vertices; let $i, j \in V$, then e_{ij} is the number of edges from vertex i to vertex j. In our scenario, sand can move in any direction so the graph G is undirected and simple. If vertex i is connected to vertex j we call them *neighbors*.

Recall from our scenario that when sand falls off the table it disappears forever. We represent the void extending pass the boundary of the table by a vertex s, which we call the *global sink* or *sink*. A boundary vertex i of V is a vertex that has a directed edge to the sink s, we denote number of directed edges from vertex i to the sink s as s_i. The *sandpile graph S(G)* is the triple (V, E, s) (for an example, see Figure 1).
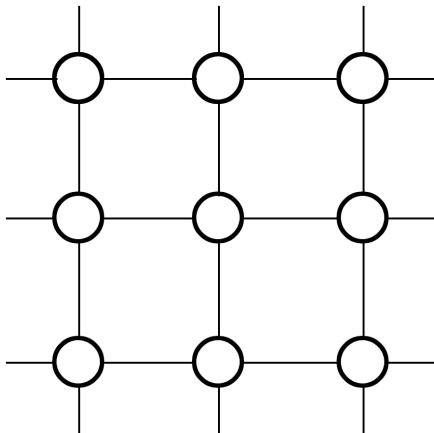

Figure 1: In this figure, we have a *sandpile graph* on a 3×3 grid. The circles represent *vertices*, the lines connecting the circles are *edges*, and the edges protruding out of the boundary vertices are directed edges to the *sink*.

It should be noted that the focus of this paper is on graphs of $p \times q$ grids. Conventionally, we label the vertices by integers $\{1, 2, ..., pq\}$ starting in the top left of the grid and proceeding from left to right then downwards; the vertex pq is the bottom right vertex of the grid. Furthermore, we have the rule that vertices $i, j \in \{1, ..., pq\}$ are connected, $e_{ij} = 1$, if

1. $|i - j| = 1$, or

2. $|i - j| = p$.

In other words, two vertices on the grid are connected, if they are (1) next to each other on the same row, or (2) next to each other on the same column. Lastly, corner vertices have two connections to the sink and all other boundary vertices have one connection (see Figure 1). Now, we proceed to define a sandpile.

Let $S(G)$ be a sandpile graph, then to create a sandpile we place grains of sand on each vertex, which can be represented by a non-negative integer vector u. We call a sandpile on a sandpile graph $S(G)$ with n vertices a *configuration* $u = (u_1, u_2, ..., u_n)$, where u_i is the number of sand grains on vertex $i \in V$; or the *weight* of vertex i. However, a vertex can only hold so many grains before the sandpile, or configuration, collapses. The weight that causes vertex i to collapse is called the *degree* of vertex i, denoted by deg_i. The degree of vertex i is equal to the number of neighbors of vertex i;

$$deg_i = s_i + \sum_{j \in V} e_{ij}.$$

Hence, the threshold of vertex i is $deg_i - 1$. Let $u = (u_1, ..., u_n)$ be a configuration, then we call u *stable* if for all vertices $i \in V$, u_i is less than the degree of i, otherwise the configuration u is *unstable* (see Figure 2). The set of all stable configurations u is denoted by Ω_n,

$$\Omega_n = \{u \in \mathbb{Z}_{\geq 0}^n \,|\, \forall i \in V, \, u_i < deg_i\}.$$

We have now defined the mathematical notion of a sandpile, and separated them into two different types: stable and unstable. In the next subsection, we mathematically define the collapse of an unstable configuration. We do so by explaining the process as a transformation.
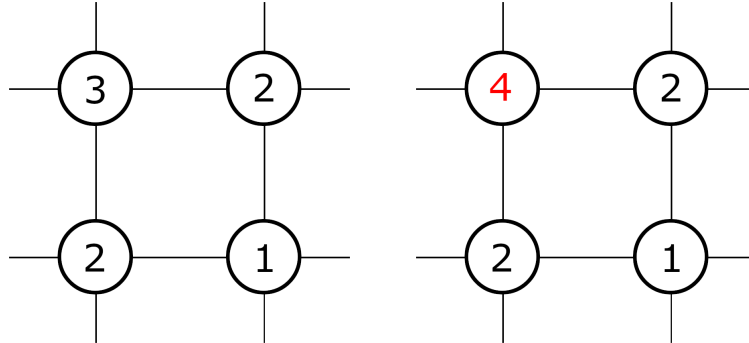

Figure 2: On the left: a stable configuration on a 2×2 grid. On the right: an unstable configuration on a 2×2 grid. The unstable vertex is highlighted in red.

2.2 The Toppling Rule

When a sandpile becomes unstable, the unstable sites send sand to their neighbors. We call this process the *toppling rule*. When an unstable vertex topples it sends a grain of sand 'through' each edge. Sand is lost to the sink if it is connected to the unstable vertex, and each neighbor gains a grain. Let $S(G)$ be a sandpile graph and $u = (u_1, ..., u_n)$ be a configuration with an unstable vertex $i \in V$. We define the toppling of vertex i as the mapping T_i from non-negative integer vectors to non-negative integers,

$$T_i : \mathbb{Z}_{\geq 0}^n \longrightarrow \mathbb{Z}_{\geq 0}^n, \ u \mapsto u' \text{ where } u'_j = \begin{cases} u_j - deg_i & \text{if } j = i, \\ u_j + e_{ij} & \text{if } j \neq i. \end{cases} \tag{1}$$

For example, if we take the unstable configuration in Figure 2, we can see the toppling rule being applied in Figure 3.
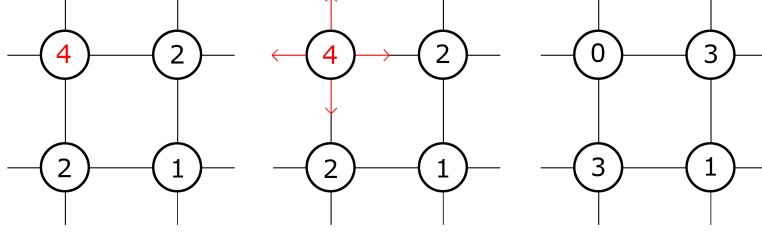

Figure 3: An unstable vertex toppling. The red arrows indicate sand grains being sent to connected vertices and sinks.

So far, we have defined the toppling rule for a single unstable vertex, however it is possible for a configuration u to have multiple unstable vertices. This brings us to our first theorem [4].

Theorem 2.1. The toppling of multiple unstable vertices commute.

Proof. To prove Theorem 2.1, it suffices to show that the toppling of two arbitrary unstable vertices commute. Then, commutativity holds for an arbitrary amount of unstable vertices.

We begin by taking $u = (u_1, ..., u_n)$ to be an unstable configuration. Let $i, j \in V$ be unstable vertices such that $i \neq j$. We want to show

$$T_j \circ T_i(u) = T_i \circ T_j(u). \tag{2}$$

Let $k \in V$, we set $T_i(u) = v$. By definition, the left-hand side of Equation 2 is

$$v_k = \begin{cases} u_k - deg_i & \text{if } k = i, \\ u_k + e_{ik} & \text{if } k \neq i. \end{cases} \tag{3}$$

Next, we set $T_j(v) = w$ and get

$$w_k = \begin{cases} v_k - deg_j & \text{if } k = j, \\ v_k + e_{jk} & \text{if } k \neq j. \end{cases}$$

We substitute v_k for its value in terms of u_k and obtain

$$w_k = \begin{cases} u_k + e_{ik} - deg_j & \text{if } k = j, \\ u_k - deg_i + e_{ji} & \text{if } k = i \\ u_k + e_{ik} + e_{jk} & \text{if } k \neq i \text{ and } k \neq j. \end{cases}$$

Using the same method for $T_i \circ T_j(u) = w'$, we get

$$w'_k = \begin{cases} u_k + e_{jk} - deg_i & \text{if } k = i, \\ u_k - deg_j + e_{ik} & \text{if } k = j \\ u_k + e_{jk} + e_{ik} & \text{if } k \neq i \text{ and } k \neq j. \end{cases}$$

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We conclude that $w_k = w'_k$ for all $k \in V$, hence $w = w'$. This proves that $T_j \circ T_i(u) = T_i \circ T_j(u)$, i.e. the toppling of multiple unstable vertices commutes.

From now on, we will denote the composition of topplings as a product, e.g.

$$T_k \circ \cdots \circ T_1(u) = \prod_{i=1}^{k} T_i(u).$$

Deepak Dhar argues in his paper, *Self-Organized Critical State of Sandpile Automation Models*, that commutativity is enough to show self-organized criticality. Dhar claims that unstable configurations 're-organize', or topple, themselves to a unique stable configuration because vertex toppling commutes [5]. For physicists this might be sufficient be a sufficient argument for proving the claim, however for mathematicians, it is not. If we look at Figure 3 we see an example where toppling all unstable vertices results in a stable configuration. However, Figure 4 shows an example where it does not; the resulting configuration has new unstable vertices.
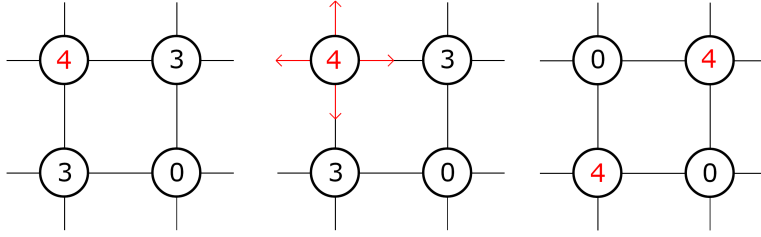

Figure 4: unstable vertex (highlighted in red) topples resulting in two more unstable vertices.

Dhar was correct that commutativity is an important aspect of self-organized criticality, but we must also show that for all unstable configurations u, (1) there exists a finite sequence of topplings such that u becomes stable and (2) this stable configuration is unique and the sequence of topplings is unique up to permutation. Then, we can conclude that sandpiles display self-organized critical behavior. Before we prove statements (1) and (2), we need the following definitions recalled from the text [6].

