ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico, 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets, in the format used below. Solutions should be received within three months of the publication date.

Contributors (in the United States) who desire acknowledgement of receipt of their contributions are asked to enclose self-addressed stamped postcards.

B-184 Proposed by Bruce W. King, Adirondack Community College, Glens Falls, New York.

Let the sequence $\{T_n\}$ satisfy $T_{n+2} = T_{n+1} + T_n$ with arbitrary initial conditions. Let

$$g(n) = T_n^2 T_{n+3}^2 + 4T_{n+1}^2 T_{n+2}^2 .$$

Show the following:

(i) $$g(n) = (T_{n+1}^2 + T_{n+2}^2)^2 .$$

(ii) If T_n is the Lucas number L_n, $g(n) = 25F_{2n+3}^2$.

(See Fibonacci Quarterly Problems H-101, October, 1968, and B-160, April, 1969.)

B-185 Proposed by L. Carlitz, Duke University, Durham, N. Carolina.

Show that

$$L_{5n}/L_n = L_{2n}^2 - (-1)^n L_{2n} - 1 .$$

B-186 Proposed by L. Carlitz, Duke University, Durham, N. Carolina.

Show that

$$L_{5n}/L_n = [L_{2n} - (-1)^n 5]^2 + (-1)^n 25F_n^2 .$$

(For n even, this result has been given by D. Jarden in the Fibonacci Quarterly, Vol. 5 (1967), p. 346.)

B-187 Proposed by Carl Gronemeijer, Saranac Lake, N. York

Find positive integers x and y, with x even, such that

$$(x^2 + y^2)(x^2 + x + y^2)(x^2 + \tfrac{3}{2}x + y^2) = 1,608,404 .$$

B-188 Proposed by A. G. Shannon, University of Papua and New Guinea, Boroko, Papua.

Two circles are related so that there is a trapezoid ABCD inscribed in one and circumscribed in the other. AB is the diameter of the larger circle which has center O, and AB is parallel to CD. θ is half of angle AOD. Prove that $\sin\theta = (-1 + \sqrt{5})/2$.

B-189 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Let $a_0 = 1$, $a_1 = 7$, and $a_{n+2} = a_{n+1}a_n$ for $n \geq 0$. Find the last digit (i.e., units digit) of a_{999}.

SOLUTIONS

GENERALIZATIONS OF SECOND-ORDER RECURRENCES

B-166 Suggested by David Zeitlin's solutions to B-148, B-149, and B-150

Let a and b be distinct numbers, $U_n = (a^n - b^n)/(a - b)$, and $V_n = a^n + b^n$. Establish generalizations of the formulas

(a) $$F_{(2^t n)} = F_n L_n L_{2n} \cdots L_{(2^{t-1}n)}$$

(b) $$L_{n+1}L_{n+3} + 4(-1)^{n+1} = 5F_n F_{n+4}$$

of B-148 and B-149 in which one deals with U_n and V_n instead of F_n and L_n.

Solution by C. B. A. Peck, State College, Pennsylvania.

(a) Since $U_nV_n = U_{2n}$, induction on j from 1 yields

$$U_{(2^t n)} = V_{(2^{t-1}n)} \cdots V_{(2^{t-j}n)} U_{(2^{t-j}n)} ,$$

which, for $j = t$, is the desired extension.

(b) $$V_{n+1}V_{n+3} + x = (a^{n+1} + b^{n+1})(a^{n+3} + b^{n+3}) + x$$

$$= a^{2n+4} + (ab)^{n+1}(b^2 + a^2) + b^{2n+4} + x ,$$

while

$$yU_nU_{n+4} = y(a^n - b^n)(a^{n+4} - b^{n+4})/(a - b)^2$$

$$= [a^{2n+4} - (ab)^n(b^4 + a^4) + b^{2n+4}]y/(a - b)^2 .$$

We can take, for instance, $y = (a - b)^2$ and

$$x = -(ab)^n(b^4 + a^4) - (ab)^{n+1}(b^2 + a^2) = -(ab)^n[b^4 + a^4 + ab(b^2 + a^2)]$$

$$= (ab)^n(a^2b^2 - U_5) .$$

Then our generalization is, for instance,

$$V_{n+1}V_{n+3} + (ab)^n[(ab)^2 - U_5] = (a - b)^2U_nU_{n+4} ,$$

which with $a = (1 + \sqrt{5})/2$ and $b = (1 - \sqrt{5})/2$, simplifies to the Fibonacci case in (b).

