EXCERPTED FROM

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A NEW KIND OF SCIENCE

NOTES FOR CHAPTER 6:

Starting from Randomness

Starting from Randomness

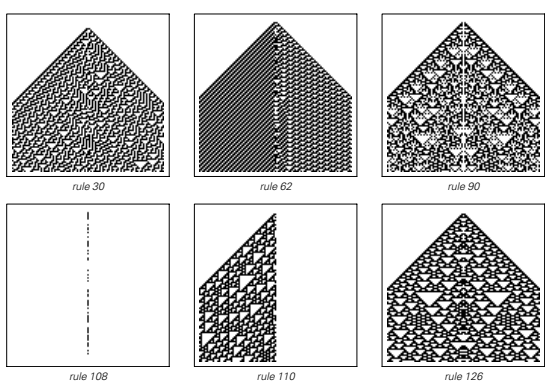
The Emergence of Order

▪ **Page 226 · Properties of patterns.** For a random initial condition, the average density of black cells is exactly 1/2. For rule 126, the density after many steps is still 1/2. For rule 22, it is approximately 0.35095. For rule 30 and rule 150 it is exactly 1/2, while for rule 182 it is 3/4. And insofar as rule 110 converges to a definite density, the density is 4/7. (See page 953 for a method of estimating these densities.)

Even after many steps, individual lines in the patterns produced by rules 30 and 150 remain in general completely random. But in rule 126, black cells always tend to appear in pairs, while in rule 182, every white cell tends to be surrounded by black ones. And in rule 22, there are more complicated conditions involving blocks of 4 cells.

The density of triangles of size n goes roughly like 2^{-n} for rules 126, 30 (see also page 871), 150 and 182 and roughly like 1.3^{-n} for rule 22.

In the algebraic representation discussed on page 869, rule 22 is $Mod[p + q + r + p\,q\,r, 2]$, rule 126 is $Mod[(p + q)(q + r) + (p + r), 2]$, rule 150 is $Mod[p + q + r, 2]$ and rule 182 is $Mod[p\,r\,(1 + q) + (p + q + r), 2]$.



▪ **Continual injection of randomness.** In the main text we discuss what happens when one starts from random initial conditions and then evolves according to a definite cellular automaton rule. As an alternative one can consider starting with very simple initial conditions, such as all cells white, and then at each step randomly changing the color of the center cell. Some examples of what happens are shown at the bottom of the previous column. The results are usually very similar to those obtained with random initial conditions.

▪ **History.** The fact that despite initial randomness processes like friction can make systems settle down into definite configurations has been the basis for all sorts of engineering throughout history. The rise of statistical mechanics in the late 1800s emphasized the idea of entropy increase and the fundamental tendency for systems to become progressively more disordered as they evolve to thermodynamic equilibrium. Theories were nevertheless developed for a few cases of spontaneous pattern formation—notably in convection, cirrus clouds and ocean waves. When the study of feedback and stability became popular in the 1940s, there were many results about how specific simple fixed or repetitive behaviors in time could emerge despite random input. In the 1950s it was suggested that reaction-diffusion processes might be responsible for spontaneous pattern formation in biology (see page 1012)—and starting in the 1970s such processes were discussed as prime examples of the phenomenon of self-organization. But in their usual form, they yield essentially only rather simple repetitive patterns. Ever since around 1900 it tended to be assumed that any fundamental theory of systems with many components must be based on statistical mechanics. But almost all work in the field of statistical mechanics concentrated on systems in or very near thermal equilibrium—in which in a sense there is almost complete disorder. In the 1970s there began to be more discussion of phenomena far from equilibrium, although typically it got no further than to consider how external forces could lead to reaction-diffusion-like phenomena. My own work on cellular

automata in 1981 emerged in part from thinking about self-gravitating systems (see page 880) where it seemed conceivable that there might be very basic rules quite different from those usually studied in statistical mechanics. And when I first generated pictures of the behavior of arbitrary cellular automaton rules, what struck me most was the order that emerged even from random initial conditions. But while it was immediately clear that most cellular automata do not have the kind of reversible underlying rules assumed in traditional statistical mechanics, it still seemed initially very surprising that their overall behavior could be so elaborate—and so far from the complete orderlessness one might expect on the basis of traditional ideas of entropy maximization.

Four Classes of Behavior

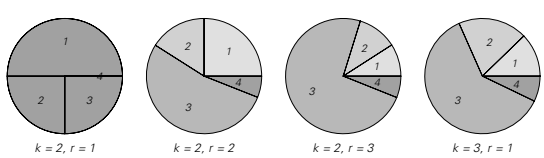
■ **Different runs.** The qualitative behavior seen with a given cellular automaton rule will normally look exactly the same for essentially all different large random initial conditions—just as it does for different parts of a single initial condition. And as discussed on page 597 any obvious differences could in effect be thought of as revealing deviations from randomness in the initial conditions.

■ **Page 232 · Elementary rules.** The examples shown have rule numbers n for which *IntegerDigits[n, 2, 8]* matches *{_, i_, _, j_, i_, _, j_, 0}*.

■ **Page 235 · States of matter.** As suggested by pages 944 and 1193, working out whether a particular substance at a particular temperature will be a solid, liquid or gas may in fact be computationally comparable in difficulty to working out what class of behavior a particular cellular automaton will exhibit.

■ **Page 235 · Class 4 rules.** Other examples of class 4 totalistic rules with $k = 3$ colors include 357 (page 282), 438, 600, 792, 924, 1038, 1041, 1086, 1329 (page 282), 1572, 1599 (see page 70), 1635 (see page 67), 1662, 1815 (page 236), 2007 (page 237) and 2049 (see page 68).

■ **Frequencies of classes.** The pie charts below show results for 1D totalistic cellular automata with k colors and range r. Class 3 tends to become more common as the number of elements in the rule increases because as soon as any of these elements yield class 3 behavior, that behavior dominates the system.



■ **History.** I discovered the classification scheme for cellular automata described here late in 1983, and announced it in January 1984. Much work has been done by me and others on ways to make the classification scheme precise. The notion that class 4 can be viewed as intermediate between class 2 and class 3 was studied particularly by Christopher Langton, Wentian Li and Norman Packard in 1986 for ordinary cellular automata, by Hyman Hartman in 1985 for probabilistic cellular automata and by Hugues Chaté and Paul Manneville in 1990 for continuous cellular automata.

■ **Subclasses within class 4.** Different class 4 systems can show localized structures with strikingly similar forms, and this may allow subclasses within class 4 to be identified. In addition, class 4 systems show varying levels of activity, and it is possible that there may be discrete transitions—perhaps analogous to percolation—that can be used to define boundaries between subclasses.

■ **Page 240 · Undecidability.** Almost any definite procedure for determining the class of a particular rule will have the feature that in borderline cases it can take arbitrarily long, often formally showing undecidability, as discussed on page 1138. (An example would be a test for class 1 based on checking that no initial pattern of any size can survive. Including probabilities can help, but there are still always borderline cases and potential undecidability.)

■ **Page 244 · Continuous cellular automata.** In ordinary cellular automata, going from one rule to the next in a sequence involves some discrete change. But in continuous cellular automata, the parameters of the rule can be varied smoothly. Nevertheless, it still turns out that there are discrete transitions in the overall behavior that is produced. In fact, there is often a complicated set of transitions that depends more on the digit sequence of the parameter than its size. And between these transitions there are usually ranges of parameter values that yield definite class 4 behavior. (Compare page 922.)

■ **Nearby cellular automaton rules.** In a range r cellular automaton the new color of a particular cell depends only on cells at most a distance r away. One can make an equivalent cellular automaton of larger range by having a rule in which cells at distance more than r have no effect. One can then define nearby cellular automata to be those where the differences in the rule involve only cells close to the edge of the range. With larger and larger ranges one can then construct closer approximations to continuous sequences of cellular automata.

■ **2D class 4 cellular automata.** No 5- or 9-neighbor totalistic rules nor 5-neighbor outer totalistic ones appear to yield

class 4 behavior with a white background. But among 9-neighbor outer totalistic rules there are examples with codes 224 (Game of Life), 226, 4320 (sometimes called HighLife), 5344, 6248, 6752, 6754 and 8416, etc. It turns out that the simplest moving structures are the same in codes 224, 226 and 4320.

■ **Page 249 · Game of Life.** Invented by John Conway around 1970 (see page 877), the Life 2D cellular automaton has been much studied in recreational computing, and as described on page 964 many localized structures in it have been identified. Each step in its evolution can be implemented using

```
LifeStep[a_List] :=
  MapThread[If[#1 == 1 && #2 == 4 || #2 == 3, 1, 0] &,
    {a, Sum[RotateLeft[a, {i, j}], {i, -1, 1}, {j, -1, 1}]}, 2]
```

A more efficient implementation can be obtained by operating not on a complete array of black and white cells but rather just on a list of positions of black cells. With this setup, each step then corresponds to

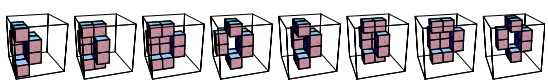
```
LifeStep[list_] :=
  With[{p = Flatten[Array[List, {3, 3}, -1], 1]},
    With[{u = Split[Sort[Flatten[Outer[Plus, list, p, 1], 1]]]},
      Union[Cases[u, {x_, _, _} → x],
        Intersection[Cases[u, {x_, _, _, _} → x], list]]]]
```

(A still more efficient implementation is based on finding runs of length 3 and 4 in *Sort[u]*.)

■ **3D class 4 rules.** With a cubic lattice of the type shown on page 183, and with updating rules of the form

```
LifeStep3D[{p_, q_, r_}, a_List] := MapThread[If[
  #1 == 1 && p ≤ #2 ≤ q || #2 == r, 1, 0] &, {a, Sum[RotateLeft[
    a, {i, j, k}], {i, -1, 1}, {j, -1, 1}, {k, -1, 1}] - a}, 3]
```

Carter Bays discovered between 1986 and 1990 the three examples *{5, 7, 6}*, *{4, 5, 5}*, and *{5, 6, 5}*. The pictures below show successive steps in the evolution of a moving structure in the second of these rules.



■ **Random initial conditions in other systems.** Whenever the initial conditions for a system can involve an infinite sequence of elements these elements can potentially be chosen at random. In systems like mobile automata and Turing machines the colors of initial cells can be random, but the active cell must start at a definite location, and depending on the behavior only a limited region of initial cells near this location may ever be sampled. Ordinary substitution systems can operate on infinite sequences of elements chosen at random. Sequential substitution systems, however, rely on scanning limited sequences of elements, and so cannot readily be given infinite random initial conditions. The same is true of ordinary and cyclic tag systems. Systems based on continuous numbers involve infinite sequences of digits which can readily be chosen at random (see page 154). But systems based on integers (including register machines) always deal with finite sequences of digits, for which there is no unique definition of randomness. (See however the discussion of number representations on page 1070.) Random networks (see pages 963 and 1038) can be used to provide random initial conditions for network systems. Multiway systems cannot meaningfully be given infinite random initial conditions since these would typically lead to an infinite number of possible states. Systems based on constraints do not have initial conditions. (See also page 920.)

