EXCERPTED FROM

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A NEW KIND OF SCIENCE

SECTION 9.4

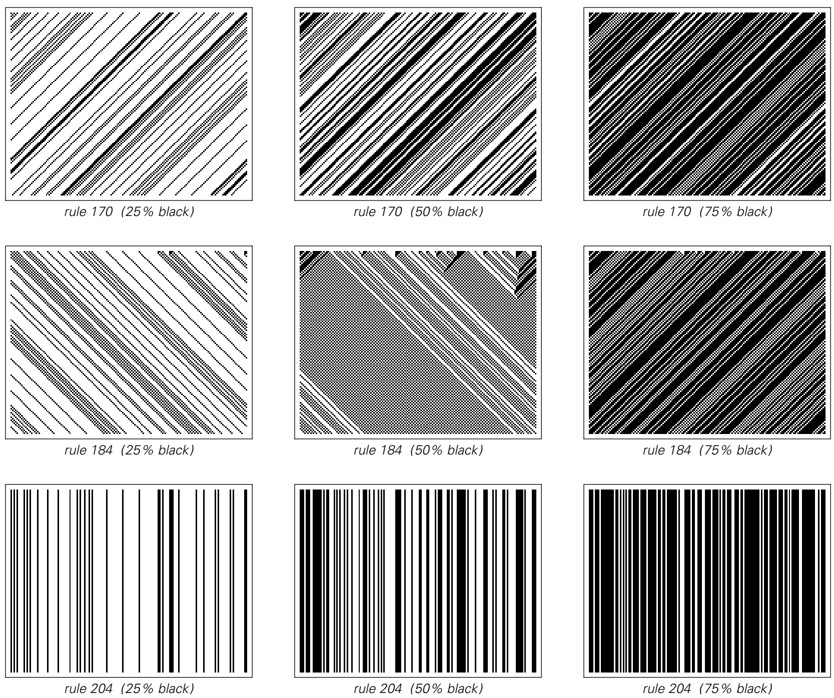
Conserved Quantities and Continuum Phenomena

Conserved Quantities and Continuum Phenomena

Reversibility is one general feature that appears to exist in the basic laws of physics. Another is conservation of various quantities—so that for example in the evolution of any closed physical system, total values of quantities like energy and electric charge appear always to stay the same.

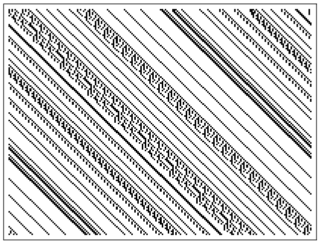
With most rules, systems like cellular automata do not usually exhibit such conservation laws. But just as with reversibility, it turns out to be possible to find rules that for example conserve the total number of black cells appearing on each step.

Among elementary cellular automata with just two colors and nearest-neighbor rules, the only types of examples are the fairly trivial ones shown in the pictures below.

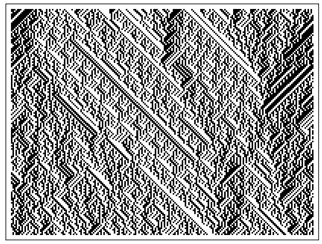


Elementary cellular automata whose evolution conserves the total number of black cells. The behavior of the rules shown here is simple enough that in each case it is fairly obvious how the number of black cells manages to stay the same on every step.

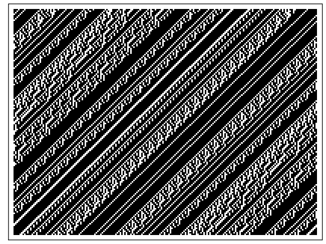
But with next-nearest-neighbor rules, more complicated examples become possible, as the pictures below demonstrate.



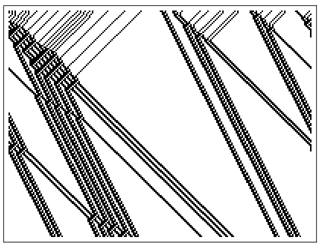
rule 3450663328 (25% black)



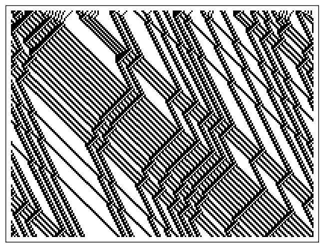
rule 3450663328 (50% black)