Also solved by Wray G. Brady and David Zeitlin.

A LUCAS INEQUALITY

B-167 Proposed by A. G. Shannon, University of Papua and New Guinea, Boroko, Papua.

Let L_n be the n^{th} Lucas number defined by $L_1 = 1$, $L_2 = 3$, and $L_{n+2} = L_{n+1} + L_n$ for $n \geq 1$. For which values of n is

$$nL_{n+1} > (n + 1)L_n \quad ?$$

Solution by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

The inequality holds for $n = 1$ and $n = 3$. Let it hold for $n = k \geq 3$. Then

$$(k + 1)L_{k+2} = (k + 1)(L_{k+1} + L_k) > (k + 1)L_{k+1} + 2L_k > (k + 1)L_{k+1} + L_k + L_{k-1} = (k + 2)L_{k+1},$$

since $L_k > 0$ and $L_k > L_{k-1}$ for $k \geq 3$. This proves the inequality for $n \geq 3$ by mathematical induction; hence it holds for all positive integers except 2.

Also solved by Herta T. Freitag, Peter A. Lindstrom, C. B. A. Peck, Gerald Satlow, John Wessner, and the Proposer.

AN APPLICATION OF 1/7

B-168 Proposed by S. H. L. Kung, Jacksonville University, Jacksonville, Florida.

Using each of six of the nine positive digits $1, 2, \cdots, 9$ exactly once, form an integer z such that each of z, 2z, 3z, 4z, 5z, and 6z contains the same six digits once and once only.

Solution by Warren Cheves, Littleton, North Carolina.

The solution is $z = 142857$. This was obtained as follows:

Obviously, the first digit of z has to be 1. Otherwise, 6z would contain more than 6 digits.

Now consider the last digit of z. It cannot be a 1. It cannot be a 2, 4, 5, 6, or 8, because these numbers when multiplied by 5, 5, 4, 5, and 5, respectively, produce a last digit of 0. This leaves only 3, 7, and 9 as possible candidates for the last digit of z.

Consider

$1 \cdot 7 = 7$	$1 \cdot 3 = 3$	$1 \cdot 9 = 9$
$2 \cdot 7 = 14$	$2 \cdot 3 = 6$	$2 \cdot 9 = 18$
$3 \cdot 7 = 21$	$3 \cdot 3 = 9$	$3 \cdot 9 = 27$
$4 \cdot 7 = 28$	$4 \cdot 3 = 12$	$4 \cdot 9 = 36$
$5 \cdot 7 = 35$	$5 \cdot 3 = 15$	$5 \cdot 9 = 45$
$6 \cdot 7 = 42$	$6 \cdot 3 = 18$	$6 \cdot 9 = 54$

here, the multiples of both 3 and 9 have for their last digits 6 different numbers, none of which is the number 1. Hence, 7 must be the last digit of z. Furthermore, by looking at the last digits of the multiples of 7 (above), we see that the six digits of z must be 1, 2, 4, 5, 7, 8, with 1 being the first and 7 the last.

The order of these six digits was found mainly by trial and error. In other words, multiples of different combinations of the six digits were computed until certain eliminations could be made. (I did find one hint: the "8" could not appear immediately after the "1" or else 6z would contain more than 6 digits.) After my trial and error method, I found that z = 142857 fitted the requirements of B-168.

Also solved by Ed and Martha Clarke, Peter A. Lindstrom, John W. Milsom, C. B. A. Peck, and the Proposer.