Let $u = (u_1, ..., u_n)$ be a configuration on the sandpile graph $S(G)$, we define the *degree* of u, or $deg(u)$, as the total number of grains the configuration holds,

$$deg(u) = \sum_{i \in V} u_i,$$

We use the degree of a configuration to define two partial orderings on the set of all configurations. The first we denote as $\leq$ and define as follows: let u and v be configuration on $S(G)$, then $u \leq v$ if $u_i \leq v_i$ for all $i \in V$. It is easy to see that the partial ordering $\leq$ has the following property:

1. For all configurations u, it holds that $0 \leq u$ where 0 is the configuration with all zeros,

2. If $u \leq v$, then $deg(u) \leq deg(v)$.

If we take the configurations in Figure 2 as an example, then the configuration on the left is 'less' than the configuration on the right. We write the two configurations in vector form, $u = (3, 2, 2, 1)$ and $v = (4, 2, 2, 1)$, then $u < v$. Now, we define our second partial ordering on the set of all configurations.

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Definition 2.1. Let $i_1, i_2, ..., i_n$ be an ordering of the vertices in V depending on their minimal distance to the sink. Let $j, k \in \{1, ..., n\}$ and $j < k$, then the minimal distance from vertex i_j to the sink is less than or equal to the minimal distance from vertex i_k to the sink, i.e. $d(i_j, s) \leq d(i_k, s)$. After permuting the vertices in V, we define the *sandpile ordering* $\prec$ of configurations on a sandpile graph as follows: given distinct configurations u and v equal to $(u_{i_1}, ..., u_{i_n})$ and $(v_{i_1}, ..., v_{i_n})$ respectively. Then, $u \prec v$ if

1. $deg(u) < deg(v)$, or

2. $deg(u) = deg(v)$ and $u_{i_k} - v_{i_k} > 0$ for the smallest i_k such that $u_{i_k} - v_{i_k} \neq 0$.

The sandpile ordering allows us to order two configurations u and v such that either (1) the degree of u is less than v or (2) they have the same degree but u has grains closer to a sink. When we write $u \preceq v$ we mean $u \prec v$ or $u = v$. An important property of sandpile ordering is the following.

Property 1. Let $\prec$ be a sandpile ordering and u be an arbitrary configuration. Then, there are only finitely many configurations u' such that $u' \preceq u$.

Property 1 follows directly from the fact that the degree of a configuration u and the number of vertices on a sandpile graph are finite. We will use this property of sandpile ordering to prove Theorem 2.2. Now, we have the information necessary to prove that given an unstable configuration u there exists a finite sequence of topplings of u such that the result is stable.

Theorem 2.2 (Existence). Let (i_k) be a sequence of vertices in V indexed by $k \in \{1, ..., m\}$. Then, for all unstable configurations u there exists a finite sequence of topplings (T_{i_k}) such that

$$\prod_{k=1}^{m} T_{i_k}(u) = u'$$

where u' is a stable configuration.

Proof. We begin by taking an arbitrary unstable configuration u. To prove Theorem 2.2, we need to first show that if we topple any unstable vertex of u, then the resulting configuration $\tilde{u}$ satisfies $\tilde{u} \prec u$. Meaning, the act of toppling either sends sand out of the system or moves it closer to a sink.

Lemma 2.1. Let $\prec$ be a sandpile ordering, and let u and $\tilde{u}$ be configurations. If $T_j(u) = \tilde{u}$ for some $j \in V$, then $\tilde{u} \prec u$.

Proof. Let $\prec$ be a sandpile ordering with vertex ordering $i_1, ..., i_n$. We assume u is a configuration with an unstable vertex i_j such that $T_{i_j}(u) = \tilde{u}$. To show $\tilde{u} \prec u$, we need to prove two cases depending on the positioning of vertex i_j on the sandpile graph.

(1) If i_j is a boundary vertex (i.e. connected to a sink), then toppling i_j sends sand out of the system, meaning $deg(\tilde{u}) < deg(u)$. So by (1) of Definition 2.1, we have $\tilde{u} \prec u$.

(2) If i_j is not connected to a sink, then toppling it sends sand to its neighbors meaning $deg(\tilde{u}) = deg(u)$. However, the sink is global so every vertex on the graph has some path to the sink. There exists an i_l such that $l < j$ and $e_{jl} = 1$. Meaning, i_j is connected to a vertex closer to a sink. When u is toppled, all vertices remain unchanged, except for vertex i_j and its neighbors, which change according to the toppling rule. Without loss of generality, assume the vertex i_l is the neighbor of i_j closest to a sink. Then, we have $\tilde{u}_{i_l} > u_{i_l}$ and

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$\tilde{u}_{i_m} - u_{i_m} = 0$ for all $m < l$. By (2) of Definition 2.1, we have $\tilde{u} \prec u$.

As a result, we have shown that the toppling of an unstable vertex $T_{i_j}(u) = \tilde{u}$ implies $\tilde{u} \prec u$.

We have shown that the act of toppling results in sand being lost or sand being pushed closer to a sink. Meaning, all there is left to show is that there exists a **finite** sequence of topplings that results in a stable configuration. Now, we use Lemma 2.1 and Property 1 to prove our main problem: given an unstable configuration u there exists a finite sequence of topplings (T_{i_k}) for u such that it results in a stable configuration u'. Let $\prec$ be a sandpile ordering and u be an unstable configuration. Then by Lemma 2.1, there exists a sequence of topplings (T_{i_k}) such that

$$u \succ T_{i_1}(u) \succ \prod_{j=1}^{2} T_{i_j}(u) \succ \cdots \succ \prod_{j=1}^{k} T_{i_j}(u) = u'. \tag{4}$$

and by Property 1 we have that for all $x \in V$

$$T_x \circ \prod_{j=1}^{k} T_{i_j}(u) = T_x(u') = u'.$$

Meaning, we cannot topple any other vertex of u', hence u' is stable. We have thus proven our desired result.

It has been shown that an unstable configuration can be *stabilized* in a finite amount of steps, i.e. toppled through a finite sequence of topplings to a stable configuration. Now, we can define the following transformation.

Definition 2.2. Let $u \in \mathbb{Z}_{\geq 0}^n$ be an unstable configuration and $(i_1, i_2, ..., i_m)$ be the set of vertices that need to be toppled for u to be stable. We define the *toppling transformation T* as the map from all configurations $\mathbb{Z}_{\geq 0}^n$ to the set of stable configurations Ω_n

$$T : \mathbb{Z}_{\geq 0}^n \longrightarrow \Omega_n, \ T(u) = \prod_{j=1}^{m} T_{i_j}(u).$$

Our last step is to show that the mapping T is well-defined. Meaning, if we have have two sequences of topplings that stabilize an unstable configuration, then their respective stabilization is unique and the sequence is unique up to permutation [4].

Theorem 2.3 (Uniqueness of Stabilization). The toppling transformation T is well-defined.

Proof. We assume $u \in \mathbb{Z}_{\geq 0}^n$ is an unstable configuration and $(x_1, ..., x_j)$ and $(y_1, ..., y_k)$ be two sets of vertices such that

$$\prod_{i=1}^{j} T_{x_i}(u) = v, \qquad \prod_{i=1}^{m} T_{y_i}(u) = w$$

where $v, w \in \Omega_n$. We need to show $v = w$ and the sets $(x_1, ..., x_j)$ and $(y_1, ..., y_k)$ are unique up to permutation. We proceed using induction on j.

Base Step: For $j = 1$, the configuration u has one unstable vertex x_1 that when toppled results in a stable configuration. As a result, there exists one sequence of topplings T_{x_1} that results in a unique stable configuration. This proves the base case.

Induction Step: Assume for all configurations that require k topplings there exists a unique sequence of topplings up to permutation that result in a unique stable configuration. Let $j = k+1$ and $x = (x_1, ..., x_j)$ be the sequence of vertices that need to be toppled for u to be stabilized. Assume, the unstable configuration u has another sequence of vertex topplings $y = (y_1, ..., y_m)$. We need to show that $m = k+1$ and y is a permutation of x. The vertex x_1 is unstable and the first to be toppled in x so it must also be toppled in the sequence y; let $k \in \{1, ..., m\}$ be the smallest such that $x_1 = y_k$. Since vertex x_1 is unstable for u and toppling is commutative, we have

$$(\prod_{i=k+1}^{m} T_{y_i}) \circ T_{x_1} \circ (\prod_{i=1}^{k-1} T_{y_i}(u)) = (\prod_{i=1, i \neq k}^{m} T_{y_i}) \circ T_{x_1}(u).$$

Let $T_{x_1}(u) = v$, then v is an unstable configuration that needs k vertices to be toppled. From our assumption, every configuration needing k vertices to be toppled has a unique stabilization and unique sequence of vertex toppling up to permutation. Thus, the indices $j-1$ and $m-1$ must be equal for v, and the sequences $(x_2, ..., x_j)$ and $(y_1, ..., y_{k-1}, y_{k+1}, ..., y_m)$ are unique up to permutation.

Conclusion: By induction, if u is an unstable configuration with a sequence of $k+1 \in \mathbb{N}$ vertex topplings, then u has a unique stabilization and the sequence is unique up to permutation. We have proven that the toppling transformation is well-defined.

From Theorem 2.2 and 2.3, we have shown existence and uniqueness of stabilization. It is quite beautiful that the abstract notion of the toppling transformation can explain such a complex physical process like avalanches. With algebra, we can see why toppling unstable sandpiles self-organize to a critical state. It is because the sandpile's site-to-site, or vertex-to-vertex, collapse commutes, and furthermore its transformation is well-defined. To see the code for toppling unstable configurations, see Appendix A. We now move on to one of the most important components of this paper.

2.3 The Toppling Matrix Δ

In this section, we define a matrix associated to a sandpile graph. The purpose of this matrix will become more apparent later on. For this subsection, we use the matrix to redefine the toppling transformation.

Definition 2.3. Let $S(G)$ be a sandpile graph with n vertices. The *toppling matrix* $\Delta \in \mathbb{Z}^{n \times n}$, associated to $S(G)$, or *Laplacian matrix* of G, is defined as follows

1. For all $i \in V$, $\Delta_{ii} = deg_i$,

2. For all $i, j \in V$ with $i \neq j$, $\Delta_{ij} = -e_{ij}$.

The toppling matrix $\Delta = D - A$, where D is the degree matrix and A is the adjacency matrix of the graph G.