Sensitivity to Initial Conditions

■ **Page 251 · Properties.** In rule 126, the outer edges of the region of change always expand by exactly one cell per step. The same is true of the right-hand edge in rule 30—though the left-hand edge in this case expands only about 0.2428 cells on average per step. In rule 22, both edges expand about 0.7660 cells on average per step.

The motion of the right-hand edge in rule 30 can be understood by noting that with this rule the color of a particular cell will always change if the color of the cell to its left is changed on the previous step (see page 601). Nothing as simple is true for the left-hand edge, and indeed this seems to execute an essentially random walk—with an average motion of about 0.2428 cells per step. Note that in the approximation that the colors of all cells in the pattern are assumed completely independent and random there should be motion by 0.25 cells per step. Curiously, as discussed on page 871, the region of non-repetitive behavior in evolution from a single black cell according to rule 30 seems to grow at a similar but not identical rate of about 0.252 cells per step. (For rule 45, the left-hand edge of the difference pattern moves about 0.1724 cells per step; for rule 54 both edges move about 0.553 cells per step.)

■ **Difference patterns.** The maximum rate at which a region of change can grow is determined by the range of the underlying cellular automaton rule. If the rule involves up to r nearest neighbors, then at each step a change in the color of a given cell can affect cells up to r away—so that the edge of the region of change can move by r cells.

For most class 3 rules, once one is inside the region of change, the colors of cells usually become essentially uncorrelated.

However, for additive rules the pattern of differences is just exactly the pattern that would be obtained by evolution from an initial condition consisting only of the changes made. In general the pattern of probabilities for changes can be thought of as being somewhat like a Green's function in mathematical physics—though the nonadditivity of most cellular automata makes this analogy less useful. (Note that the pattern of differences between two initial conditions in a rule with k possible colors can always be reproduced by looking at the evolution from a single initial condition of a suitable rule with $2k$ colors.) In 2D class 3 cellular automata, the region of change usually ends up having a roughly circular shape—a result presumably related to the Central Limit Theorem (see page 976).

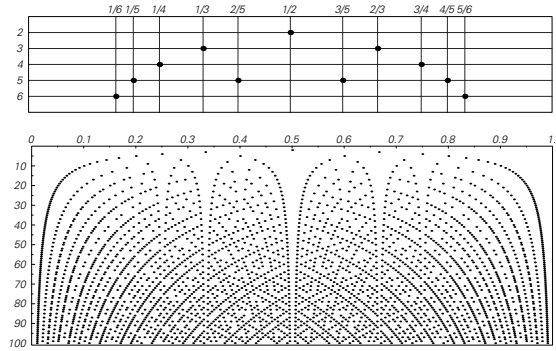
For any additive or partially additive class 3 cellular automaton (such as rule 90 or rule 30) any change in initial conditions will always lead to expanding differences. But in other rules it sometimes may not. And thus, for example, in rule 22, changing the color of a single cell has no effect after even one step if the cell has a ■ block on either side. But while there are a few other initial conditions for which differences can die out after several steps most forms of averaging will say that the majority of initial conditions lead to growing patterns of differences.

■ **Lyapunov exponents.** If one thinks of cells to the right of a point in a 1D cellular automaton as being like digits in a real number, then linear growth in the region of differences associated with a change further to the right is analogous to the exponentially sensitive dependence on initial conditions shown on page 155. The speed at which the region of differences expands in the cellular automaton can thus be thought of as giving a Lyapunov exponent (see page 921) that characterizes instability in the system.

Systems of Limited Size and Class 2 Behavior

■ **Page 255 · Cyclic addition.** After t steps, the dot will be at position *Mod[m t, n]* where n is the total number of positions, and m is the number of positions moved at each step. The repetition period is given by *n/GCD[m, n]*. The picture on page 613 shows the values of m and n for which this is equal to n.

An alternative interpretation of the system discussed here involves arranging the possible positions in a circle, so that at each step the dot goes a fraction m/n of the way around the circle. The repetition period is maximal when m/n is a fraction in lowest terms. The picture below shows the repetition periods as a function of the numerical size of the quantity m/n.



■ **Page 257 · Cyclic multiplication.** With multiplication by k at each step the dot will be at position *Mod[k^t, n]* after t steps. If k and n have no factors in common, there will be a t for which *Mod[k^t, n] == 1*, so that the dot returns to position 1. The smallest such t is given by *MultiplicativeOrder[k, n]*, which always divides *EulerPhi[n]* (see page 1093), and has a value between *Log[k, n]* and *n - 1*, with the upper limit being attained only if n is prime. (This value is related to the repetition period for the digit sequence of *1/n* in base k, as discussed on page 912). When *GCD[k, n] == 1* the dot can never visit position 0. But if $n == k^s$, the dot reaches 0 after s steps, and then stays there. In general, the dot will visit position *m = k ^ IntegerExponent[n, k]* every *MultiplicativeOrder[k, n/m]* steps.

■ **Page 260 · Maximum periods.** A cellular automaton with n cells and k colors has k^n possible states, but if the system has cyclic boundary conditions, then the maximum repetition period is smaller than k^n. The reason is that different states of the cellular automaton have different symmetry properties, and thus cannot be on the same cycle. In particular, if a state of a cellular automaton has a certain spatial period, given by the minimum positive m for which *RotateLeft[list, m] == list*, then this state can never evolve to one with a larger spatial period. The number of states with spatial period m is given by

```
s[m_, k_] :=
  k^m - Apply[Plus, Map[s[#, k] &, Drop[Divisors[m], -1]]]
```

or equivalently

```
s[m_, k_] := Apply[Plus,
    (MoebiusMu[m/#] k^# &)[Divisors[m]]]
```

In a cellular automaton with a total of n cells, the maximum possible repetition period is thus *s[n, k]*. For *k = 2*, the maximum periods for n up to 10 are: *{2, 2, 6, 12, 30, 54, 126, 240, 504, 990}*. In all cases, *s[n, k]* is divisible by n. For prime n, *s[n, k]* is $k^n - k$. For large n, *s[n, k]* oscillates between about $k^n - k^{n/2}$ and $k^n - k$. (See page 963.)

■ **Additive cellular automata.** In the case of additive rules such as rule 90 and rule 60, a mathematical analysis of the repetition periods can be given (as done by Olivier Martin, Andrew Odlyzko and me in 1983). One starts by converting the list of cell colors at each step to a polynomial *FromDigits[list, x]*. Then for the case of rule 60 with n cells and cyclic boundary conditions, the state obtained after t steps is given by

PolynomialMod[(1 + x)^t z, {x^n - 1, 2}]

where z is the polynomial representing the initial state, and $z = 1$ for a single black cell in the first position. The state $z = 1$ evolves after one step to the state $z = 1 + x$, and for odd n this latter state always eventually appears again. Using the result that $1 + x^{2^m} == (1 + x)^{2^m}$ modulo 2 for any m, one then finds that the repetition period always divides the quantity *p[n] = 2^MultiplicativeOrder[2, n] - 1*, which in turn is at most $2^{n-1} - 1$. The actual periods are often smaller than *p[n]*, with the following ratios occurring:

n	11	13	19	25	27	29	37	41	43	53
ratio	3	5	27	41	19	565	21255	25	3	1266205

There appears to be no case for $n > 5$ where the period achieves the absolute maximum $2^{n-1} - 1$.

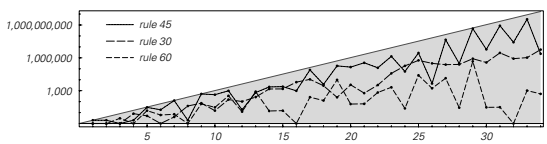
In the case of rule 90 a similar analysis can be given, with the $1 + x$ used at each step replaced by $1/x + x$. And now the repetition period for odd n divides

q[n] = 2^MultiplicativeOrder[2, n, {1, -1}] - 1

The exponent here always lies between *Log[k, n]* and $(n - 1)/2$, with the upper bound being attained only if n is prime. Unlike for the case of rule 60, the period is usually equal to *q[n]* (and is assumed so for the picture on page 260), with the first exception occurring at $n = 37$.

■ **Rules 30 and 45.** Maximum periods are often achieved with initial conditions consisting of a single black cell. Particularly for rule 30, however, there are quite a few exceptions. For $n = 13$, for example, the maximum period is 832 but the period with a single black cell is 260. For rule 45, the maximum possible period discussed above is achieved for $n = 9$, but does not appear to be achieved for any larger n. (See page 962.)

■ **Comparison of rules.** Rules 45, 30 and 60, together with their conjugates and reflections, yield the longest repetition periods of all elementary rules (see page 1087). The picture below compares their periods as a function of n.

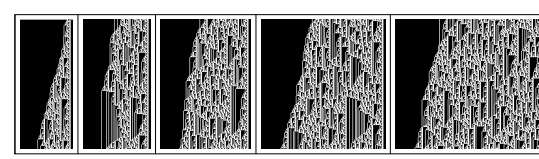


■ **Implementing boundary conditions.** In the bitwise representation discussed on page 865, 0's outside of a width n can be implemented by applying *BitAnd[a, 2^n - 1]* at each step. Cyclic boundary conditions can be implemented efficiently in assembler on computers that support cyclic shift instructions.

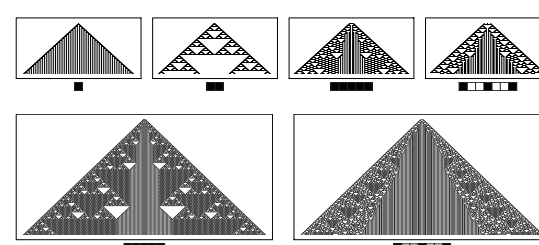
Randomness in Class 3 Systems

■ **Page 263 · Rule 22.** Randomness is obtained with initial conditions consisting of two black squares $4m$ positions apart for any $m \geq 2$. The base 2 digit sequences for 19, 25, 37, 39, 41, 45, 47, 51, 57, 61, … also give initial conditions that yield randomness. Despite its overall randomness there are some regularities in the pattern shown at the bottom of the page. The overall density of black cells is not 1/2 but is instead approximately 0.35, just as for random initial conditions. And if one looks at the center cell in the pattern one finds that it is never black on two successive steps, and the probability for white to follow white is about twice the probability for black to follow white. There is also a region of repetitive behavior on each side of the pattern; the random part in the middle expands at about 0.766 cells per step—the same speed that we found on page 949 that changes spread in this rule.

■ **Rule 225.** With initial conditions consisting of a single black cell, this class 3 rule yields a regular nested pattern, as shown on page 58. But with the initial condition ■■■, it yields the much more complicated pattern shown below. With a background consisting of repetitions of the block ■□, insertion of a single initial white cell yields a largely random pattern that expands by one cell per step. Rule 225 can be expressed as $\neg p \veebar (q \vee r)$.



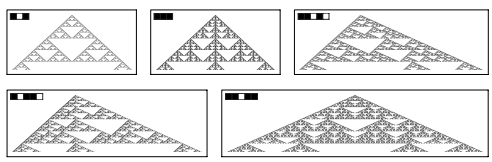
■ **Rule 94.** With appropriate initial conditions this class 2 rule can yield both nested and random behavior, as shown below.