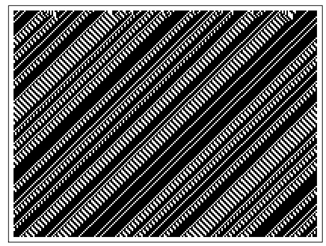
rule 3450663328 (75% black)



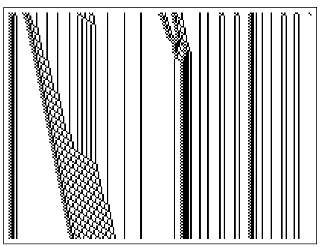
rule 3484741764 (25% black)



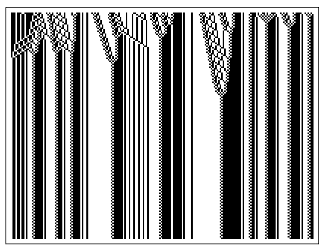
rule 3484741764 (50% black)



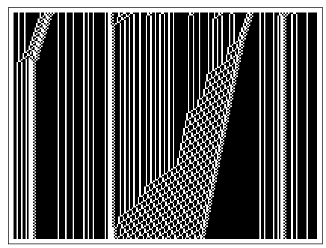
rule 3484741764 (75% black)



rule 3822644248 (25% black)



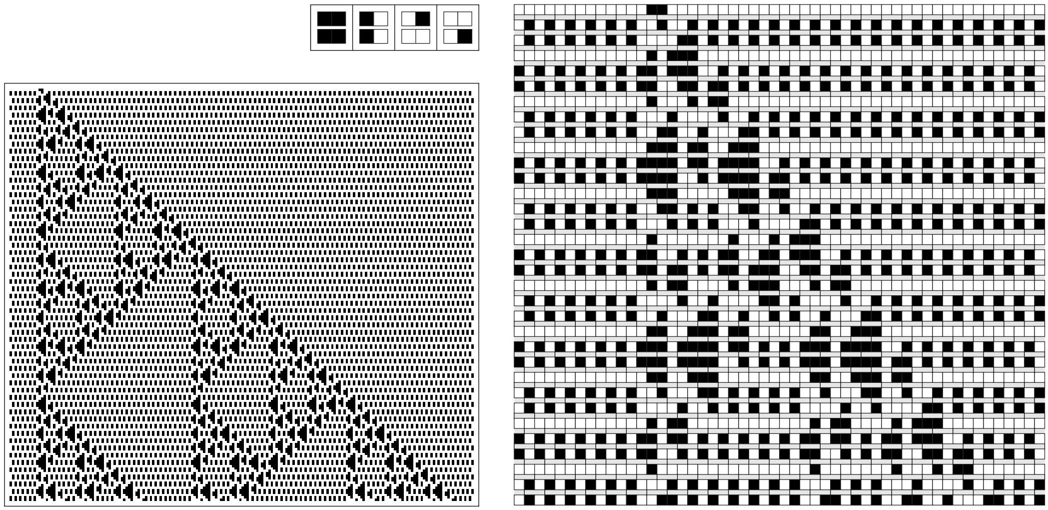
rule 3822644248 (50% black)



rule 3822644248 (75% black)

Examples of cellular automata with next-nearest-neighbor rules whose evolution conserves the total number of black cells. Even though it is not immediately obvious by eye, the total number of black cells stays exactly the same on each successive step in each picture. Among the 4,294,967,296 possible next-neighbor rules, only 428 exhibit the kind of conservation property shown here.

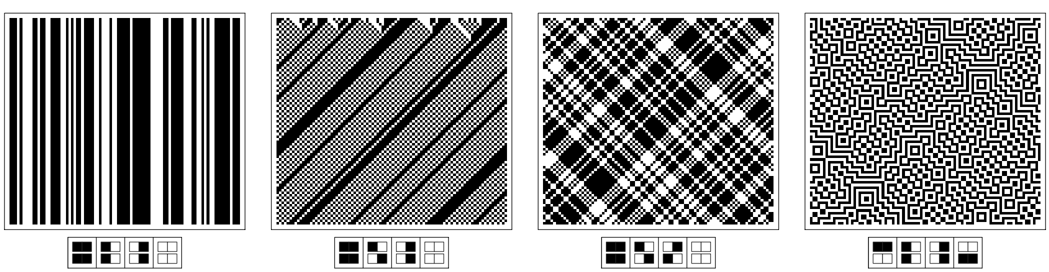
One straightforward way to generate collections of systems that will inevitably exhibit conserved quantities is to work not with ordinary cellular automata but instead with block cellular automata. The basic idea of a block cellular automaton is illustrated at the top of the next page. At each step what happens is that blocks of adjacent cells are replaced by other blocks of the same size according to some definite rule. And then on successive steps the alignment of these blocks shifts by one cell.