For the code to create the toppling matrix, see Appendix A. It should be noted that the toppling matrix Δ is not the Laplacian matrix of $S(G)$ because it does not take into account the sink. In Figure 5, we see an example of a toppling matrix associated to a sandpile graph. In this paper, we are working with undirected graphs G so we have the following properties of the toppling matrix.

Property 2. For all $i, j \in V$, $\Delta_{ij} = \Delta_{ji}$.

Property 3. For all $i \in V$, $\sum_{j \in V} \Delta_{ij} \geq 0$.

Property 4. $\sum_{i,j \in V} \Delta_{ij} > 0$.

$$\Delta = \begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix}$$
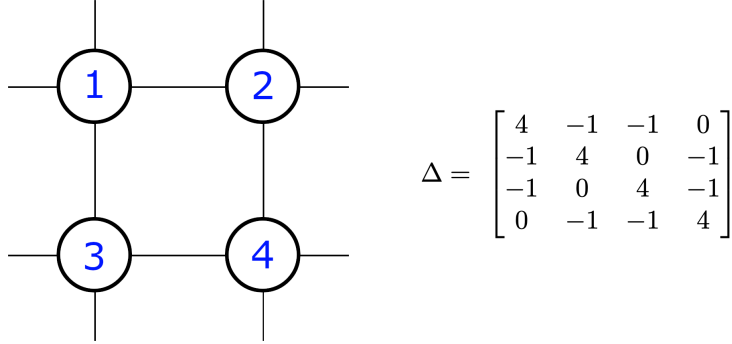
Figure 5: On the left, we have a 2×2 sandpile graph with vertices labeled in blue. On the right, we have its associated toppling matrix Δ.

Now, we can redefine the toppling transformation. Let $u \in \mathbb{Z}_{\geq 0}^n$ be an unstable configuration with unstable vertex $i \in V$. If we topple vertex i, i.e. $T_i(u) = u'$ then we have

$$u'_j = \begin{cases} u_j - deg_i & \text{if } i = j, \\ u_j + e_{ij} & \text{if } j \neq i \end{cases}$$

which is equivalent to

$$u'_j = u_j - \Delta_{ij}, \text{ for all } j \in V.$$

To be more precise, let $x \in \mathbb{Z}^n$ such that $x_i = 1$ and $x_j = 0$ for all $j \in V/\{i\}$, then

$$u' = u - \Delta x.$$

If we take Figure 3 as an example, we have $T((4,2,2,1)) = (0,3,3,1)$. We can use the toppling matrix Δ from Figure 5 to rewrite it in the following vector form

$$\begin{bmatrix} 0 \\ 3 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 2 \\ 1 \end{bmatrix} - \Delta \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{5}$$

From here, we state the following proposition.

Proposition 2.1. The configuration u' is obtained from the configuration u through a series of vertex topplings if and only if $u' = u - \Delta x$ where $x \in \mathbb{Z}^n$ and x_i is the number of topplings of vertex i.

Proposition 2.1 comes directly from the definition of toppling and the construction of the toppling matrix Δ. If we topple an unstable vertex i say k times, then this is equivalent to subtracting the ith column k times from the unstable configuration.

Remark. Proposition 2.1 can be rephrased to $u' \equiv u \bmod \Delta \mathbb{Z}^n$. The reason for this rephrasing will become more apparent in the next section.

In this section, we have defined what a sandpile is and its associated sandpile graph. Furthermore, we have defined a transformation which acts on all unstable sandpiles as a stabilizer. We can now move on to the main focus of this paper. In the next section, we will prove that within the set of all stable configurations Ω_n there exists a group. First, we need an operation that acts on Ω_n. Then, we can show existence of the group, as well as its "non-trivial" identity.

3 The Abelian Sandpile Group

In this subsection, we show existence of the main focus of this paper, the abelian sandpile group. We do this by showing that there exists a subset of the set of stable configurations Ω_n that when coupled with an operation is a group. This comes from a unique structure of Ω_n: the set of stable configurations Ω_n is a *monoid*.

3.1 Existence of the Group

Before we define a *monoid*, we need to define the *toppling operation* $\oplus$ that acts on all configuration $\mathbb{Z}_{\geq 0}^n$. We will see that this operation coupled with Ω_n satisfies the definition of a monoid. Let $u = (u_1, ..., u_n)$ and $v = (v_1, ..., v_n)$ be configurations, we define the operation $+$ to be vertex-wise addition,

$$u + v = (u_1 + v_1, ..., u_n + v_n).$$

The toppling operation $\oplus$ is a mapping from $\mathbb{Z}_{\geq 0}^n$ to Ω_n as follows

$$u \oplus v = T(u + v),$$

Trivially, the toppling operation is commutative because vertex-wise addition is commutative. Now, we define a *monoid*.

Definition 3.1. A *monoid* is a pair (M, b), where M is a set and b is a mapping such that for all $m_1, m_2 \in M$

$$b : M \times M \longrightarrow M, \ (m_1, m_2) \mapsto m_1 \oplus m_2$$

which satisfies the following two properties

1. (associativity) for any $m_1, m_2, m_3 \in M$ the following equality holds

$$(m_1 \oplus m_2) \oplus m_3 = m_1 \oplus (m_2 \oplus m_3),$$

2. (identity) there exists and element $e \in M$ such that for any $m \in M$ the following equality holds

$$e \oplus m = m \oplus e = m.$$

Essentially, a monoid is a group without inverses. We want to show that the set of stable configurations Ω_n, paired with the toppling operation, is a monoid. First, the identity of Ω_n is trivially the zero configuration 0. Meaning, all there is left to prove is that the toppling operation is associative.

Theorem 3.1. The toppling operation $\oplus$ is associative.

Proof. Let $u, v, w \in \mathbb{Z}_{\geq 0}$, we want to show

$$(u \oplus v) \oplus w = u \oplus (v \oplus w).$$

There exists integer vectors $x_1, x_2, x_3, x_4 \in \mathbb{Z}^n$ such that the following holds

$$u \oplus v = u + v - \Delta x_1, \quad v \oplus w = v + w - \Delta x_2,$$
$$(u \oplus v) \oplus w = (u \oplus v) + w - \Delta x_3, \quad u \oplus (v \oplus w) = u + (v \oplus w) - \Delta x_4.$$

Now, we observe the following

$$u + v + w = (u \oplus v) + w + \Delta x_1$$
$$= (u \oplus v) \oplus w + \Delta(x_1 + x_3),$$

and

$$u + v + w = u \oplus (v \oplus w) + \Delta(x_2 + x_4).$$

Let $x \in \mathbb{Z}^n$ such that $T(u + v + w) = u + v + w - \Delta x$, then by uniqueness of stabilization $x = x_1 + x_3$ and $x = x_2 + x_4$ which implies

$$(u \oplus v) \oplus w = u \oplus (v \oplus w).$$

Hence, the toppling operation is associative.

As a result, the set of stable configurations Ω_n coupled with the toppling operation $\oplus$ is a monoid. In fact, Ω_n is a finite commutative monoid. This particular fact is the reason why there exists a subset of Ω_n that is a group. Without it, there would be no abelian sandpile group. We can now prove the existence of this unique subset [7].

Theorem 3.2. The minimal ideal of a finite commutative monoid is a group.

Before we begin the proof, we need to define the *minimal ideal* of a monoid. Let M be a monoid and $I \subset M$ with $I \neq \emptyset$, the subset I is an ideal of M if for all $m \in M$ and $i \in I$ the sum $m \oplus i \in I$, which is equivalent to

$$m \oplus I := \{m \oplus i \mid i \in I\}, \quad m \oplus I \subset I.$$

If $\oplus$ is commutative, like in our case, then $m \oplus I = I \oplus m$. The *minimal ideal I* of a monoid M is the intersection of all the ideals of M, i.e. the smallest ideal of M and for all $m \in M$, $m \oplus I = I$. Now, we can prove Theorem 3.2.

Proof. Let M be a finite commutative monoid with the operation $\oplus$ and identity 0. Let I be its minimal ideal. To begin, if the minimal ideal $I = M$, then M is a group because for all $x \in M$, we have $x \oplus M = M$ meaning there exists an element $y \in M$ such that $x \oplus y = 0$. Now, assume $I \neq M$.

We claim I is nonempty. Every finite set has a finite number of subsets. So, M must have a finite number of ideals I_m, and $I = \bigcap_{i=1}^m I_i$. If we take an element x_j from every ideal I_j then by definition

$$x_1 \oplus x_2 \oplus \cdots \oplus x_m = \bigoplus_{i=1}^m x_i \in \bigcap_{i=1}^m I_i = I.$$

Hence, the minimal ideal is nonempty. Now, we show that there exists an idempotent element in I. We need to show there exists an element $a \in I$ such that $\bigoplus_{i=1}^k a = a$ for some positive integer k. Then, we have that $\bigoplus_{i=1}^{k-1} a$ is idempotent;

$$(\bigoplus_{i=1}^{k-1} a) \oplus (\bigoplus_{i=1}^{k-1} a) = a \oplus (\bigoplus_{i=1}^{k-2} a) = \bigoplus_{i=1}^{k-1} a.$$

Let $m \in I$ be arbitrary, I is finite and closed under $\oplus$, so the set

$$\{m, \bigoplus_{i=1}^2 m, \bigoplus_{i=1}^4 m, \bigoplus_{i=1}^8 m, ...\}$$

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is also finite. There exists positive integers s and t such that $s > t \geq 1$ and $\bigoplus_{i=1}^{2^s} m = \bigoplus_{i=1}^{2^t} m$. We observe the following

$$\bigoplus_{i=1}^{2^t} m = \bigoplus_{j=1}^{2^{s-t}} \left(\bigoplus_{i=1}^{2^t} m \right) = \bigoplus_{i=1}^{2^s} m.$$

Hence, take $a = \bigoplus_{i=1}^{2^t} m$ and $k = 2^{s-t}$ giving us the idempotent element $\bigoplus_{i=1}^{k-1} a \in I$.

We denote the idempotent element of I as e, and show that e is the identity of I. For all $x \in I$ and $m \in M$, the element $x \oplus m \in I$. So, the ideal $x \oplus M \subset I$ thus $x \oplus M = I$. Let $x \in I$ be arbitrary, from the previous statement, $x \in e \oplus M$, so $x = e \oplus y$ for some $y \in M$. We observe that

$$x \oplus e = e \oplus x = e \oplus e \oplus y = e \oplus y = x$$

hence for all $x \in I$ the idempotent element e of I acts as an identity on x. Lastly, we show the existence of inverses in I.