■ **Rule 218.** If pairs of adjacent black cells appear anywhere in its initial conditions this class 2 rule gives uniform black, but if none do it gives a rule 90 nested pattern.

■ **Additive rules.** Of the 256 elementary cellular automata 8 are additive: *{0, 60, 90, 102, 150, 170, 204, 240}*. All of these are either trivial or essentially equivalent to rules 90 or 150.

Of all $k^{k^{2r+1}}$ rules with k colors and range r it turns out that there are always exactly k^{2r+1} additive ones—each obtained by taking the cells in the neighborhood and adding them modulo k with weights between *0* and $k-1$. As discussed on page 955, any rule based on addition modulo k must yield a nested pattern, and it therefore follows that any rule that is additive must give a nested pattern, as in the examples below. (See also page 870.)



Note that each step in the evolution of any additive cellular automaton can be computed as

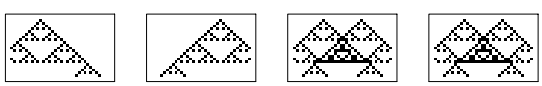
```
Mod[ListCorrelate[w, list, Ceiling[Length[w]/2]], k]
```

(See page 1087 for a discussion of partial additivity.)

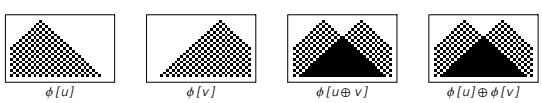
■ **Page 264 · Generalized additivity.** In general what it means for a system to be additive is that some addition operation $\oplus$ can be used to combine any set of evolution histories to yield another possible evolution history. If ϕ is the rule for the system, this then requires for any states u and v the distributive property

$$\phi[u \oplus v] == \phi[u] \oplus \phi[v]$$

(In mathematical terms this is equivalent to the statement that ϕ is conjugate to itself under the action of $\oplus$—or alternatively that ϕ defines a homomorphism with respect to the $\oplus$ operation.) In the usual case, $u \oplus v$ is just *Mod[u + v, k]*, yielding say for rule 90 the results below.

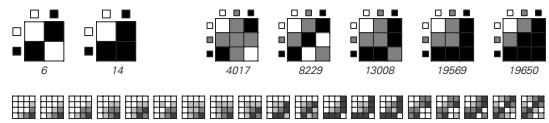


But it turns out that some elementary rules show additivity with respect to other addition operations. An example as shown below is rule 250 with $u \oplus v$ taken as *Max[u, v]* (*Or*).



If a system is additive it means that one can work out how the system will behave from any initial condition just by combining the patterns ("Green's functions") obtained from certain basic initial conditions—say ones containing a single black cell. To get all the familiar properties of additivity one needs an addition operation that is associative (*Flat*) and commutative (*Orderless*), and has an identity element (white or *0* in the cases above)—so that it defines a commutative monoid. (Usually it is also convenient to be able to get all possible elements by combining a small number of basic generator elements.)

The inequivalent commutative monoids with up to $k = 4$ colors are (in total there are 1, 2, 5, 19, 78, 421, 2637, … such objects):



For $k = 2$, $r = 1$ the number of rules additive with respect to these is respectively: *{8, 9}*; for $k = 2$, $r = 2$: *{32, 33}*; for $k = 3$, $r = 1$: *{28, 27, 35, 244, 28}*; for $k = 4$, $r = 1$:

{1001, 65, 540, 577, 126, 4225, 540, 9065, 757, 408, 65, 133, 862, 224, 72, 72, 91, 4096, 64}

It turns out to be possible to show that any rules ϕ additive with respect to some addition operation $\oplus$ must work by applying that operation to values associated with cells in their neighborhood. The values are obtained by applying to cells at each position one of the unary operations (endomorphisms) σ that satisfy $\sigma[a \oplus b] == \sigma[a] \oplus \sigma[b]$ for individual cell values a and b. (For *Xor*, there are 2 possible σ, while for *Or* there are 3.)

The basic examples are then rules of the form *RotateLeft[a] ⊕ RotateRight[a]*—analogs of rule 90, but with other addition operations (compare page 886). The σ can be used to give analogs of the weights that appear in the note above. And rules that involve more than two cells can be obtained by having several instances of $\oplus$—which can always be flattened. But in all cases the general results for associative rules on page 956 show that the patterns obtained must be at most nested.

If instead of an ordinary cellular automaton with a limited number of possible colors one considers a system in which every cell can have any integer value then additivity with respect to ordinary addition becomes just traditional linearity. And the only way to achieve this is to have a rule in which the new value of a cell is given by a linear form such as $a\,x + b\,y$. If the values of cells are allowed to be any real number then

linear forms such as $a x + b y$ again yield additivity with respect to ordinary addition. But in general one can apply to each cell value any function σ that obeys the so-called Cauchy functional equation $\sigma[x + y] == \sigma[x] + \sigma[y]$. If $\sigma[x]$ is required to be continuous, then the only form it can have is $c x$. But if one allows σ to be discontinuous then there can be some other exotic possibilities. It is inevitable that within any rationally related set of values x one must have $\sigma[x] = c x$ with fixed c. But if one assumes the Axiom of Choice then in principle it becomes possible to construct $\sigma[x]$ which have different c for different sets of x values. (Note however that I do not believe that such σ could ever actually be constructed in any explicit way by any real computational system—or in fact by any system in our universe.)

In general $\oplus$ need not be ordinary addition, but can be any operation that defines a commutative monoid—including an infinite one. An example is ordinary numbers modulo an irrational. And indeed a cellular automaton whose rule is based on $Mod[x + y, \pi]$ will show additivity with respect to this operation (see page 922). If $\oplus$ has an inverse, so that it defines a group, then the only continuous (Lie group) examples turn out to be combinations of ordinary addition and modular addition (the group U(1)). This assumes, however, that the underlying cellular automaton has discrete cells. But one can also imagine setting up systems whose states are continuous functions of position. ϕ then defines a mapping from one such function to another. To be analogous to cellular automata one can then require this mapping to be local, in which case if it is continuous it must be just a linear differential operator involving $Derivative[n]$—and at some level its behavior must be fairly simple. (Compare page 161.)

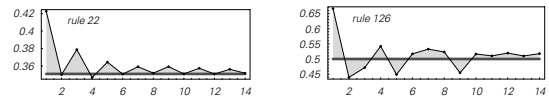
■ **Probabilistic estimates.** One way to get estimates for density and other properties of class 3 cellular automata is to make the assumption that the color of each cell at each step is completely random. And with this assumption, if the overall density of black cells at a particular step is p, then each cell at that step should independently have probability p to be black. This means that for example the probability to find a black cell followed by two white cells is $p (1 - p)^2$. And in general, the probabilities for all 8 possible combinations of 3 cells are given by

```
probs = Apply[Times, Table[IntegerDigits[8 - i, 2, 3],
        {i, 8}] /. {1 → p, 0 → 1 - p}, {1}]
```

In terms of these probabilities the density at the next step in the evolution of cellular automaton with rule number m is then given by

```
Simplify[probs . IntegerDigits[m, 2, 8]]
```

For rule 22, for example, this means that if the density at a particular step is p, then the density on the next step should be $3 p (1 - p)^2$, and the densities on subsequent steps should be obtained by iterating this function. (At least for the 256 elementary cellular automata this iterated map is never chaotic.) The stable density after many steps is then given by $Solve[3 p (1 - p)^2 == p, p]$, so that $p \to 1 - 1/\sqrt{3}$ or approximately 0.42. The actual density for rule 22 is however 0.35095. The reason for the discrepancy is that the probabilities for different cells are in fact correlated. One can systematically include more such correlations by looking at more steps of evolution at once. For two steps, one must consider probabilities for all 32 combinations of 5 cells, and for rule 22 the function becomes $p (1 - p)^2 (2 + 3 p^2)$, yielding density 0.35012; for three steps it is $p (1 - p)^2 (p^4 - 18 p^3 + 41 p^2 - 22 p + 6)$ yielding density 0.379. The plot below shows what happens with more steps: the results seem to converge slowly to the exact result indicated by the gray line.



(For rules 90 and 30 the functions obtained after one step are respectively $2 p (1 - p)$ and $p (2 p^2 - 5 p + 3)$, both of which turn out to imply correct final densities of $1/2$).

Probabilistic approximation schemes like this are often used in statistical physics under the name of mean field theories. In general, such approximations tend to work better for systems in larger numbers of dimensions, where correlations tend to be less important.

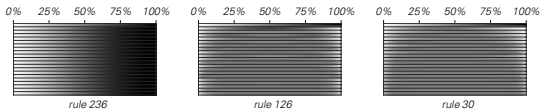
Probabilistic estimates can also be used for other quantities, such as growth rates of difference patterns (see page 949). In most cases, however, buildup of correlations tends to prevent systematic improvement of such approximations.

■ **Density in rule 90.** From the superposition principle above and the number of black cells at step t in a pattern starting from a single black cell (see page 870) one can compute the density after t steps in the evolution of rule 90 with initial conditions of density p to be (see also page 602)

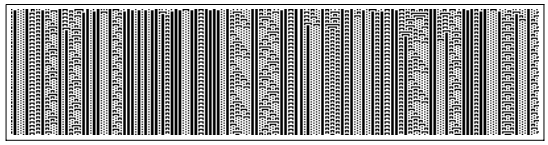
```
1/2 (1 - (1 - 2 p))^(2 ^ DigitCount[t, 2, 1])
```

■ **Densities in other rules.** The pictures below show how the densities on successive steps depend on the initial density. Densities are indicated by gray levels. Initial densities are shown across each picture. Successive steps are shown down the page. Rule 236 is class 2, and the density retains a memory of its initial value. But in the class 3 rules 126 and 30, the densities converge quickly to a fixed value.

Page 339 shows a cellular automaton with very different behavior.

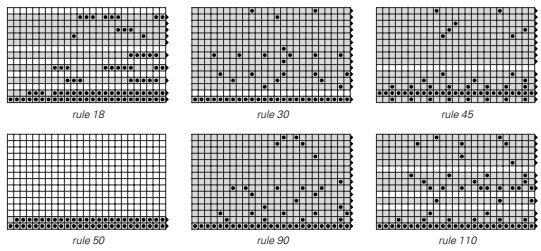


■ **Density oscillations in rule 73.** Although there are always some fluctuations, most rules yield densities that converge more or less uniformly to their final values. One exception is rule 73, which yields densities that continue to oscillate with a period of 3 steps forever. The origin of this phenomenon is that with completely random initial conditions rule 73 evolves to a collection of independent regions, as in the picture below, and many of these regions contain patterns that repeat with period 3. The boundaries between regions come from blocks of even numbers of black cells in the initial conditions, and if one does not allow any such blocks, the density oscillations no longer occur. (See also page 699.)



Special Initial Conditions

■ **Page 267 · Repeating blocks.** The discussion in the main text is mostly about repetition strictly every p steps, and no sooner. (If a system repeats for example every 3 steps, then it is inevitable that it will also repeat in the same way every 6, 9, 12, 15, etc. steps.) Finding configurations in a 1D cellular automaton that repeat with a particular period is equivalent to satisfying the kind of constraints we discussed on page 211. And as described there, if such constraints can be satisfied at all, then it must be possible to satisfy them with a configuration that consists of a repetition of identical blocks. Indeed, for period p, the length of blocks required is at most 2^{2p} (or 2^{2pr} for range r rules).