A SEQUENCE OF IDENTITIES

B-169 Proposed by C. C. Yalavigi, Government College, Mercara, India.

Prove the following identities:

(a) $$F_n^4 + F_{n-1}^4 + F_{n+1}^4 = 2(F_nF_{n-1} - F_{n+1})^2$$

(b) $$F_n^5 + F_{n-1}^5 - F_{n+1}^5 = 5F_nF_{n-1}F_{n+1}(F_nF_{n-1} - F_{n+1}^2) ,$$

where $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$. Show that these are two cases of an infinite sequence of identities.

Solution by L. Carlitz, Duke University, Durham, North Carolina.

The first identity should read

(a) $$F_n^4 + F_{n-1}^4 + F_{n+1}^4 = 2(F_{n+1}^2 - F_nF_{n-1})^2 .$$

This follows from

$$\begin{aligned} F_n^4 + F_{n-1}^4 + F_{n+1}^4 &= F_n^4 + F_{n-1}^4 + (F_n + F_{n-1})^4 \\ &= 2(F_n^4 + 2F_n^3F_{n-1} + 3F_n^2F_{n-1}^2 + 2F_nF_{n-1}^3 + F_{n-1}^4) \\ &= 2(F_n^2 + F_nF_{n-1} + F_{n-1}^2)^2 \\ &= 2(F_{n+1}^2 - F_nF_{n-1}) . \end{aligned}$$

Similarly, to prove (b), we have

$$\begin{aligned} F_{n+1}^5 - F_n^5 - F_{n-1}^5 &= (F_n + F_{n-1})^5 - F_n^5 - F_{n-1}^5 \\ &= 5F_nF_{n-1}(F_n^3 + 2F_n^2F_{n-1} + 2F_nF_{n-1}^2 + F_{n-1}^3) \\ &= 5F_nF_{n-1}(F_n + F_{n-1})(F_n^2 + F_nF_{n-1} + F_{n-1}^2) \\ &= 5F_nF_{n-1}F_{n+1}(F_{n+1}^2 - F_nF_{n-1}) . \end{aligned}$$

To get a general result, we recall that Cauchy (see P. Bachmann, Das Fermatproblem in seiner bisherigen Entwickelung, Berlin, 1919, p. 31) has proven that if p is a prime >3, then

(1) $$(x + y)^p - x^p - y^p = pxy(x + y)(x^2 + xy + y^2)f_p(x,y) ,$$

where $f_p(x,y)$ is a polynomial with integral coefficients. For $p \equiv 1 \pmod 3$ there is the stronger result,

(2) $$(x + y)^p - x^p - y^p = pxy(x + y)(x^2 + xy + y^2)g_p(x,y) ,$$

where $g_p(x,y)$ is a polynomial with integral coefficients.

Substituting $x = F_n$, $y = F_{n-1}$ in (1) or (2), we get identities of the required kind. In particular, for $p = 7$,

$$F_{n+1}^7 - F_n^7 - F_{n-1}^7 = 7F_nF_{n-1}F_{n+1}(F_{n+1}^2 - F_nF_{n-1}) .$$

For further results of this kind, see "Sums of Powers of Fibonacci and Lucas Numbers," by L. Carlitz and J. A. H. Hunter, Fibonacci Quarterly, December, 1969, p. 467.

Also solved by the Proposer.

A PERIODIC SEQUENCE

B170 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Let the binomial coefficient $\binom{m}{r}$ be zero when m r and let

$$S_n = \sum_{j=0}^{\infty} (-1)^j \binom{n-j}{j} .$$

Show that $S_{n+2} - S_{n+1} + S_n = 0$, and hence $S_{n+3} = -S_n$ for $n = 0, 1, 2, \cdots$.

Solution by F. D. Parker, St. Lawrence University, Canton, New York.