Let $x \in I$, the idempotent element $e \in x \oplus M$ so $e = x \oplus z$ for some $z \in M$. Then, we have

$$x \oplus (e \oplus z) = e \oplus (x \oplus z) = e \oplus e = e$$

and $e \oplus z \in I$, meaning $e \oplus z$ must be the inverse of $x \in I$. As a result, we have shown that the minimal ideal of a commutative finite monoid is a group.

As a result, the monoid Ω_n, consisting of stable configurations, contains a group under the toppling operation. This group is called the *abelian sandpile group*. It should be noted that the proof underlines the non-triviality of our identity in the minimal ideal. We need to find an idempotent configuration that is not the zero configuration. To understand the elements in the abelian sandpile group, we define two types of configurations in Ω_n [8].

Definition 3.2. Let $S(G)$ be a sandpile graph and Ω_n be the set of stable configuration on $S(G)$. Then, a stable configuration $u \in \Omega_n$ is *recurrent* if there exists a configuration $v \neq 0$ such that $u \oplus v = u$. We denote the set of recurrent configurations as $\mathscr{R}_n$. If a stable configuration u is not recurrent, then we call it *transient*.

The recurrent configurations are the sandpiles that can be returned to once enough sand has been dropped on them and stabilized. The set of recurrent configurations is the abelian sandpile group. In order to show this, we need to prove that $\mathscr{R}_n$ is a group, i.e, $\mathscr{R}_n$ is the minimal ideal of Ω_n.

Theorem 3.3. The set of recurrent configurations $\mathscr{R}_n$ is the minimal ideal of Ω_n.

Proof. To prove that $\mathscr{R}_n$ is the minimal ideal of Ω_n, we need to show that for all configurations $u \in \Omega_n$ it holds that $u \oplus \mathscr{R}_n = \mathscr{R}_n$. Let u be a recurrent configuration, then there exists a configuration $v \neq 0$ such that $u \oplus v = u$. Take an arbitrary configuration $w \in \Omega_n$, and observe

$$(u \oplus w) \oplus v = (u \oplus v) \oplus w = u \oplus w$$

hence $u \oplus w \in \mathscr{R}_n$. The set of recurrent configurations $\mathscr{R}_n$ is the minimal ideal of Ω_n.

As a result, the set of recurrent configurations $\mathscr{R}_n$ on a sandpile graph $S(G)$ is a group. Recurrent configurations are not always easy to spot. However, the *maximal configuration* $u_{max} = (deg_1 - 1, ..., deg_n - 1)$

is always recurrent. Let $u = (u_1,...,u_n)$ be recurrent, then if we add any configuration to u the sum is also recurrent. Take $u' = (deg_1 - 1 - u_1,...,deg_n - 1 - u_n)$, then $u \oplus u' \in \mathscr{R}_n$ and $u + u' = u_{max}$. In Figure 7, we see examples of transient and recurrent configurations.
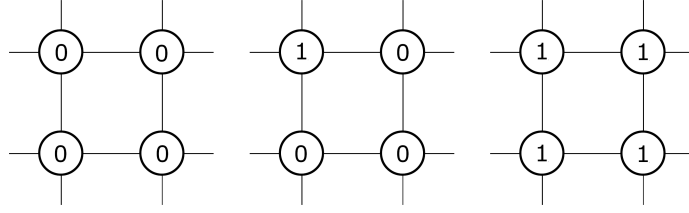

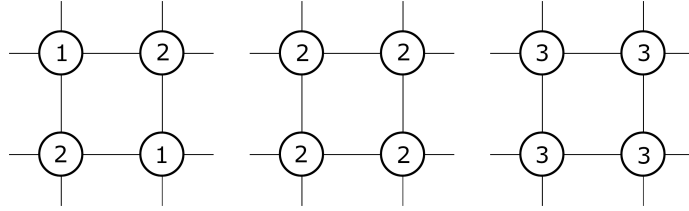Figure 6: examples of transient configurations on a 2×2 sandpile graph.

Figure 7: examples of recurrent configurations on a 2×2 sandpile graph.

The last configuration in Figure 7 is the maximal configuration on a 2×2 sandpile graph. The maximal configuration is important because it is the easiest recurrent configuration to find. Since the group of recurrent configurations is closed under the toppling operation, we use the maximal configuration to find other recurrent configurations - as we will see in the next theorem [6].

Theorem 3.4. Let $u \in \mathbb{Z}_{\geq 0}^n$, the configuration u is recurrent if and only if for all $v \in \Omega_n$ there exists a configuration $w \neq 0$ such that $v \oplus w = u$.

Proof. The proof requires us to prove both directions. So, we begin by proving left-to-right.

($\implies$): Assume u is a recurrent configuration, then there exists a configuration $u' \neq 0$ such that $u \oplus u' = u$. Let v be a stable configuration, if $v \leq u + u'$ then take $w = u + u' - v$ to get $v \oplus w = u \oplus u' = u$. Otherwise, let b be a configuration such that $v + b = u_{max}$. Since u_{max} is recurrent, there exists a configuration c such that $u_{max} \oplus c = u$. We take the configuration $w = b + c$, which gives us $v \oplus w = T(v + b + c) = T(u_{max} + c) = u$. Hence, for all $v \in \Omega_n$ there exists a configuration w such that $v \oplus w = u$.

($\impliedby$): Now, we take an arbitrary $v \in \Omega_n$ there exists a configuration $w \neq 0$ such that $v \oplus w = u$ is recurrent. If we take $w = u_{max}$, then u is recurrent since $u_{max} \in \mathscr{R}_n$ and $\mathscr{R}_n$ is the minimal ideal of Ω_n. This proves the desired result.

We have shown that a recurrent sandpile can be reached from any other configuration. Now, we proceed by finding the order of $\mathscr{R}_n$. Obviously, the order of Ω_n is the product of the degrees of its vertices, however this is rarely the case for $\mathscr{R}_n$. In the next subsection, we show the order of $\mathscr{R}_n$ is equal to the absolute value of the determinant of the toppling matrix. We do this by proving it is isomorphic to $\mathbb{Z}^n / \Delta \mathbb{Z}^n$.

3.2 The Order of $\mathscr{R}_n$

In this subsection, we show that the set of recurrent configurations $\mathscr{R}_n$ is isomorphic to $\mathbb{Z}^n / \Delta\mathbb{Z}^n$. We use this isomorphism to easily find the order of $\mathscr{R}_n$, but more importantly, the isomorphism between $\mathscr{R}_n$ and $\mathbb{Z}^n / \Delta\mathbb{Z}^n$ tells us a lot about the identity of $\mathscr{R}_n$. It is the only recurrent configuration that is a linear combination of the rows of the toppling matrix Δ. We prove this via the explicit mapping ϕ,

$$\phi : \mathscr{R}_n \longrightarrow \mathbb{Z}^n / \Delta\mathbb{Z}^n, \quad u \mapsto u \bmod \Delta\mathbb{Z}^n.$$

Since addition is preserved by the mapping ϕ, it is a homomorphism. We show that ϕ is bijective by proving the following statements:

1. (Surjectivity) each element of $\mathbb{Z}^n / \Delta\mathbb{Z}^n$ is represented by a recurrent configuration,

2. (Injectivity) there exists a unique recurrent configuration in each equivalence class of $\Delta\mathbb{Z}^n$.

We define the following configurations to help us in this endeavour. Let $\mathbf{1}$ denote the configuration with one on each vertex, $\mathbf{1} = (1, 1, ..., 1)$, then we have the *degree* and *null* configurations defined respectively as such

1. $u_{deg} := u_{max} + \mathbf{1} = (deg_1, deg_2, ..., deg_n)$,

2. $u_{null} := u_{deg} - T(u_{deg})$.

The degree configuration u_{deg} is recurrent once stabilized because it can be reached from u_{max}. On the other hand, the null configuration is a linear combination of the rows of the toppling matrix. Let $x \in \mathbb{Z}^n$ such that $T(u_{deg}) = u_{deg} - \Delta x$, then

$$u_{null} = u_{deg} - T(u_{deg}) = \Delta x \implies u_{null} = 0 \bmod \Delta\mathbb{Z}^n.$$

Furthermore, the vertex i of a stable configuration cannot exceed $deg_i - 1$ meaning $\mathbf{1} \leq u_{deg} - T(u_{deg}) = u_{null}$. We can now prove that the the homomorphism ϕ is indeed an isomorphism. We begin by proving the mapping is surjective [6].

Theorem 3.5. Each element of $\mathbb{Z}^n / \Delta\mathbb{Z}^n$ is represented by a recurrent configuration.

Proof. Let $u \in \mathbb{Z}^n / \Delta\mathbb{Z}^n$ such that if $u = u' + \Delta x$ then $x = 0$ and $u = u'$. Now, we want to find a recurrent configuration v such that $v \equiv u \bmod \Delta\mathbb{Z}^n$. It should be noted that if $k \in \mathbb{Z}$ and $u = (u_1, ..., u_n)$ is a configuration then we define $ku = (ku_1, ..., ku_n)$. With this in mind, we take an integer $k > 0$ such that $u_{max} \leq u + ku_{null}$, which is possible because $\mathbf{1} \leq u_{null}$. We claim the configuration $u \oplus ku_{null}$ is recurrent because of our choice of k. We observe that

$$T(u + ku_{null}) = T(u_{max} + (u + ku_{null} - u_{max})).$$

Since $u + ku_{null}$ can be reached by adding a configuration to u_{max} and toppling, this means $u \oplus ku_{null}$ is recurrent.

Lastly, the recurrent configuration $u \oplus ku_{null}$ is mapped to $u \bmod \Delta\mathbb{Z}^n$. Let $v \in \mathbb{Z}^n$ be the vector such that $u \oplus ku_{null} = u + ku_{null} - \Delta v$, and recall that $u_{null} = \Delta x$, then

$$\begin{aligned}
T(u + ku_{null}) &= u + ku_{null} - \Delta v \\
&= u + k(\Delta x) - \Delta v \\
&= u + \Delta(kx + v) \\
&\equiv u \bmod \Delta\mathbb{Z}^n.
\end{aligned}$$

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Hence, each element of $\mathbb{Z}^n/\Delta\mathbb{Z}^n$ is represented by a recurrent configuration. This proves our homomorphism is surjective.