The pictures at the bottom of the previous column summarize which periods can be obtained with various rules. Periods from 1 to 15 are represented by different rows, with period 1 at the bottom. Within each row a gray bar indicates that a particular period can be obtained with blocks of some length. The black dots indicate specific block sizes up to 25 that work.

In rule 90 (as well as other additive rules such as 60 and 150) any period can occur, but all configurations that repeat must consist of a sequence of identical blocks. For periods up to 10, examples of such blocks in rule 90 are given by the digits of

{0, 40, 24, 2176, 107904, 640, 96, 8421376, 7566031296234863392, 15561286137}

For period 1 the possible blocks are □ and ■□; for period 2 ■□■□□□ and ■■■□□□. The total number of configurations in rule 90 that repeat with any period that divides p is always 4^p.

Rules 30 and 45 (as well as other one-sided additive rules) also have the property that all configurations that repeat must consist of a sequence of identical blocks. The total number of configurations in rule 30 that repeat with periods that divide 1 through 10 are *{3, 3, 15, 10, 8, 99, 18, 14, 30, 163}*. In general for one-sided additive rules the number of such configurations increases for large p like $k^{h_{tx} p}$, where h_{tx} is the spacetime entropy of page 960. (This is the analog of a standard result in dynamical systems theory about expansive homeomorphisms.)

For rules that do not show at least one-sided additivity there can be an infinite number of configurations that repeat with a given period. To find them one considers all possible blocks of length $2pr+1$ and picks out those that after p steps evolve so that their center cell ends up the same color as it was originally. The possible configurations that repeat with period p then correspond to the finite complement language (see page 958) obtained by stringing together these blocks. For $p = 2$, rule 18 leaves 20 of the 32 possible length 5 blocks invariant, but these blocks can only be strung together to yield repetitions of *{a, b, 0, 0}*, where now a and b are not fixed, but in every case can each be either *{1}* or *{0, 1}*.

(See also page 700.)

■ **Localized structures.** See pages 281 and 1118.

■ **2D cellular automata.** As expected from the discussion of constraints on page 942, the problem of finding repeating configurations is much more difficult in two dimensions than in one dimension. Thus for example unlike in 1D there is no guarantee in 2D that among repeating configurations of a particular period there is necessarily one that consists just of a repetitive array of fixed blocks. Indeed, as discussed on page 1139, it is in a sense possible that the only repeating configurations are arbitrarily complex. Note that if one

considers configurations in 2D that consist only of infinitely long stripes, then the problem reduces again to the 1D case. (See also page 349.)

■ **Systems based on numbers.** An iterated map of the kind discussed on page 150 with rule $x \to Mod[a\,x, 1]$ (with rational a) will yield repetitive behavior when its initial condition is a rational number. The same is true for higher-dimensional generalizations such as so-called Anosov maps $\{x, y\} \to Mod[m . \{x, y\}, 1]$. The continued fraction map $x \to Mod[1/x, 1]$ discussed on page 914 becomes repetitive whenever its initial condition is a solution to a quadratic equation.

For a map $x \to f[x]$ where $f[x]$ is a polynomial such as $a\,x\,(1-x)$ the real initial conditions that yield period p are given by

```
Select[x /. Solve[Nest[f, x, p] == x, x], Im[#] == 0 &]
```

For $x \to a\,x\,(1-x)$ the results usually cannot be expressed in terms of explicit radicals beyond period 2. (See page 961.)

■ **Sarkovskii's theorem.** For any iterated map based on a continuous function such as a polynomial it was shown in 1962 that if an initial condition exists that gives period 3, then other initial conditions must exist that give any other period. In general, if a period m is possible then so must all periods n for which $p = \{m, n\}$ satisfies

```
OrderedQ[(Transpose[If[MemberQ[p/#, 1], Map[Reverse,
    {p/#, #}], {#, p/#}]] &)[2 ^ IntegerExponent[p, 2]]]
```

Extensions of this to other types of systems seem difficult to find, but it is conceivable that when viewed as continuous mappings on a Cantor set (see page 869) at least some cellular automata might exhibit similar properties.

■ **Page 269 · Rule emulations.** See pages 702 and 1118.

■ **Renormalization group.** The notion of studying systems by seeing the effect of changing the scale on which one looks at them has been widely used in physics since about 1970, and there is some analogy between this and what I do here with cellular automata. In the lattice version in physics one typically considers what happens to averages over all possible configurations of a system if one does a so-called blocking transformation that replaces blocks of elements by individual elements. And what one finds is that in certain cases—notably in connection with nesting at critical points associated with phase transitions (see page 981)—certain averages turn out to be the same as one would get if one did no blocking but just changed parameters ("coupling constants") in the underlying rules that specify the weighting of different configurations. How such effective parameters change with scale is then governed by so-called renormalization group differential equations. And when one looks at large scales the versions of these equations that arise in practice essentially always show fixed points, whose properties do not depend much on details of the equations—leading to certain universal results across many different underlying systems (see page 983).

What I do in the main text can be thought of as carrying out blocking transformations on cellular automata. But only rarely do such transformations yield cellular automata whose rules are of the same type one started from. And in most cases such rules will not suffice even if one takes averages. And indeed, so far as I can tell, only in those cases where there is fairly simple nested behavior is any direct analog of renormalization group methods useful. (See page 989.)

■ **Page 271 · Self-similarity of additive rules.** The fact that rule 90 can emulate itself can be seen fairly easily from a symbolic description of the rule. Given three cells $\{a_1, a_2, a_3\}$ the rule specifies that the new value of the center cell will be $Mod[a_1 + a_3, 2]$. But given $\{a_1, 0, a_2, 0, a_3, 0\}$ the value after one step is $\{Mod[a_1 + a_2, 2], 0, Mod[a_2 + a_3, 2], 0\}$ and after two steps is again $\{Mod[a_1 + a_3, 2], 0\}$. It turns out that this argument generalizes (by interspersing $k-1$ 0's and going for k steps) to any additive rule based on reduction modulo k (see page 952) so long as k is prime. And it follows that in this case the pattern generated after a certain number of steps from a single non-white cell will always be the same as one gets by going k times that number of steps and then keeping only every k^{th} row and column. And this immediately implies that the pattern must always have a nested form. If k is not prime the pattern is no longer strictly invariant with respect to keeping only every k^{th} row and column—but is in effect still a superposition of patterns with this property for factor of k. (Compare page 870.)

■ **Fractal dimensions.** The total number of nonzero cells in the first t rows of the pattern generated by the evolution of an additive cellular automaton with k colors and weights w (see page 952) from a single initial 1 can be found using

```
g[w_, k_, t_] := Apply[Plus, Sign[NestList[Mod[
    ListCorrelate[w, #, {-1, 1}, 0], k] &, {1}, t - 1]], {0, 1}]
```

The fractal dimension of this pattern is then given by the large m limit of

```
Log[k, g[w, k, k^(m+1)]/g[w, k, k^m]]
```

When k is prime it turns out that this can be computed as

```
d[w_, k_ : 2] := Log[k, Max[Abs[Eigenvalues[With[
  {s = Length[w] - 1}, (Map[Function[u, Map[Count[u, #] &,
    #1]], Map[Flatten[Map[Partition[Take[#, k + s - 1], s, 1] &,
     NestList[Mod[ListConvolve[w, #], k] &, #, k - 1]], 1] &,
     Map[Flatten[Map[{Table[0, {k - 1}], #} &, Append[#,
      0]]] &, #]]] &)[Array[IntegerDigits[#, k, s] &, k^s - 1]]]]]]]
```

For rule 90 one gets $d[\{1, 0, 1\}] = Log[2, 3] \approx 1.58$. For rule 150 $d[\{1, 1, 1\}] = Log[2, 1 + \sqrt{5}\,] \approx 1.69$. (See page 58.) For the other rules on page 952:

```
d[{1, 1, 0, 1, 0}] =
  Log[2, Root[4 + 2 # - 2 #^2 - 3 #^3 + #^4 &, 2]] ≃ 1.72
d[{1, 1, 0, 1, 1}] =
  Log[2, Root[-4 + 4 # + #^2 - 4 #^3 + #^4 &, 2]] ≃ 1.8
```

Other cases include (see page 870):

```
d[{1, 0, 1}, k] = 1 + Log[k, (k + 1)/2]

d[{1, 1, 1}, 3] = Log[3, 6] ≃ 1.63

d[{1, 1, 1}, 5] = Log[5, 19] ≃ 1.83

d[{1, 1, 1}, 7] = Log[7, Root[-27136 + 23280 # -
                  7288 #^2 + 1008 #^3 - 59 #^4 + #^5 &, 1]] ≃ 1.85
```

■ **General associative rules.** With a cellular automaton rule in which the new color of a cell is given by $f[a_1, a_2]$ (compare page 886) it turns out that the pattern generated by evolution from a single non-white cell is always nested if the function f has the property of being associative or *Flat*. In fact, for a system involving k colors the pattern produced will always be essentially just one of the patterns obtained from an additive rule with k or less colors. In general, the pattern produced by evolution for t steps is given by

```
NestList[
  Inner[f, Prepend[#, 0], Append[#, 0], List] &, {a}, t]
```

so that the first few steps yield

```
{a}
{f[0, a], f[a, 0]}
{f[0, f[0, a]], f[f[0, a], f[a, 0]], f[f[a, 0], 0]}
{f[0, f[0, f[0, a]]], f[f[0, f[0, a]], f[f[0, a], f[a, 0]]],
  f[f[f[0, a], f[a, 0]], f[f[a, 0], 0]], f[f[f[a, 0], 0], 0]}
```

If f is *Flat*, however, then the last two lines here become

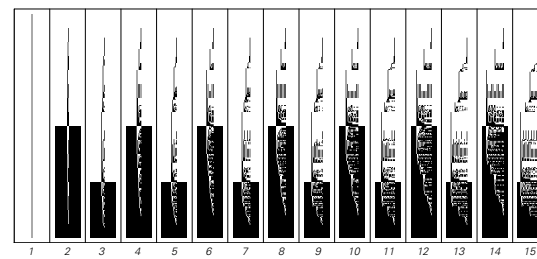
```
{f[0, 0, a], f[0, a, a, 0], f[a, 0, 0]}
{f[0, 0, 0, a], f[0, 0, a, 0, a, a, 0],
  f[0, a, a, 0, a, 0, 0], f[a, 0, 0, 0]}
```

and in general the number of a's that appear in a particular element is given as in Pascal's triangle by a binomial coefficient. If f is commutative (*Orderless*) then all that can ever matter to the value of an element is its number of a's. Yet since there are a finite set of possible values for each element it immediately follows that the resulting pattern must be essentially Pascal's triangle modulo some integer. And even if f is not commutative, the same result will hold so long as $f[0, a] == a$ and $f[a, 0] == a$—since then any element can be reduced to $f[a, a, a, \ldots]$. The result can also be generalized to cellular automata with basic rules involving more than two elements—since if f is *Flat*, $f[a_1, a_2, a_3]$ is always just $f[f[a_1, a_2], a_3]$.