If

$$S_n = \sum_{j=0}^{\infty} (-1)^j \binom{n-j}{j} ,$$

then

$$S_{n+2} - S_{n+1} + S_n = \sum_{j=0}^{\infty} (-1)^j \left\{ \binom{n-j+2}{j} - \binom{n-j+1}{j} + \binom{n-j}{j} \right\} .$$

But

$$\binom{n-j+2}{j} - \binom{n-j+1}{j} = \binom{n-j+1}{j-1} ,$$

so that

$$S_{n+2} - S_{n+1} + S_n = \sum_{j=0}^{\infty} (-1)^j \left\{ \binom{n+1-j}{j-1} + \binom{n-j}{j} \right\} .$$

Changing indices, we have

$$\sum_{j=0}^{\infty} (-1)^j \binom{n+1-j}{j-1} = \sum_{j=-1}^{\infty} (-1)^{j+1} \binom{n-j}{j} = \sum_{j=0}^{\infty} (-1)^{j+1} \binom{n-j}{j} ,$$

and therefore

$$S_{n+2} - S_{n+1} + S_n = \sum_{j=0}^{\infty} (-1)^j \left\{ -\binom{n-j}{j} + \binom{n-j}{j} \right\} = 0 .$$

Using this identity, we have

$$0 = S_{n+3} - S_{n+2} + S_{n+1} = S_{n+3} - (S_{n+1} - S_n) + S_{n+1} ,$$

and so $S_{n+3} = -S_n$.

Also solved by A. K. Gupta, C. B. A. Peck, John Wessner, David Zeitlin, and the Proposer.

Zeitlin noted the following:

The Chebyshev polynomial of the second kind, $U_n(x)$, satisfies

$$U_{n+2}(x) = 2xU_{n+1}(x) - U_n(x) ,$$

and is defined by

$$U_n(s) = \sum_{j=0}^{\infty} (-1)^j \binom{n-j}{j} (2x)^{n-2j} .$$

Thus,

$$S_n \equiv U_n(1/2),$$

i.e.,

$$S_{n+2} = S_{n+1} - S_n .$$

AVERAGING EIBONACCI AND PERIODIC SEQUENCES

B-171 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Let $\binom{m}{r} = 0$ for $m < r$, and let

$$T_n = \sum_{j=0}^{\infty} \binom{n-2j}{2j} .$$

Obtain a fourth-order homogeneous linear recurrence formula for T_n.

Solution by A. K. Gupta, University of Arizona, Tuscon, Arizona.

$$T_{n+3} = \sum_{j=0}^{\infty} \binom{n+3-2j}{2j}$$

$$= \binom{n+3}{0} + \sum_{j=1}^{\infty} \left[\binom{n+2-2j}{2j} + \binom{n+2-2j}{2j-1}\right] ,$$

since

$$\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1} ,$$

$$T_{n+3} = T_{n+2} + \sum_{j=1}^{\infty} \binom{n+2-2j}{2j-1}$$

$$= T_{n+2} + \sum_{j=1}^{\infty} \left[\binom{n+3-2j}{2j-1} - \binom{n+2-2j}{2j-2} \right]$$

$$= T_{n+2} - T_n + \sum_{j=1}^{\infty} \binom{n+3-2j}{2j-1}$$

$$= T_{n+2} - T_n + \sum_{j=1}^{\infty} \left[\binom{n+4-2j}{2j} - \binom{n+3-2j}{2j} \right]$$

$$= T_{n+2} - T_n + (T_{n+4} - T_{n+3}) .$$

Thus we get

$$T_{n+4} - 2T_{n+3} + T_{n+2} - T_n = 0 .$$

Also solved by C. B. A. Peck, John Wessner, David Zeitlin, and the Proposer.

[Continued from page 310.]

20. Servius, Aeneid, IV.
21. For example, titles of standard sizes, Vitruvius De Architectura V.
22. C. R. Lepsius, die Langenmasse der Alten, Berlin (1884).
23. A. Bosio, Roma Sotterranea, Rome (1632).
24. J. Greaves, A Discourse of the Romane foot and denarius, from whence the measures and weights used by the ancients may be deduced, London (1647).

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