We have seen that every element of $\mathbb{Z}^n/\Delta\mathbb{Z}^n$ can be represented by a recurrent configuration. Now, we need to prove that this recurrent element is unique. We conclude the proof that $\mathscr{R}_n$ is isomorphic to $\mathbb{Z}^n/\Delta\mathbb{Z}^n$ by proving ϕ is injective [6].

Theorem 3.6. There exists a unique recurrent configuration in each equivalence class of $\Delta\mathbb{Z}^n$

Proof. By Theorem 3.2, in each equivalence class of $\Delta\mathbb{Z}^n$ there exists a $u \in \mathscr{R}_n$. Suppose there exists two recurrent configurations $u', u'' \in \mathscr{R}_n$ such that $u' = u'' \bmod \Delta\mathbb{Z}^n$. We want to show $u' = u''$.

By our assumption, we have $u' = u'' + \Delta v$ for some $v \in \mathbb{Z}^n$. Now, set $v = v^+ + v^-$ where v^+ are all the positive elements of v and v^- are all the negative elements of v. We define u to be the following configuration

$$u = u' - \Delta v^- = u'' + \Delta v^+.$$

Now, we want to use the configuration u to construct a configuration that we can show stabilizes to u' and u''. Then, we can conclude by uniqueness of stabilization $u' = u''$. We need the following lemma in order to do this.

Lemma 3.1. If u is a recurrent configuration then $u \oplus u_{null} = u$.

Proof. Let u be a recurrent configuration, then there exists a configuration $v \neq 0$ such that $u_{max} \oplus v = u$. Set $a = v - \mathbf{1}$, then we get

$$u = T(u_{max} + v) = T(u_{max} + \mathbf{1} + v - \mathbf{1}) = T(a + u_{deg}).$$

Now, we use uniqueness of stabilization to show $u \oplus u_{null} = u$. Note the following

$$\begin{aligned} T(a + u_{deg} + u_{null}) &= T(T(a + u_{deg}) + u_{null}) \\ &= T(u + u_{null}), \end{aligned}$$

and furthermore

$$\begin{aligned} T(a + u_{deg} + u_{null}) &= T(a + u_{deg} + u_{deg} - T(u_{deg})) \\ &= T(a + u_{deg} + u_{deg} - u_{deg}) \\ &= T(a + u_{deg}) \\ &= u. \end{aligned}$$

Hence, if u is recurrent then $u \oplus u_{null} = u$.

The configurations u' and u'' are recurrent so u_{null} acts as an identity on them. As a result, we construct a configuration using u and u_{null} to ensure it topples to u' and u''. We do this by choosing an integer $k > 0$ such that for all vertices $i \in V$

$$(u + k u_{null})_i \geq \max_{j \in V}\{|v_j| deg_j\}. \tag{6}$$

The right-hand side of Equation 6 is the maximum of the absolute value of each v_i times its respective degree. The reason for choosing such a k is because each vertex i of $u + k u_{null}$ can be toppled $|v_i|$ times, i.e.

$T(u + ku_{null}) = T(u + ku_{null} - \Delta v)$. Now, we show that $u + ku_{null}$ stabilizes to u' by toppling each vertex i of the unstable configuration $|v_i|$ times if $v_i < 0$,

$$
\begin{aligned}
T(u + ku_{null}) &= T(u + ku_{null} + \Delta v^-) \\
&= T(u' - \Delta v^- + ku_{null} + \Delta v^-) \\
&= T(u' + ku_{null}).
\end{aligned} \tag{7}
$$

The configuration u' is recurrent, so by Lemma 3.1 we have

$$u' \oplus ku_{null} = (u' \oplus u_{null}) \oplus (k-1)u_{null} = u' \oplus (k-1)u_{null} = \cdots = u' \oplus u_{null} = u'.$$

Using Equation 7 and 3.2, we get $T(u + ku_{null}) = u'$. Now, if we take Equation 7 and instead topple each vertex $|v_i|$ times if $v_i > 0$ we get

$$T(u + ku_{null}) = u''.$$

By uniqueness of stabilization, we have $u' = u''$ thus each equivalence class of $\Delta \mathbb{Z}^n$ is represented by a unique recurrent configuration.

As a result, we have shown that the mapping ϕ is an isomorphism; $\mathscr{R}_n \cong \mathbb{Z}^n / \Delta \mathbb{Z}^n$. A formula for the order of $\mathbb{Z}^n / \Delta \mathbb{Z}^n$ is proven in [9]. It states in our case that

$$\text{ord}(\mathscr{R}_n) = \text{ord}(\mathbb{Z}^n / \Delta \mathbb{Z}^n) = |\det(\Delta)|.$$

Remark. It should be noted that, by the properties of the toppling matrix, it is positive definite so the determinant of the toppling matrix exists.

An interesting fact is that for sandpile graphs on an $(n-1) \times (n-1)$ grid, there is a closed formula for the determinant of the toppling matrix;

$$\det(\Delta) = \frac{2^{n^2-1}}{n^2} \prod_{n_1=0}^{n} \prod_{n_2=0}^{n} (2 - \cos(\frac{\pi n_1}{n}) - \cos(\frac{\pi n_2}{n})) \text{ and } (n_1, n_2) \neq (0, 0).$$

This remarkable formula can be found on the *Online Encyclopedia of Integer Sequences* with sequence code *A*007341. The proof for this closed formula or that the right hand side is an integer is outside the scope of this paper. However, the proof comes from the fact that the $n \times n$ grid graph is the dual graph of the sandpile graph on an $(n-1) \times (n-1)$ grid. The order of the first five $n \times n$ grids are:

1. $\text{ord}(R_{1 \times 1}) = 4$,

2. $\text{ord}(R_{2 \times 2}) = 192$,

3. $\text{ord}(R_{3 \times 3}) = 100,352$,

4. $\text{ord}(R_{4 \times 4}) = 557,568,000$.

As a result, a brute force search method for finding the identity of $R_{n \times n}$ becomes impossible on large grids. We turn back to algebra to help us in this endeavour.

The isomorphism ϕ has given us a method for finding the order of the group of recurrent configurations. Furthermore, isomorphisms preserve identities. The identity of $\mathscr{R}_n$ is the only recurrent configuration of the form Δx for some $x \in \mathbb{Z}^n$. In the next subsection, we use this fact to give two explicit methods for finding the identity.

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4 The Identity of $\mathscr{R}_n$

In this section, we produce two different methods for calculating the identity of $\mathscr{R}_n$. Both methods come from the fact that the identity is preserved by the isomorphism between $\mathscr{R}_n$ and $\mathbb{Z}^n/\Delta\mathbb{Z}^n$. The first method constructs the identity using two recurrent configurations [6], and the second method is an iterative solution using transient configurations [10].

Theorem 4.1 (Method 1). The identity of $\mathscr{R}_n$ is

$$T(2u_{max} - T(2u_{max})).$$

Proof. We need to show that the configuration $T(2u_{max} - T(2u_{max}))$ is a linear combination of the rows of the toppling matrix Δ, and furthermore it is recurrent. We begin by proving the former statement. Let $x \in \mathbb{Z}^n$ such that $T(2u_{max}) = 2u_{max} - \Delta x$, then

$$
\begin{aligned}
T(2u_{max} - T(2u_{max})) &= T(2u_{max} - 2u_{max} + \Delta x) \\
&= T(\Delta x) \\
&= \Delta(x - x')
\end{aligned}
$$

for some $x' \in \mathbb{Z}^n$. As a result, we have

$$T(2u_{max} - T(2u_{max})) = 0 \bmod \Delta\mathbb{Z}^n.$$

We now prove that $T(2u_{max} - T(2u_{max}))$ is recurrent. We observe that the configuration $T(2u_{max} - T(2u_{max}))$ can be rearranged

$$T(2u_{max} - T(2u_{max})) = T(u_{max} + (u_{max} - T(2u_{max}))).$$

For all stable configurations $u \in \Omega_n$, trivially $u \le u_{max}$ meaning $u_{max} - T(2u_{max})$ has no vertices with negative integers. Any configuration added to the maximal configuration and toppled is recurrent, hence $T(2u_{max} - T(2u_{max}))$ is recurrent. As a result, the configuration $T(2u_{max} - T(2u_{max}))$ must be identity of $\mathscr{R}_n$. ∎

It is quite apparent that computing the identity on a very large grid is not possible by hand. So, we will do so on a small grid outlined in the following example.

Example 4.1. In this example, we will compute the identity of $\mathscr{R}_{2\times2}$ on the sandpile graph of a 2×2 grid. We need to calculate $T(u_{max} - T(u_{max}))$, so we begin by adding the maximal recurrent configuration of $\Omega_{2\times2}$ to itself, $2u_{max}$ can be seen in the following figure.
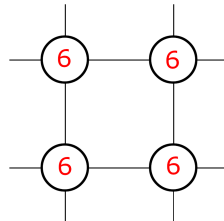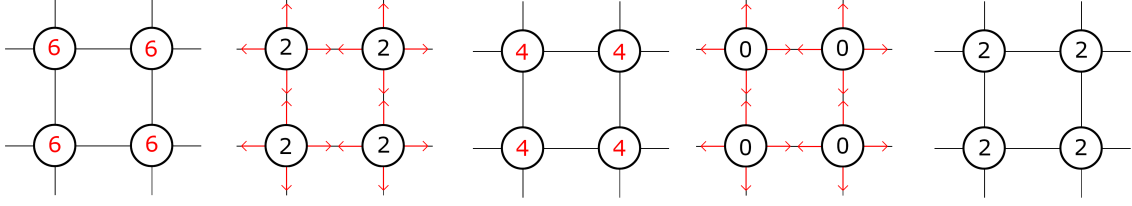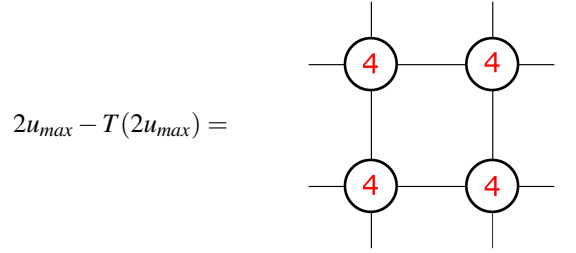

Figure 8: The maximal recurrent configuration of $\Omega_{2\times2}$ on a 2×2 grid added to itself

Now, we need calculate the stable configuration $T(2u_{max})$. We will topple all four vertices at the same time to minimize the calculation.