If one starts from more than a single non-0 element, then it is still true that a nested pattern will be produced if f is both associative and commutative. And from the discussion on page 952 this means that any rule that shows generalized additivity must always yield a nested pattern. But if f is not commutative, then even if it is associative, non-nested patterns can be produced. And indeed page 887 shows an example of this based on the non-commutative group S_3. (In general f can correspond to an almost arbitrary semigroup, but with a single initial element only a cyclic subgroup of it is ever explored.)

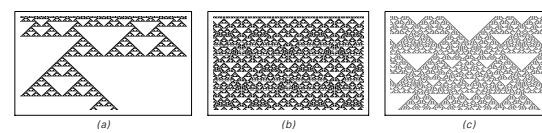
■ **Nesting in rule 45.** As illustrated on page 701, starting from a single black cell on a background of repeated ■□ blocks, rule 45 yields a slanted version of the nested rule 90 pattern.

■ **Uniqueness of patterns.** Starting from a particular initial condition, different rules can often yield the same pattern. The picture below shows in sorted order the configurations obtained at each successive step in the evolution of all 256 elementary cellular automata starting from a single black cell. After a large number of steps, between 94 and 105 distinct individual configurations are obtained, together with 143 distinct complete patterns. (Compare page 1186.)



■ **Square root of rule 30.** Although rule 30 cannot apparently be decomposed into other $k = 2$, $r = 1$ cellular automata, it can be viewed as the square of the $k = 3$, $r = 1/2$ cellular automata with rule numbers 11736, 11739 and 11742.

■ **Page 272 · Nested initial conditions.** The pictures below show patterns generated by rule 90 starting from the nested sequences on page 83. (See page 1091.)



The Notion of Attractors

■ **Page 275 · Discrete systems.** In traditional mathematics mechanical and other systems are assumed continuous, so that for example a pendulum may get exponentially close to

the attractor state where it has stopped, but it will never strictly reach this attractor. In discrete systems like cellular automata, however, there is no problem in explicitly reaching at least simple attractors.

■ **Implementation.** One can represent a network by a list such as *{{1 → 2}, {0 → 3, 1 → 2}, {0 → 3, 1 → 1}}* where each element represents a node whose number corresponds to the position of the element, and for each node there are rules that specify to which nodes arcs with different values lead. Starting with a list of nodes, the nodes reached by following arcs with value *a* for one step are given by

```
NetStep[net_, i_, a_] :=
  Union[ReplaceList[a, Flatten[net[[i]]]]]
```

A list of values then corresponds to a path in the network starting from any node if

```
Fold[NetStep[net, #1, #2] &,
    Range[Length[net]], list] =!= {}
```

Given a set of sequences of values represented by a particular network, the set obtained after one step of cellular automaton evolution is given by

```
NetCAStep[{k_, r_, rtab_}, net_] := Flatten[
  Map[Table[# /. (a_ → s_) :→ rtab[[i k + a + 1]] → k^(2 r) (s - 1) +
        1 + Mod[i k + a, k^(2 r)], {i, 0, k^(2 r) - 1}] &, net], 1]
```

where here elementary rule 126 is specified for example by *{2, 1, Reverse[IntegerDigits[126, 2, 8]]}*. Starting from the set of all possible sequences, as given by

```
AllNet[k_ : 2] := {Thread[Range[k] - 1 → 1]}
```

this then yields for rule 126 the network

```
{{0 → 1, 1 → 2}, {1 → 3, 1 → 4}, {1 → 1, 1 → 2}, {1 → 3, 0 → 4}}
```

It is always possible to find a minimal network that represents a set of sequences. This can be done by first creating a "deterministic" network in which at most one arc of each value comes out of each node, then combining equivalent nodes. The whole procedure can be performed using

```
MinNet[net_, k_ : 2] := Module[{d = DSets[net, k], q, b},
 If[First[d] =!= {}, AllNet[k], q = ISets[b = Map[Table[
     Position[d, NetStep[net, #, a]][[1, 1]], {a, 0, k - 1}] &, d]];
  DeleteCases[MapIndexed[#2[[2]] - 1 → #1 &, Rest[
    Map[Position[q, #][[1, 1]] &, Transpose[Map[#[[Map[
      First, q]]] &, Transpose[b]]], {2}]] - 1, {2}], _ → 0, {2}]]]

DSets[net_, k_ : 2] :=
 FixedPoint[Union[Flatten[Map[Table[NetStep[net, #, a],
       {a, 0, k - 1}] &, #], 1]] &, {Range[Length[net]]}]

ISets[list_] := FixedPoint[Function[g, Flatten[Map[
   Map[Last, Split[Sort[Transpose[{Map[Position[g, #][[1,
     1]] &, list, {2}], Range[Length[list]]}][[#]]], First[#1] ==
    First[#2] &], {2}] &, g], 1]], {{1}, Range[2, Length[list]]}]
```

If *net* has *q* nodes, then in general *MinNet[net]* can have as many as $2^q - 1$ nodes. The form of *MinNet* given here can take up to about n^2 steps to generate a result with *n* nodes; an *n Log[n]* procedure is known. The result from *MinNet* for rule 126 is *{{1 → 3}, {0 → 2, 1 → 1}, {0 → 2, 1 → 3}}*.

In general *MinNet* will yield a network with the property that any allowed sequence of values corresponds to a path which starts from node 1. In the main text, however, the networks allow paths that start at any node. To obtain such trimmed networks one can apply the function

```
TrimNet[net_] :=
 With[{m = Apply[Intersection, Map[FixedPoint[
       Union[#, Flatten[Map[Last, net[[#]], {2}]]] &,
       #] &, Map[List, Range[Length[net]]]]]},
  net[[m]] /. Table[(a_ → m[[i]]) → a → i, {i, Length[m]}]]
```

■ **Finite automata.** The networks discussed in the main text can be viewed as finite automata (also known as finite state machines). Each node in the network corresponds to a state in the automaton, and each arc represents a transition that occurs when a particular value is given as input. *NetCAStep* above in general produces a non-deterministic finite automaton (NDFA) for which a particular sequence of values does not determine a unique path through the network. *MinNet* creates an equivalent DFA, then minimizes this. The Myhill-Nerode theorem ensures that a unique minimal DFA can always be found (though to do so is in general a PSPACE-complete problem).

The total number of distinct minimal finite automata with *k = 2* possible labels for each arc grows with the number of nodes as follows: 3, 7, 78, 1388, … (The simple result $(n + 1)^{n k}$ based on the number of ways to connect up *n* nodes is a significant overestimate because of equivalence between automata with different patterns of connections.)

■ **Regular languages.** The set of sequences obtained by following possible paths through a finite network is often called a regular language, and appears in studies of many kinds of systems. (See page 939.)

■ **Regular expressions.** The sequences in a regular language correspond to those that can be matched by *Mathematica* patterns that use no explicit pattern names. Thus for example *{(0|1)…}* corresponds to all possible sequences of *0*'s and *1*'s, while *{1, 1, (1)…, 0, (0)…}…* corresponds to the sequences that can occur after 2 steps in rule 126 and *{(0)…, 1, {0, (0)…, 1, 1}|{1, (1)…, 0}}…* to those that can occur after 2 steps in rule 110 (see page 279).

■ **Generating functions.** The sequences in a regular language can be thought of as corresponding to products of non-commuting variables that appear as coefficients in a formal power series expansion of a generating function. A basic result is that for regular languages this generating function

is always rational. (Compare the discussion of entropies below.)

■ **History.** Simple finite automata have implicitly been used in electromechanical machines for over a century. A formal version of them appeared in 1943 in McCulloch-Pitts neural network models. (An earlier analog had appeared in Markov chains.) Intensive work on them in the 1950s (sometimes under the name sequential machines) established many basic properties, including interpretation as regular languages and equivalence to regular expressions. Connections to formal power series and to substitution systems (see page 891) were studied in the 1960s. And with the development of the Unix operating system in the 1970s regular expressions began to be widely used in practical computing in lexical analysis (lex) and text searching (ed and grep). Regular languages also arose in dynamical systems theory in the early 1970s under the name of sofic systems.

■ **Page 278 · Network properties.** The number of nodes and connections at step $t > 1$ are: rule 108: 8, 13; rule 128: $2t$, $2t+2$; rule 132: $2t+1$, $3t+3$; rule 160: $(t+1)^2$, $(t+1)(t+3)$; rule 184: $2t$, $3t+1$. For rule 126 the first few cases are

```
{{1, 2}, {3, 5}, {13, 23}, {106, 196}, {2866, 5474}}
```

and for rule 110 they are

```
{{1, 2}, {5, 9}, {20, 38}, {206, 403}, {1353, 2666}}
```

The maximum size of network that can possibly be generated after t steps of cellular automaton evolution is $2^{k^{2rt}} - 1$. For $t = 1$ the maximum of 15 (with 29 connections) is achieved for 16 out of the 256 possible elementary rules, including 22, 37, 73, 94, 104, 122, 146 and 164. For $t = 2$, rule 22 gives the largest network, with 280 nodes and 551 arcs. The $k = 2$, $r = 2$ totalistic rule with code 20 gives a network with 65535 nodes after just 1 step. Note that rules which yield maximal size networks are in a sense close to allowing all possible sequences. (The shortest excluded block for code 20 is of length 36.)

■ **Excluded blocks.** As the evolution of a cellular automaton proceeds, the set of sequences that can appear typically shrinks, with progressively more blocks being excluded. In some cases the set of allowed sequences forms a so-called finite complement language (or subshift of finite type) that can be characterized completely just by saying that some finite set of blocks are excluded. But whenever the overall behavior is at all complex, there tend to be an infinite set of blocks excluded, making it necessary to use a network of the kind discussed in the main text. If there are n nodes in such a network, then if any blocks are excluded, the shortest one of them must be of length less than n. And if there are going to be an infinite number of excluded blocks, there must be additional excluded blocks with lengths between n and $2n$. In rule 126, the lengths of the shortest newly excluded blocks on successive steps are 0, 3, 12, 13, 14, 14, 17, 15. It is common to see such lengths progressively increase, although in principle they can decrease by as much as $2r$ from one step to the next. (As an example, in rule 54 they decrease from 9 to 7 between steps 4 and 5.)

■ **Entropies and dimensions.** There are 2^n sequences possible for n cells that are each either black or white. But as we have seen, in most cellular automata not all these sequences can occur except in the initial conditions. The number of sequences s_n of length n that can actually occur is given by

```
Apply[Plus, Flatten[MatrixPower[m, n]]]
```

where the adjacency matrix m is given by

```
MapAt[1 + # &, Table[0, {Length[net]}, {Length[net]}],
  Flatten[MapIndexed[{First[#2], Last[#1]} &, net, {2}], 1]]
```

For rule 32, for example, s_n turns out to be $\mathrm{Fibonacci}[n+3]$, so that for large n it is approximately $\mathrm{GoldenRatio}^n$. For any rule, s_n for large n will behave like κ^n, where κ is the largest eigenvalue of m. For rule 126 after 1 step, the characteristic polynomial for m is $x^3 - 2x^2 + x - 1$, giving $\kappa \simeq 1.755$. After 2 steps, the polynomial is

$$x^{13} - 4x^{12} + 6x^{11} - 5x^{10} + 3x^9 - 3x^8 + 5x^7 - 3x^6 - x^5 + 4x^4 - 2x^3 + x^2 - x + 1$$

giving $\kappa \simeq 1.732$. Note that κ is always an algebraic number—or strictly a so-called Perron number, obtained from a polynomial with leading coefficient 1. (Note that any possible Perron number can be obtained for example from some finite complement language.)

It is often convenient to fit s_n for large n to the form 2^{hn}, where h is the so-called spatial (topological) entropy (see page 1084), given by $\mathrm{Log}[2, \kappa]$. The value of this for successive t never increases; for the first 3 steps in rule 126 it is for example approximately 1, 0.811, 0.793. The exact value of h after more steps tends to be very difficult to find, and indeed the question of whether its limiting value after infinitely many steps satisfies a given bound—say even being nonzero—is in general undecidable (see page 1138).

If one associates with each possible sequence of length n a number $\mathrm{Sum}[a_i 2^{-i}, \{i, n\}]$, then the set of sequences that actually occur at a given step form a Cantor set (see note below), whose Hausdorff dimension turns out to be exactly h.

■ **Cycles and zeta functions.** The number of sequences of n cells that can occur repeatedly, corresponding to cycles in the network, is given in terms of the adjacency matrix m by $\mathrm{Tr}[\mathrm{MatrixPower}[m, n]]$. These numbers can also be obtained as the coefficients of x^n in the series expansion of

$x\,\partial_x \text{Log}[\zeta[m, x]]$, with the so-called zeta function, which is always a rational function of x, given by

```
ζ[m_, x_] := 1/Det[IdentityMatrix[Length[m]] - m x]
```

and corresponds to the product over all cycles of $1/(1-x^n)$.

■ **2D generalizations.** Above 1D no systematic method seems to exist for finding exact formulas for entropies (as expected from the discussion at the end of Chapter 5). Indeed, even working out for large n how many of the 2^{n^2} possible configurations of a $n \times n$ grid of black and white squares contain no pair of adjacent black cells is difficult. Fitting the result to $2^{h n^2}$ one finds $h \approx 0.589$, but no exact formula for h has ever been found. With hexagonal cells, however, the exact solution of the so-called hard hexagon lattice gas model in 1980 showed that $h \approx 0.481$ is the logarithm of the largest root of a degree 12 polynomial. (The solution of the so-called dimer problem in 1961 also showed that for complete coverings of a square grid by 2-cell dominoes $h = \text{Catalan}/(\pi\,\text{Log}[2]) \approx 0.421$.)

■ **Probability-based entropies.** This section has concentrated on characterizing what sequences can possibly occur in 1D cellular automata, with no regard to their probability. It turns out to be difficult to extend the discussion of networks to include probabilities in a rigorous way. But it is straightforward to define versions of entropy that take account of probabilities—and indeed the closest analog to the usual entropy in physics or information theory is obtained by taking the probabilities $p[i]$ for the k^n blocks of length n (assuming k colors), then constructing

```
-Limit[Sum[p[i] Log[k, p[i]], {i, k^n}]/n, n → ∞]
```

I have tended to call this quantity measure entropy, though in other contexts, it is often just called entropy or information, and is sometimes called information dimension. The quantity

```
Limit[Sum[UnitStep[p[i]], {i, k^n}]/n, n → ∞]
```

is the entropy discussed in the notes above, and is variously called set entropy, topological entropy, capacity and fractal dimension. An example of a generalization is the quantity given for blocks of size n by

```
h[q_, n_] := Log[k, Sum[p[i]^q, {i, k^n}]]/(n (q - 1))
```

where $q = 0$ yields set entropy, the limit $q \to 1$ measure entropy, and $q = 2$ so-called correlation entropy. For any q the maximum $h[q, n] == 1$ occurs when all $p[i] == k^{-n}$. It is always the case that $h[q+1, n] \le h[q, n]$. The $h[q]$ have been introduced in almost identical form several times, notably by Alfréd Rényi in the 1950s as information measures for probability distributions, in the 1970s as part of the thermodynamic formalism for dynamical systems, and in the 1980s as generalized dimensions for multifractals. (Related objects have also arisen in connection with Hölder exponents for discontinuous functions.)

■ **Entropy estimates.** Entropies $h[n]$ computed from blocks of size n always decrease with n; the quantity $n\,h[n]$ is always convex (negative second difference) with respect to n. At least at a basic level, to compute topological entropy one needs in effect to count every possible sequence that can be generated. But one can potentially get an estimate of measure entropy just by sampling possible sequences. One problem, however, is that even though such sampling may give estimates of probabilities that are unbiased (and have Gaussian errors), a direct computation of measure entropy from them will tend to give a value that is systematically too small. (A potential way around this is to use the theory of unbiased estimators for polynomials just above and below $p\,\text{Log}[p]$.)

■ **Nested structure of attractors.** Associating with each sequence of length n (and k possible colors for each element) a number $\text{Sum}[a[i]\,k^{-i}, \{i, n\}]$, the set of sequences that occur in the limit $n \to \infty$ forms a Cantor set. For $k = 3$, the set of sequences where the second color never occurs corresponds to the standard middle-thirds Cantor set. In general, whenever the possible sequences correspond to paths through a finite network, it follows that the Cantor set obtained has a nested structure. Indeed, constructing the Cantor set in levels by considering progressively longer sequences is effectively equivalent to following successive steps in a substitution system of the kind discussed on page 83. (To see the equivalence first set up s kinds of elements in the substitution system corresponding to the s nodes in the network.) Note that if the possible sequences cannot be described by a network, then the Cantor set obtained will inevitably not have a strictly nested form.

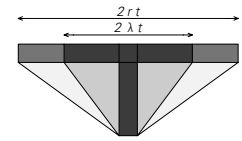
■ **Surjectivity and injectivity.** One can think of a cellular automaton rule as a mapping (endomorphism) from the space of possible states of the cellular automaton to itself. (See page 869.) Usually this mapping is contractive, so that not all the states which appear as input to the mapping can also appear as output. But in some cases, the mapping is surjective or onto, meaning that any state which appears as input can also appear as output. Among $k = 2$, $r = 1$ elementary cellular automata it turns out that this happens precisely for those 30 rules that are additive with respect to at least the first or last position on which they depend (see pages 601 and 1087); this includes both rules 90 and 150 and rules 30 and 45. With $k = 2$, $r = 2$ there are a total of 4,294,967,296 possible rules. Out of these 141,884 are onto—and 11,388 of these turn out not to be additive with respect to any position. The easiest way to test whether a particular rule is onto seems to be essentially just to construct the minimal finite automaton discussed on page 957. The onto

$k = 2$, $r = 2$ rules were found in 1961 in a computer study by Gustav Hedlund and others; they later apparently provided input in the design of S-boxes for DES cryptography (see page 1085).

Even when a cellular automaton mapping is surjective, it is still often many-to-one, in the sense that several input states can yield the same output state. (Thus for example additive rules such as 90 and 150, as well as one-sided additive rules such as 30 and 45 are always 4-to-1.) But some surjective rules also have the property of being injective, so that different input states always yield different output states. And in such a case the cellular automaton mapping is one-to-one or bijective (an automorphism). This is equivalent to saying that the rule is reversible, as discussed on page 1017.

(In 2D such properties are in general undecidable; see page 1138.)

■ **Temporal sequences.** So far we have considered possible sequences of cells that can occur at a particular step in the evolution of a cellular automaton. But one can also consider sequences formed from the color of a particular cell on a succession of steps. For class 1 and 2 cellular automata, there are typically only a limited number of possible sequences of any length allowed. And when the length is large, the sequences are almost always either just uniform or repetitive. For class 3 cellular automata, however, the number of sequences of length n typically grows rapidly with n. For additive rules such as 60 and 90, and for partially additive rules such as 30 and 45, any possible sequence can occur if an appropriate initial condition is given. For rule 18, it appears that any sequence can occur that never contains more than one adjacent black cell. I know of no general characterization of temporal sequences analogous to the finite automaton one used for spatial sequences above. However, if one defines the entropy or dimension h_t for temporal sequences by analogy with the definition for spatial sequences above, then it follows for example that $h_t \le 2 \lambda h_x$, where λ is the maximum rate at which changes grow in the cellular automaton. The origin of this inequality is indicated in the picture below. The basic idea is that the size of the region that can affect a given cell in the course of t steps is $2 \lambda t$. But for large sizes x the total number of possible configurations of this region is $k^{h_x x}$. (Inequalities between entropies and Lyapunov exponents are also common in dynamical systems based on numbers, but are more difficult to derive.) Note that in effect, h_x gives the information content of spatial sequences in units of bits per unit distance, while h_t gives the corresponding quantity for temporal sequences in units of bits per unit time. (One can also define directional entropies based on sequences at different slopes; the values of such entropies tend to change discontinuously when the slope crosses λ.)



Different classes of cellular automata show characteristically different entropy values. Class 1 has $h_x = 0$ and $h_t = 0$. Class 2 has $h_x \neq 0$ but $h_t = 0$. Class 3 has $h_x \neq 0$ and $h_t \neq 0$. Class 4 tends to show fluctuations which prevent definite values of h_x and h_t from being found.

■ **Spacetime patches.** One can imagine defining entropies and dimensions associated with regions of any shape in the spacetime history of a cellular automaton. As an example, one can consider patches that extend x cells across in space and t cells down in time. If the color of every cell in such a patch could be chosen independently then there would be $k^{t x}$ possible configurations of the complete patch. But in fact, having just specified a block of length $x + 2 r t$ in the initial conditions, the cellular automaton rule then uniquely determines the color of every cell in the patch, allowing a total of at most $s[t, x] = k^{x+2 r t}$ configurations. One can define a topological spacetime entropy h_{tx} as

$$Limit[Limit[Log[k, s[t, x]]/t, t \to \infty], x \to \infty]$$

and a measure spacetime entropy h_{tx}^{μ} by replacing s with $p\,Log[p]$. In general, $h_t \le h_{tx} \le 2 \lambda h_x$ and $h \le 2 r h_t$. For additive rules like rule 90 and rule 150 every possible configuration of the initial block leads to a different configuration for the patch, so that $h_{tx} = 2 r = 2$. But for other rules many different configurations of the initial block can lead to the same configuration for the patch, yielding potentially much smaller values of h_{tx}. Just as for most other entropies, when a cellular automaton shows complicated behavior it tends to be difficult to find much more than upper bounds for h_{tx}. For rule 30, $h_{tx}^{\mu} < 1.155$, and there is some evidence that its true value may actually be 1. For rule 18 it appears that $h_{tx}^{\mu} = 1$, while for rule 22, $h_{tx}^{\mu} < 0.915$ and for rule 54 $h_{tx}^{\mu} < 0.25$.