Hence, $2u_{max} - T(2u_{max})$ is equal to the configuration of all fours

$$2u_{max} - T(2u_{max}) =$$

.

To conclude our calculation of the identity, we need to stabilize the configuration of all fours. If we look at the calculation of $T(2u_{max})$ we can see that that the identity of $\mathscr{R}_{2\times 2}$ is the configuration of all twos,

$$T(2u_{max} - T(2u_{max})) =$$

.
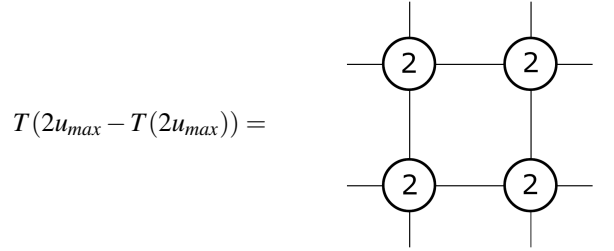
We have given one method for calculating the identity of $\mathscr{R}_n$ (for code see Appendix B). Now, we give an iterative approach.

Theorem 4.2 (Method 2). Let $I_0 = \Delta \mathbf{1}$ and $I_k = I_{k-1} \oplus I_{k-1}$ for all $k \in \mathbb{N}$. Then, there exists $n \in \mathbb{N}$ such that

$$I_n = I_{n+1} = I_{n+2} = \cdots$$

and I_n is the identity of $\mathscr{R}_n$

Proof. Set $I_0 = \Delta \mathbf{1}$, then for $n \in \mathbb{N}$ we have iteration $I_n = I_{n-1} \oplus I_{n-1}$. Note, the configuration $I_0 \equiv 0 \mod \Delta \mathbb{Z}^n$ meaning for all $n \in \mathbb{N}$,

$$I_n = \bigoplus_{j=1}^{2^n} I_0 \equiv 0 \mod \Delta_n \mathbb{Z}^n$$

Furthermore, I_0 is a stable configuration. Recall from the properties of the toppling matrix, that for all $i \in V$ the sum of the rows of Δ are non-negative, $(I_0)_i = \sum_{j \in V} \Delta_{ij} \geq 0$.

Now, the set of iterations $\{I_0, I_1, I_2, ...\}$ is finite since Ω_n is finite. So, there exists positive integers $k > m$ such that $I_m = I_k$. The configuration I_m must be recurrent because we can find a nontrivial configuration I'

such that $I_m \oplus I' = I_k = I_m$. Also, I_m must be the identity of $\mathscr{R}_n$ because it is recurrent and $I_m = 0 \mod \Delta \mathbb{Z}^n$. Since I_m is the identity, trivially $I_{m+1} = I_m \oplus I_m = I_m$ thus for all $k > m$, we have $I_k = I_m$. This concludes our proof.

Example 4.2. In this example, we will use method 2 on the same sandpile graph as our previous example. We begin by calculating $\Delta \mathbf{1}$,

$$
\begin{bmatrix}
4 & -1 & -1 & 0 \\
-1 & 4 & 0 & -1 \\
-1 & 0 & 4 & -1 \\
0 & -1 & -1 & 4
\end{bmatrix}
\begin{bmatrix}
1 \\
1 \\
1 \\
1
\end{bmatrix}
=
\begin{bmatrix}
2 \\
2 \\
2 \\
2
\end{bmatrix}.
$$

As a result, we have our identity. In Figure 9, we see the identity element of the group of recurrent configurations on some small grids.
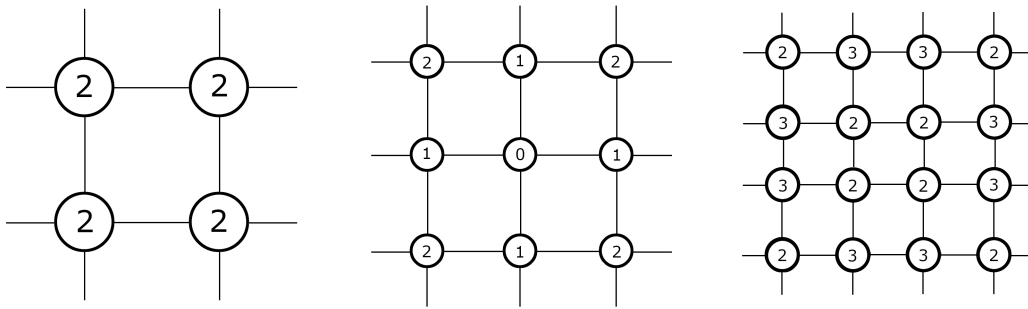

Figure 9: The identity of the group of recurrent configurations on a 2×2, 3×3 and 4×4 grid sandpile graph.

Our iterative example for the 2×2 grid seems like less calculations are needed, however its efficiency does not cut down calculations that much on larger grids (for code see Appendix B). The number of iterations k depending on the length of the grid n grows quadratically [11], see Figure 10.
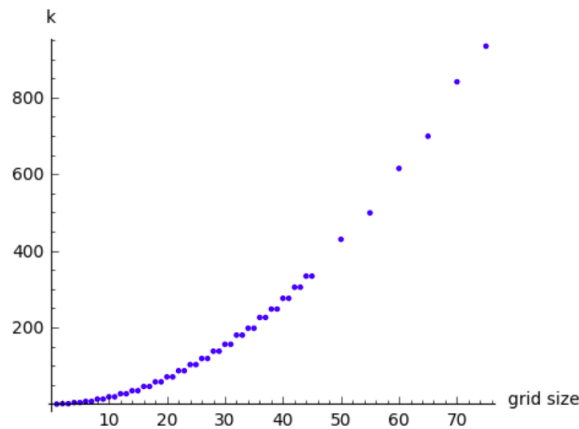

Figure 10: Grid size refers to side length of grid. [11]

In addition, Takumi Shibazaki and Takao Namiki prove in their paper, *Efficient computation method for identity element of Abelian Sandpile Model*, that the iterative method on an $n \times n$ grid runs in $O(n^4 \log n)$

time. The bulk of the run time is spent on toppling which takes $O(n^4)$ time each iteration [12]. As a result, if a faster algorithm is found, then it would have to limit the amount of topplings.

On larger grids, we use colors to represent the integers $\{0, 1, 2, 3\}$, as numbers become too difficult to see. This can be seen in Figure 11. Furthermore, these methods are not restricted to rectangular grids. Method 1 and 2 can be used on any sandpile graph. A triangular grid example can be seen in Figure 11.
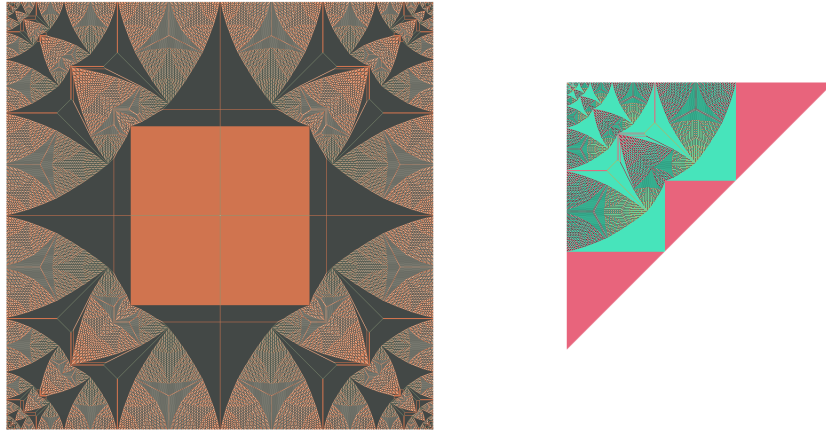
Figure 11: On the left, the identity on a 801×801 grid (White $= 0$, Green $= 1$, Orange $= 2$, Dark Gray $= 3$). On the right, the identity on a 500×500 triangle grid (Black $= 0$, Orange $= 1$, Pink $= = 2$, Light Blue $= 3$).

Calculating the identity of the recurrent configurations is a time consuming task. There is no closed formula solution for grids, but there are open problems which will be discussed in the conclusion. In the next subsection, we discuss the symmetric sandpile subgroup. The motivation comes from the structure of the identity on square grids. Its fractal-like structure presents symmetries under rotation and reflection.

5 The Symmetric Sandpile Subgroup

In the previous section, we presented the identity element of the set of recurrent configurations $\mathscr{R}_n$ on various $n \times n$ grids. These identity configurations have a fractal-like structure and possess the characteristic that under rotation and reflection their structure is preserved. Furthermore, there is another configuration we have seen that displays this behavior, the maximal recurrent configuration u_{max}. In this section, we define a group action on the sandpile graph, which partitions its set of vertices into equivalence classes. Then, we show that this group action induces a group action on the set of recurrent configurations; producing a subgroup of symmetric recurrent configurations. The findings in this section mainly come from Natalie Durgin's paper, *Abelian Sandpile Model on Symmetric Graphs*. [13]

5.1 The Group Action

Let $S(G)$ be a sandpile graph with a square grid structure, so the vertex set $V = V' \times V'$ where $V' = \{1, ..., n\}$. We define the group $\mathbb{S}_V$ to be the set of all bijections from V to itself with the composition of maps as its operation. Since the graph is a square grid, there are 8 elements in $\mathbb{S} \subset \mathbb{S}_V$ preserving the graph structure; let $(j, k) \in V$

$$
\begin{aligned}
\textbf{the identity map:} \quad & id : V \to V, \quad (j,k) \mapsto (j,k), \\
\textbf{the rotation maps:} \quad & r_1 : V \to V, \quad (j,k) \mapsto (k, n-(j-1)), \\
& r_2 : V \to V, \quad (j,k) \mapsto (n-(j-1), n-(k-1)), \\
& r_3 : V \to V, \quad (j,k) \mapsto (n-(k-1), j), \\
\textbf{the reflection maps:} \quad & m_1 : V \to V, \quad (j,k) \mapsto (j, n-(k-1)), \\
& m_2 : V \to V, \quad (j,k) \mapsto (n-(j-1), k), \\
& m_3 : V \to V, \quad (j,k) \mapsto (n-(k-1), n-(j-1)), \\
& m_4 : V \to V, \quad (j,k) \mapsto (k, j).
\end{aligned}
$$

The subgroup $\mathbb{S} \cong D_4$ and induces a group action on the set of vertices V in the following way

$$
\begin{aligned}
\mathbb{S} \times V &\longrightarrow V, \\
(s, v) &\mapsto s(v)
\end{aligned}
$$

This group action partitions the set of vertices into equivalence classes called *orbits*. The orbit of a vertex $v \in V$ is defined to be $\mathbb{S}v = \{s(v) \mid s \in S\}$; we denote the set of all orbits of V as $V^{\mathbb{S}} = \{\mathbb{S}v \mid v \in V\}$. Now, the group action $\mathbb{S}$ can be induced on configurations on an $n \times n$ grid sandpile graph.