■ **History.** The analysis of cellular automata given in this section is largely as I worked it out in the early 1980s. Parts of it, however, are related to earlier investigations, particularly in dynamical systems theory. Starting in the 1930s the idea of symbolic dynamics began to emerge, in which one partitions continuous values in a system into bins represented by discrete symbols, and then looks at the sequences of such symbols that can be produced by the evolution of the system. In connection with early work on

chaos theory, it was noted that there are some systems that act like "full shifts", in the sense that the set of sequences they generate includes all possibilities—and corresponds to what one would get by starting with any possible number, then successively shifting digits to the left, and at each step picking off the leading digit. It was noted that some systems could also yield various kinds of subshifts that are subsets of full shifts. But since—unlike in cellular automata—the symbol sequences being studied were obtained by rather arbitrary partitionings of continuous values, the question arose of what effect using different partitionings would have. One approach was to try to find invariants that would remain unchanged in different partitionings—and this is what led, for example, to the study of topological entropy in the 1960s. Another approach was to look at actual possible transformations between partitionings, and this led from the late 1950s to various studies of so-called shift-commuting block maps (or sliding-block codes)—which turn out to be exactly 1D cellular automata (see page 878). The locality of cellular automaton rules was thought of as making them the analog for symbol sequences of continuous functions for real numbers (compare page 869). Of particular interest were invertible (reversible) cellular automaton rules, since systems related by these were considered conjugate or topologically equivalent.

In the 1950s and 1960s—quite independent of symbolic dynamics—there was a certain amount of work done in connection with ideas about self-reproduction (see page 876) on the question of what configurations one could arrange to produce in 1D and 2D cellular automata. And this led for example to the study of so-called Garden of Eden states that can appear only in initial conditions—as well as to some general discussion of properties such as surjectivity.

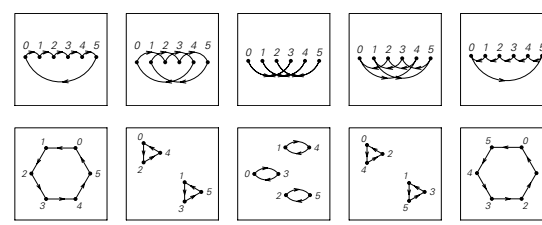
When I started working on cellular automata in the early 1980s I wanted to see how far one could get by following ideas of statistical mechanics and dynamical systems theory and trying to find global characterizations of the possible behavior of individual cellular automata. In the traditional symbolic dynamics of continuous systems it had always been assumed that meaningful quantities must be invariant under continuous invertible transformations of symbol sequences. It turns out that the spacetime (or "invariant") entropy defined in the previous note has this property. But the spatial and temporal entropies that I introduced do not—and indeed in studying specific cellular automata there seems to be no particular reason why such a property would be useful.

▪ **Attractors in systems based on numbers.** Particularly for systems based on ordinary differential equations (see page 922) a geometrical classification of possible attractors exists. There are fixed points, limit cycles and so-called strange attractors. (The first two of these were identified around the end of the 1800s; the last with clarity only in the 1960s.) Fixed points correspond to zero-dimensional subsets of the space of possible states, limit cycles to one-dimensional subsets (circles, solenoids, etc.). Strange attractors often have a nested structure with non-integer fractal dimension. But even in cases where the behavior obtained with a particular random initial condition is very complicated the structure of the attractor is almost invariably quite simple.

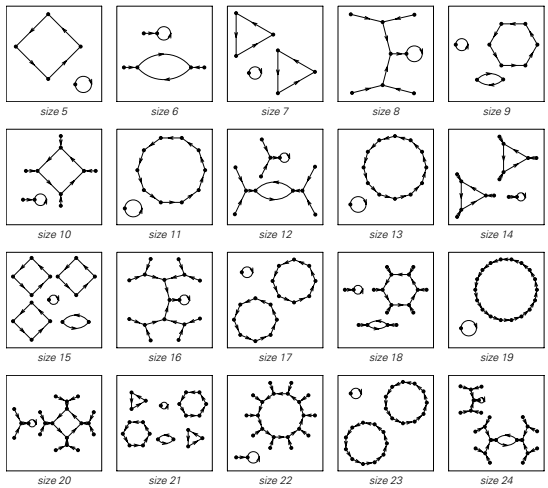
▪ **Iterated maps.** For maps of the form $x \to a\,x\,(1-x)$ discussed on page 920 the attractor for small a is a fixed point, then a period 2 limit cycle, then period 4, 8, 16, etc. There is an accumulation of limit cycles at $a \simeq 3.569946$ where the system has a special nested structure. (See pages 920 and 955.)

▪ **Attractors in Turing machines.** In theoretical studies Turing machines are often set up so that if their initial conditions follow a particular formal grammar (see page 938) then they evolve to "accept" states—which can be thought of as being somewhat like attractors.

▪ **Systems of limited size.** For any system with a limited total number of states, it is possible to create a finite network that gives a global representation of the behavior of the system. The idea of this network (which is very different from the finite automata networks discussed above) is to have each node represent a complete state of the system. At each step in the evolution of the system, every state evolves to some new state, and this process is represented in the network by an arc that joins each node to a new node. The picture below gives the networks obtained for systems of the kind shown on page 255. Each node is labelled by a possible position for the dot. In the first case shown, starting for example at position 4 the dot then visits positions 5, 0, 1, 2 and so on, at each step going from one node in the network to the next.

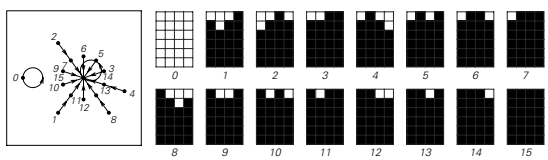


The pictures below give networks obtained from the system shown on page 257 for various values of n. For odd n, the networks consist purely of cycles. But for even n, there are also trees of states that lead to these cycles.

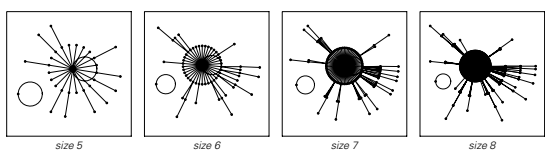


In general, any network that represents the evolution of a system with definite rules will have the same basic form. There are cycles which contain states that are visited repeatedly, and there can also be trees that represent transient states that can each only ever occur at most once in the evolution of the system.

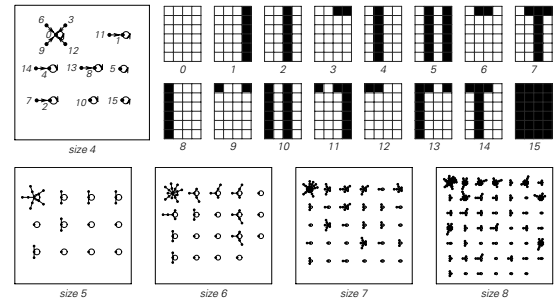
The picture below shows the network obtained from a class 1 cellular automaton (rule 254) with 4 cells and thus 16 possible states. All but one of these 16 states evolve after at most two steps to state 15, which corresponds to all cells being black.



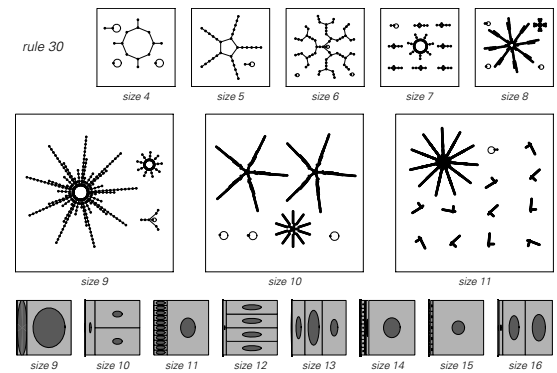
The pictures below show networks obtained when more cells are included in the cellular automaton above. The same convergence to a single fixed point is observed.

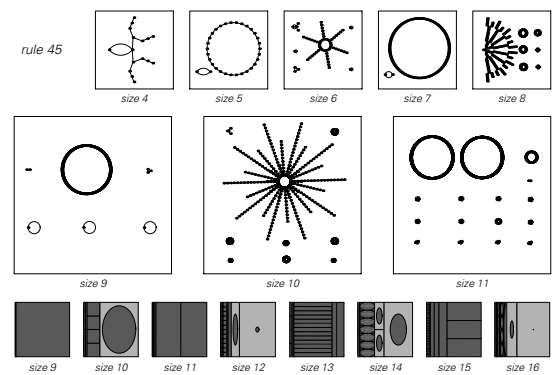


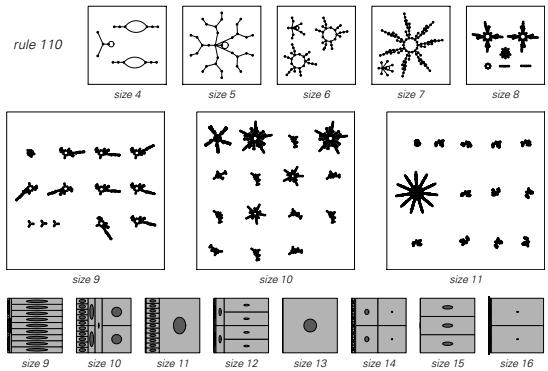
The pictures below give corresponding results for a class 2 cellular automaton (rule 132). The number of distinct cycles now increases with the size of the system. (As discussed below, identical pieces of the network are often related by symmetries of the underlying cellular automaton system.)



In class 3, larger cycles are usually obtained, and often the whole network is dominated by a single largest cycle. The second set of pictures below summarize the results for some larger cellular automata. Each distinct region corresponds to a disjoint part of the network, with the area of the region being proportional to the number of nodes involved. The dark blobs represent cycles. (See page 1087.)







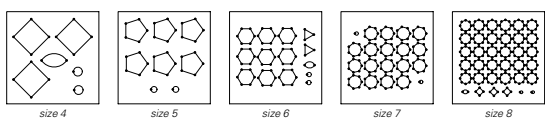
For large sizes there is a rough correspondence with the infinite size case, but many features are still different. (To recover correct infinite size results one must increase size while keeping the number of steps of evolution fixed; the networks shown above, however, effectively depend on arbitrarily many steps of evolution.)

■ **Symmetries.** Many of the networks above contain large numbers of identical pieces. Typically the reason is that the states in each piece are shifted copies of each other, and in such cases the number of pieces will be a divisor of n. (See page 950.) If the underlying cellular automaton rule exhibits an invariance—say under reflection in space or permutation of colors—this will also often lead to the presence of identical pieces in the final network, corresponding to cosets of the symmetry transformation.