The group action is induced on the indices of a configuration, as a result we must proceed with caution. Configurations are non-negative vectors indexed by the vertices of the graph. In our case, we have a sandpile graph $S(G)$ on an $n \times n$ grid, if u is a configuration on the graph then it is a non-negative vector with n^2 elements, $u \in \mathbb{Z}_{\geq 0}^{n^2}$. To ensure that our indexing stays consistent, the weight of u on vertex $(i, j) \in V$ is $u_{i+n(j-1)}$, but for consistency, we write $u_{(i,j)}$. Now, we can define the group action on the set of configurations.

Let $u \in \mathbb{Z}_{\geq 0}^{n^2}$ be a configuration on an $n \times n$ grid sandpile graph and let $s \in \mathbb{S}$, the group action of $\mathbb{S}$ on the set of configurations $\mathbb{Z}_{\geq 0}^{n^2}$ is the map

$$
\begin{aligned}
\mathbb{S} \times \mathbb{Z}_{\geq 0}^{n^2} &\longrightarrow \mathbb{Z}_{\geq 0}^{n^2}, \\
(s, u_{(i,j)}) &\mapsto u_{s((i,j))}
\end{aligned}
$$

for all $(i, j) \in V$. Hence, the group $\mathbb{S}$ acts on the indices of the configuration u; we denote an element $s \in \mathbb{S}$ acting on a configuration u as $s(u)$. With this induced group action, we can define a symmetric configuration.

5.2 The Subgroup

The group $\mathbb{S}$ induces a group action on the set of configurations. In this subsection, we use this to show that there exists a symmetric subgroup of the set of recurrent configurations. First, we define a symmetric configuration.

Definition 5.1. Let $u \in \mathbb{Z}_{\geq 0}^{n^2}$ be a configuration on an $n \times n$ grid sandpile graph, we call u a *symmetric* configuration if $s(u) = u$ for all $s \in \mathbb{S}$.

It is quite easy to see that if a configuration has the same weight on every vertex, then it is a symmetric configuration. For example, the zero configuration is symmetric. However, the main purpose of this section is on symmetric recurrent configurations. Using this reasoning, and as stated earlier, there exists at least one symmetric recurrent configuration.

Proposition 5.1. The maximal stable configuration $u_{max} \in \mathscr{R}_n$ is symmetric.

Before we discuss the symmetric sandpile subgroup, we have an example finding the symmetric configurations on the sandpile graph of a 3×3 grid. In this example, we show how the symmetric equivalence classes $V^{\mathbb{S}}$ define symmetric configurations.

Example 5.1. In this example, we will find the set of all orbits $V^{\mathbb{S}}$ for the sandpile graph on a 3×3 grid. We label the set of vertices V as such $\{v_1, v_2, ..., v_9\}$ (see Figure 12).
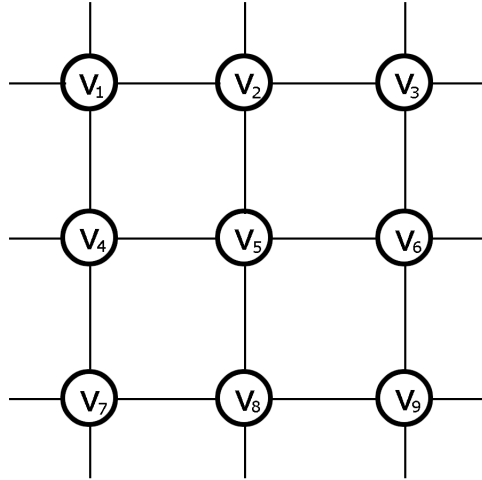
Figure 12: 3×3 sandpile graph with labelled vertices.

Under the group action $\mathbb{S}$, we find three orbits

$$V^{\mathbb{S}} = \{\mathbb{S}v_1, \mathbb{S}v_2, \mathbb{S}v_5\} = \{\{v_1, v_3, v_7, v_9\}, \{v_2, v_4, v_6, v_8\}, \{v_5\}\}.$$

Under reflection and rotation, corner vertices are mapped to corner vertices, edge vertices are mapped to edge vertices and the center vertex v_5 is mapped to itself. Meaning, all symmetric configurations u must have equal

weight on the vertices in the orbit, i.e. $a, b, c \in \mathbb{Z}/4\mathbb{Z}$

$$u = (a, b, a, a, c, a, a, b, a).$$

Now, we understand the composition of symmetric configurations; they are the configurations with equal weight on all vertices in their respective symmetric equivalence class. Keeping this in mind, we proceed with proving the existence of the symmetric sandpile subgroup.

Proposition 5.2. The elements of $\mathbb{S}$ are linear maps on $\mathbb{Z}_{\geq 0}^{n^2}$.

If we have two configurations $u, v \in \mathbb{Z}_{\geq 0}^{n^2}$ and the map $s \in \mathbb{S}$, then reflecting (or rotating) the sum of the two configurations is equivalent to reflecting (or rotating) then summing the two configurations, i.e. $s(u + v) = s(u) + s(v)$. To be more precise, let $(i, j) \in V$

$$(u + v)_{(i,j)} = u_{(i,j)} + v_{(i,j)} \implies (u + v)_{s((i,j))} = u_{s((i,j))} + v_{s((i,j))},$$

hence $s(u + v) = s(u) + s(v)$. An extension of this is that the action by elements of $\mathbb{S}$ commute with stabilization.

Theorem 5.1. Let $u \in \mathbb{Z}_{\geq 0}^{n^2}$ and $s \in \mathbb{S}$, then $s(T(u)) = T(s(u))$.

Proof. We begin by taking a configuration $u \in \mathbb{Z}_{\geq 0}^{n^2}$ and the element $s \in \mathbb{S}$. We can write the stabilization of the configuration u as follows

$$T(u) = u - \Delta x$$

for some $x \in \mathbb{Z}^{n^2}$. Then, we have

$$\begin{aligned} s(T(u)) &= s(u - \Delta x) \\ &= s(u) - \Delta s(x) \\ &= T(s(u)). \end{aligned}$$

The last line comes from uniqueness of stabilization since the map s does not affect stability. Furthermore, $\Delta s(x) = s(\Delta x)$ because the element s sends vertices to vertices with the same structure. Hence, elements of $\mathbb{S}$ commute with toppling.

As a result, if the configuration u is symmetric then its stabilization is symmetric. The orbits of the graph define the symmetric configurations since all vertices in an orbit must have equal weight. For a $2k \times 2k$ and $(2k + 1) \times (2k + 1)$ grid, the number of orbits is $\frac{k(k+1)}{2}$ since the triangular subgrid is invariant under rotations and reflections; see Figure 13.
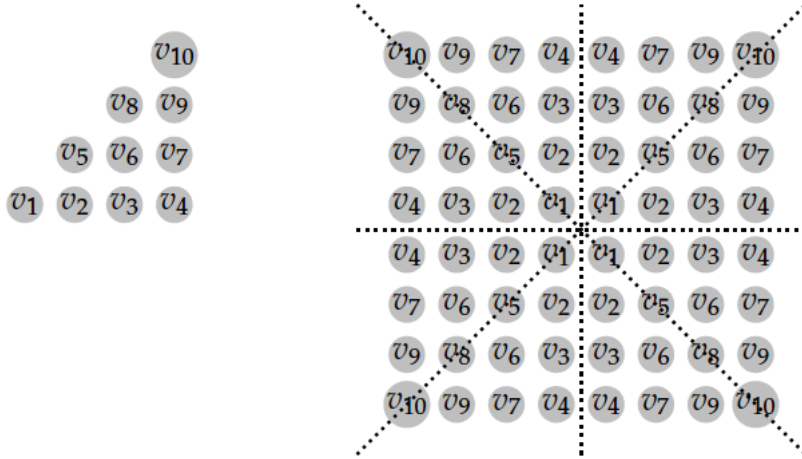
Figure 13: The triangular subgrid is invariant under $\mathbb{S}$. [13]

Proposition 5.3. If the configurations $u, v \in \mathbb{Z}_{\geq 0}^n$ are symmetric, then $T(u+v)$ is symmetric.

From Theorem 5.1, toppling commutes with the maps of $\mathbb{S}$. Let u and v be symmetric configuration and $s \in \mathbb{S}$,

$$s(T(u+v)) = T(s(u+v)) = T(u+v)$$

so $T(u+v)$ is symmetric.

Proposition 5.4. The identity of the group of recurrent configurations is symmetric.

Recall that the identity of $\mathscr{R}_n$ can be written in the following form

$$T(2u_{max} - T(2u_{max})). \tag{8}$$

From Proposition 5.1, the maximal recurrent configuration is symmetric, and by Proposition 5.3 the configuration $T(2u_{max})$ must be symmetric as well. As a result, the identity of the group of recurrent configurations is symmetric.

Theorem 5.2. The symmetric recurrent configurations forms a subgroup of the group of recurrent configurations.

Proof. To begin, the symmetric sandpile subgroup is nonempty since the maximal recurrent configuration is always symmetric. Furthermore, the identity is symmetric and thus an element of the subgroup. Toppling commutes with the elements of $\mathbb{S}$, so the addition and toppling of symmetric sandpiles preserves their symmetry. Since the group of recurrent configurations $\mathscr{R}$ is finite, this is enough to show that the set of symmetric recurrent configurations is a subgroup of $\mathscr{R}$.

To end this section, we provide the reader with a shorter alternative proof.

Theorem 5.3 (Alternative Proof). The identity of $\mathscr{R}_{n \times n}$ is symmetric and the set of symmetric recurrent configurations is a subgroup of $\mathscr{R}_{n \times n}$.

Proof. Let $S(G)$ be a sandpile graph on an $n \times n$ grid. Let $s \in \mathbb{S}$, we define ϕ_s to be the map from the set of configurations on $S(G)$ to itself as follows

$$\phi_s : \mathbb{Z}_{\geq 0}^{n^2} \longrightarrow \mathbb{Z}_{\geq 0}^{n^2},$$
$$u \mapsto s(u).$$

The map ϕ_s is linear and preserves the toppling operation (see Proposition 5.2 and Theorem 5.1), so ϕ_s restricted to $\mathscr{R}_{n \times n}$ is a group homomorphism. Furthermore, ϕ_s is bijective. This comes directly from the fact that $\mathbb{S}$ is a group so there exists an inverse $s^{-1} \in \mathbb{S}$. Hence, $\phi_s|_{\mathscr{R}_{n \times n}}$ is a group isomorphism. As a result, the identity of $\mathscr{R}_{n \times n}$ is preserved via the isomorphism $\phi_s|_{\mathscr{R}_{n \times n}}$. We have that the identity of $\mathscr{R}_{n \times n}$, denoted by e, has the property

$$s(e) = e, \quad \forall s \in \mathbb{S}.$$

Thus it follow that, by definition, the identity e is a symmetric configuration. Lastly, we prove that the set of symmetric recurrent configurations is a subgroup by taking the group homomorphisms $(\phi_s - id)|_{\mathscr{R}_{n \times n}}$, where id is the identity map of the set of configurations $\mathbb{Z}_{\geq 0}^{n^2}$. Let u be a configuration, u is symmetric if and only if $s(u) - u = 0$ for all $s \in \mathbb{S}$. Thus, the set of symmetric recurrent configurations are

$$\bigcap_{s \in \mathbb{S}} \ker\left((\phi_s - id)|_{\mathscr{R}_{n \times n}}\right).$$

The kernel of a group homomorphism is a subgroup and the finite intersection of subgroups is a subgroup. As a result, the set of recurrent symmetric configurations is a subgroup of $\mathscr{R}_{n \times n}$. This finalizes the proof. ∎

To conclude, the motivation for this subsection was to prove that the identity of the abelian sandpile group is symmetric. However, it turns out there are other recurrent configurations that possess this structure. With this in mind, we conclude the findings of this paper.

6 Conclusion and Other Remarks

The main purpose of this project was to present methods for calculating the identity of the abelian sandpile group. However, there are other findings that need to be discussed first. By algebraically studying sandpiles, or configurations, the behavior of collapsing sandpiles can be explained. The toppling of unstable vertices commutes, but furthermore, we prove that unstable sandpiles topple to a unique stable state via a (unique) finite amount of steps up to permutation. This nicely explains the self-organized critical behavior of collapsing sandpiles, which is why the sandpile model rose to prominence.

In this paper, the approach for finding the abelian sandpile group was different to other academic writing. We proved that the set of stable configurations is a finite abelian monoid, a group without inverses. This algebraic structure has the interesting property that its minimal ideal is a group. Then, we proved that the set of recurrent configurations is the minimal ideal of the set of stable configurations. Through this approach, we better understand why recurrent sandpiles occur. Since recurrent sandpiles are in the minimal ideal, any vertex-wise addition results in a recurrent configuration because the minimal ideal is closed under the toppling operation. Lastly, we showed that the abelian sandpile group is isomorphic to $\mathbb{Z}^n/\Delta\mathbb{Z}^n$. This isomorphism not only tells us the order of the abelian sandpile group, moreover it aids us in finding its identity.

Two different methods are given for calculating the identity of the group of recurrent configurations. The first uses the maximal recurrent configuration. Since, it is recurrent any addition to the maximal recurrent configuration results in a recurrent configuration. Moreover, the configuration $2u_{max} - T(2u_{max})$ is in the image of the toppling matrix. The second algorithm for calculating the identity is an iterative method. Once again, the second method uses the fact that the identity is the sole recurrent configuration in the image of the toppling matrix. In terms of efficiency, the issue with both methods is that they involve toppling. Stabilization of a sandpile on an $n \times n$ grid runs in $O(n^4)$ time. As a result, any closed-formula for the identity would not involve toppling, which does not seem feasible. However, if such a closed-formula exists it could have some important ramifications. To begin, Dhar conjectures that $2n \times 2n$ grid identities have a central square kernel consisting entirely of weight two, and $(2n+1) \times (2n+1)$ grid identities are the same except for a vertical and horizontal line running through the center (see Figure 14) [5]. Furthermore, if a closed-formula exists, then toppling could be predicted meaning it would be no effort in finding any recurrent configuration.
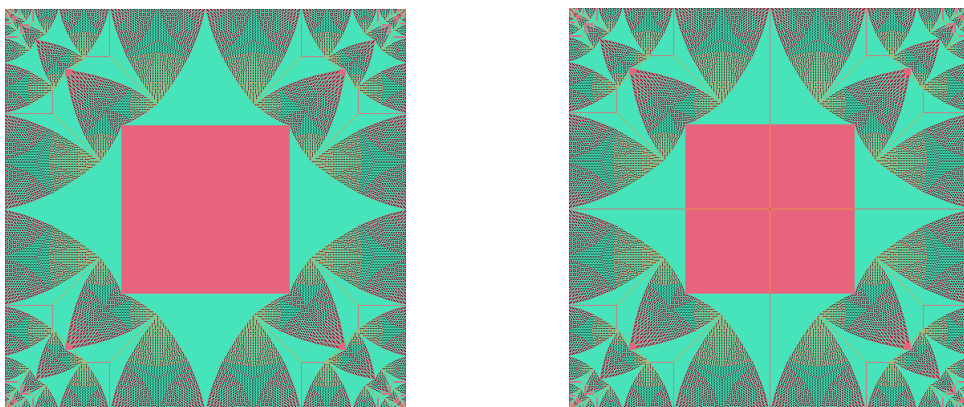

Figure 14: (Left) Identity on 500×500 grid with a central square kernel consisting of twos. (Right) Identity on 501×501 grid with a central square kernel of twos and the intersecting vertical and horizontal lines in the center.

To conclude, the identity of the group of recurrent configurations on a rectangular grid is a beautiful structure. The symmetries it possesses motivates us to look into all symmetric configurations. We proved that there exists a symmetric sandpile subgroup, hence the identity is indeed symmetric. Overall, there is more to sandpiles than meets the eye. Their complexity leaves a lot left to be discovered.

A

Preliminary Code

```matlab
function toppleConfig = topple(Config, height, width)
%{
Config is the unstable sandpile,
height and width are the height and width of the grid.
%}

    B = Config;
    check = 0; %used to check if configuration is stable.

    while check < height*width
        check = 0;
        for i = 1 : height
            for j = 1 : width
                if B(i,j) < 4 %if condition to ensure     vertex is stable
                    check = check + 1; %sum the stable vertices.
                end
                if B(i,j) >= 4 %if condition for unstable vertex

                    B(i,j) = B(i,j) - 4; %subtract 4 grains from unstable vertex

                    %{
                    the following if conditions are for adding a grain
                    to any neighboring vertex.
                    %}

                    if i+1 <= height
                        B(i+1,j) = B(i+1,j) + 1; %add a grain to neighbor vertex
                    end

                    if i-1 > 0
                        B(i-1,j) = B(i-1,j) + 1;
                    end

                    if j+1 <= width
                        B(i,j+1) = B(i,j+1) + 1;
                    end

                    if j-1 > 0
                        B(i,j-1) = B(i,j-1) + 1;
                    end
                end
            end
        end
    end

    toppledConfig = B;

end
```

Listing 1: Function for toppling configurations.

```matlab
function topplingMat = topMat(height,width)

%height and width are the height and width of the grid.

    grid = zeros(height,width);
    k = 1;

%for loop for labeling all the vertices of the grid.

    for i = 1:height
        for j = 1:width
            grid(i,j) = k;
            k = k + 1;
        end
    end

%create adjacency matrix

    [r, c] = size(grid);

    %Make the first diagonal vector (for horizontal connections).
    diagVec1 = repmat([ones(c-1, 1); 0], r, 1);

    %Remove the last value.
    diagVec1 = diagVec1(1:end-1);

    % Make the second diagonal vector (for vertical connections).
    diagVec2 = ones(c*(r-1), 1);

    % Add the diagonals to a zero matrix.
    adj = diag(diagVec1, 1)+diag(diagVec2, c);
    adj = adj+adj.';

    topplingMat = 4*eye(height*width) - adj;

end
```

Listing 2: Function for generating toppling matrix of a sandpile graph on an $n \times m$ grid.

B

Code for Methods on Calculating the Identity

```matlab
function identity = methodOne(height, width)

%height and width of grid.

    maxConfig = 6*ones(height,width); %maximal stable configurations times two.

```

```
 7      toppledMaxConfig = topple(maxConfig,height,width);
 8
 9      newConfig = maxConfig - toppledMaxConfig;
10
11      identity = topple(newConfig,height,width);
12
13  end
```

Listing 3: Function for Method 1 on an $n \times m$ grid.

```
 1  function identity = methodTwo(height,width)
 2
 3  %height and width of grid.
 4
 5  %create matrix form of the toppling matrix times a vector of all ones.
 6
 7      B = zeros(n,m);
 8
 9  %boundary vertices are one
10      for i = 1:height
11          for j = 1:width
12              if i == 1 || j == 1 || i == height || j == width
13                  B(i,j) = 1;
14              end
15          end
16      end
17
18  %corner vertices are two
19      B(1,1)=2;
20      B(1,m)=2;
21      B(n,1)=2;
22      B(n,m)=2;
23
24      check = toppling(B+B,height,width); %first iteration
25
26  %while loop to check when consecutive iterations are equal.
27
28      while isequal(check,B) == 0
29          B=check;
30          check = toppling(B+B,n,m);
31      end
32
33      identity = check;
34
35  end
```

Listing 4: Function for Method 2 on an $n \times m$ grid.

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