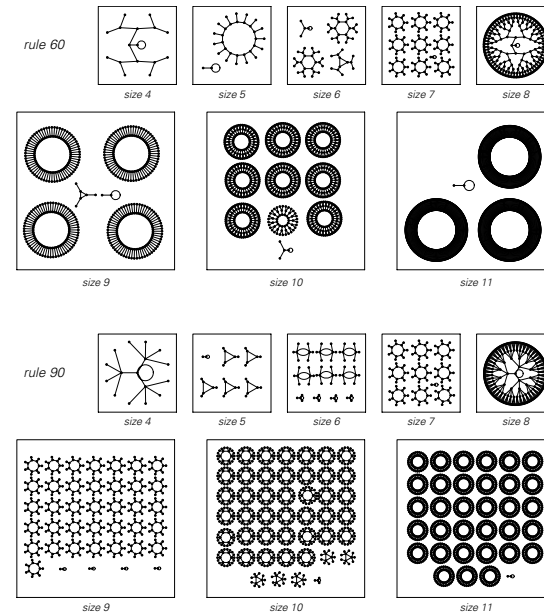
■ **Shift rules.** The pictures below show networks obtained with rule 170, which just shifts every configuration one position to the left at each step. With any such shift rule, all states lie on cycles, and the lengths of these cycles are the divisors of the size n. Every cycle corresponds in effect to a distinct necklace with n beads; with k colors the total number of these is

```
Apply[Plus, (EulerPhi[n/#] k^# &)[Divisors[n]]]/n
```

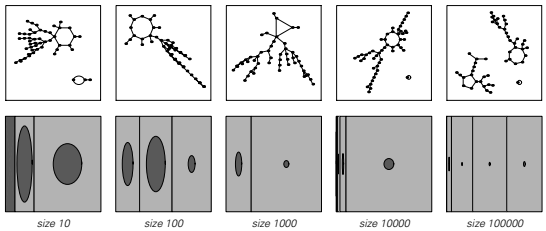
The number of cycles of length exactly m is $s[m, k]/m$, where $s[m, k]$ is defined on page 950. For prime k, each cycle (except all 0's) corresponds to a term in the product $Factor[x^{k^n-1} - 1, Modulus \to k]$. (See page 975.)



■ **Additive rules.** The pictures below show networks obtained for the additive cellular automata with rules 60 and 90. The networks are highly regular and can be analyzed by the algebraic methods mentioned on page 951. The lengths of the longest cycles are given on page 951; all other cycles must have lengths which divide these. Rooted at every state on each cycle is an identical structure. When the number of cells n is odd this structure consists of a single arc, so that half of all states lie on cycles. When n is even, the structure is a balanced tree of depth `2^IntegerExponent[n, 2]` and degree 2 for rule 60, and depth `2^IntegerExponent[n/2, 2]` and degree 4 for rule 90. The total fraction of states on cycles is in both cases `2^(-2^IntegerExponent[n, 2])`. States with a single black cell are always on the longest cycles. The state with no black cells always forms a cycle of length 1.

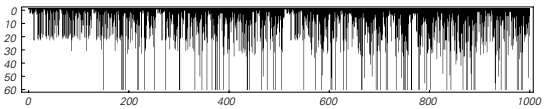


■ **Random networks.** The pictures below show networks in which each of a set of n nodes has as its successor a node that is chosen at random from the set. The total number of possible such networks is n^n. For large n, the average number of distinct cycles in all such networks is $Sqrt[\pi/2] Log[n]$, and the average length of these cycles is $Sqrt[\pi n/8]$. The average fraction of nodes that have no predecessor is $(1 - 1/n)^n$ or $1/e$ in the limit $n \to \infty$. Note that processes such as cellular automaton evolution do not yield networks whose properties are particularly close to those of purely random ones.

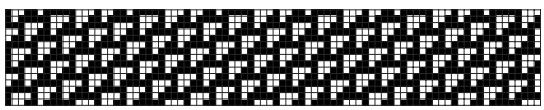


Structures in Class 4 Systems

■ **Page 283 · Survival data.** The number of steps for which the pattern produced by each of the first 1000 initial conditions in code 20 survive are indicated in the picture below. 72 of these initial conditions lead to persistent structures. Among the first million initial conditions, 60,171 lead to persistent structures and among the first billion initial conditions the number is 71,079,205.



■ **Page 290 · Background.** At every step the background pattern in rule 110 consists of repetitions of the block *b = {1, 0, 0, 1, 1, 0, 1, 1, 1, 1, 1, 0, 0, 0}*, as shown in the picture below. On step *t* the color of a cell at position *x* is given by *b[[Mod[x + 4 t, 14] + 1]]*.



■ **Page 292 · Structures.** The persistent structures shown can be obtained from the following *{n, w}* by inserting the sequences *IntegerDigits[n, 2, w]* between repetitions of the background block *b*:

```
{{152, 8}, {183, 8}, {18472955, 25}, {732, 10}, {129643, 18},
  {0, 5}, {152, 13}, {39672, 21}, {619, 15}, {44, 7},
  {334900605644, 39}, {8440, 15}, {248, 9}, {760, 11}, {38, 6}}
```

The repetition periods and distances moved in each period for the structures are respectively

```
{{4, -2}, {12, -6}, {12, -6}, {42, -14},
  {42, -14}, {15, -4}, {15, -4}, {15, -4}, {15, -4},
  {30, -8}, {92, -18}, {36, -4}, {7, 0}, {10, 2}, {3, 2}}
```

Note that the periodicity of the background forces all rule 110 structures to have periods and distances given by *{4, -2} r + {3, 2} s* where *r* and *s* are non-negative integers. Extended versions of structures (d)–(i) can be obtained by collisions with (a). Extended versions of (b) and (c) can be obtained from

```
Flatten[{IntegerDigits[1468, 2], Table[
    IntegerDigits[102524348, 2], {n}], IntegerDigits[v, 2]}]
```

where *n* is a non-negative integer and *v* is one of

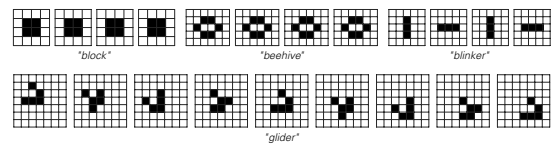
```
{1784, 801016, 410097400, 13304, 6406392, 3280778648}
```

Note that in most cases multiple copies of the same structure can travel next to each other, as seen on page 290.

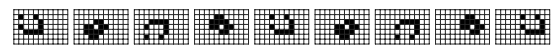
■ **Page 293 · Glider gun.** The initial conditions shown correspond to *{n, w} = {1339191737336, 41}*.

■ **Page 294 · Collisions.** A fundamental result is that the sum of the widths of all persistent structures involved in an interaction must be conserved modulo 14.

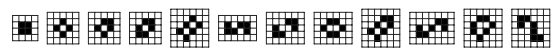
■ **The Game of Life.** The 2D cellular automaton described on page 949 supports a whole range of persistent structures, many of which have been given quaint names by its enthusiasts. With typical random initial conditions the most common structures to occur are:



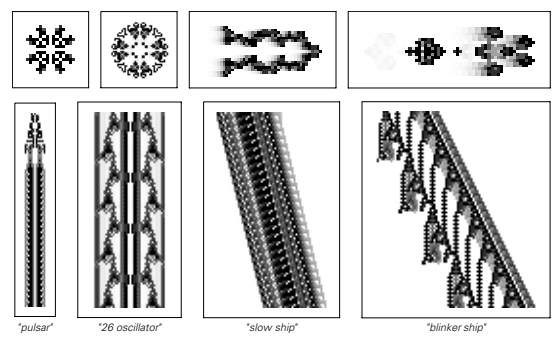
The next most common moving structure is the so-called "spaceship":



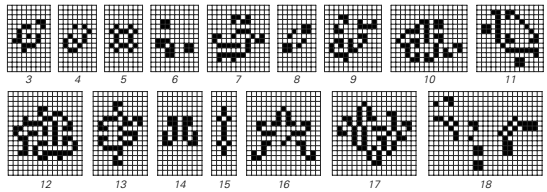
The complete set of structures with less than 8 black cells that remain unchanged at every step in the evolution are:



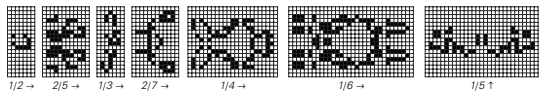
More complicated repetitive and moving structures are shown in the pictures below. If one looks at the history of a single row of cells, it typically looks much like the complete histories we have seen in 1D class 4 cellular automata.



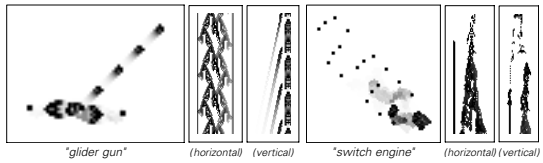
Structures with all repetition periods up to 18 have been found in Life; examples are shown in the pictures below.



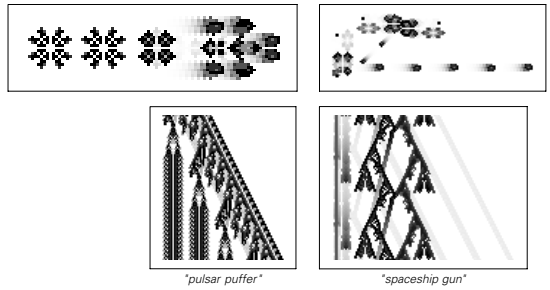
Persistent structures with various speeds in the horizontal and vertical direction have also been found, as shown below.



The first example of unbounded growth in Life was the so-called "glider gun", discovered by William Gosper in 1970 and shown below. This object emits a glider every 30 steps. The simplest known initial condition which leads to a glider gun contains 21 black cells. The so-called "switch engine" discovered in 1971 generates unbounded growth by leaving a trail behind when it moves; it is now known that it can be obtained from an initial condition with 10 black cells, or black cells in just a 5×5 or 39×1 region. It is also known that from less than 10 initial black cells no unbounded growth is ever possible.

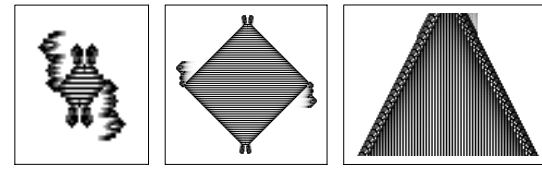


Many more elaborate structures similar to the glider gun were found in the 1970s and 1980s; two are illustrated below.

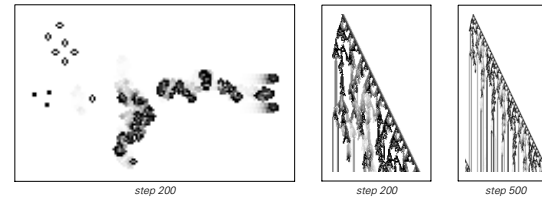


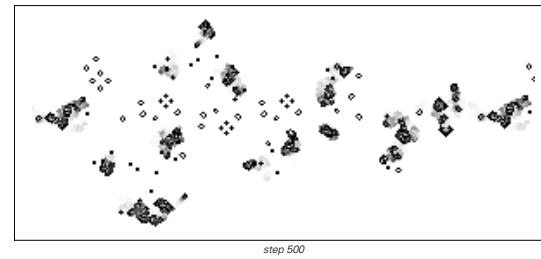
A simpler kind of unbounded growth occurs if one starts from an infinite line of black cells. In that case, the evolution is effectively 1D, and turns out to follow elementary rule 22, thus producing the infinitely growing nested pattern shown on page 263.

For a long time it was not clear whether Life would support any kind of uniform unbounded growth from a finite initial region of black cells. However, in 1993 David Bell found starting from 206 black cells the "spacefiller" shown below. This object is closely analogous to those shown for code 1329 on page 287.



As in other class 4 cellular automata, there are structures in Life which take a very long time to settle down. The so-called "puffer train" below which starts from 23 black cells becomes repetitive with period 140 only after more than 1100 steps.





■ **Other 2D cellular automata.** The general problem of finding persistent structures is much more difficult in 2D than in 1D, and there is no completely general procedure, for example, for finding all structures of any size that have a certain repetition period.

■ **Structures in Turing machines.** See page 888.