An example of a block cellular automaton. The system works by partitioning the sequence of cells that exists at each step into pairs, then replacing these pairs by other pairs according to the rule shown. The choice of whether to pair a cell with its left or right neighbor alternates on successive steps. Like many block cellular automata, the system shown is reversible, since in the rule each pair has a unique predecessor. It does not, however, conserve the total number of black cells.

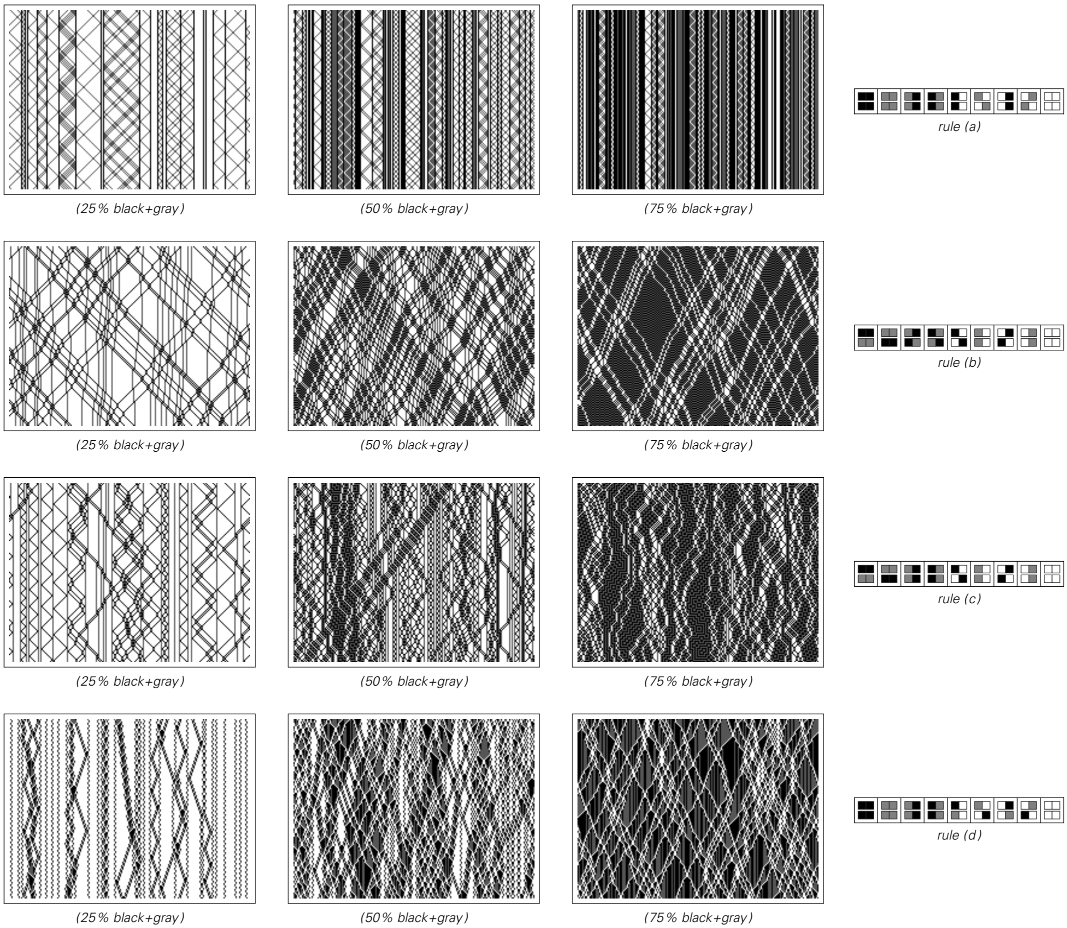
And with this setup, if the underlying rules replace each block by one that contains the same number of black cells, it is inevitable that the system as a whole will conserve the total number of black cells.

With two possible colors and blocks of size two the only kinds of block cellular automata that conserve the total number of black cells are the ones shown below—and all of these exhibit rather trivial behavior.



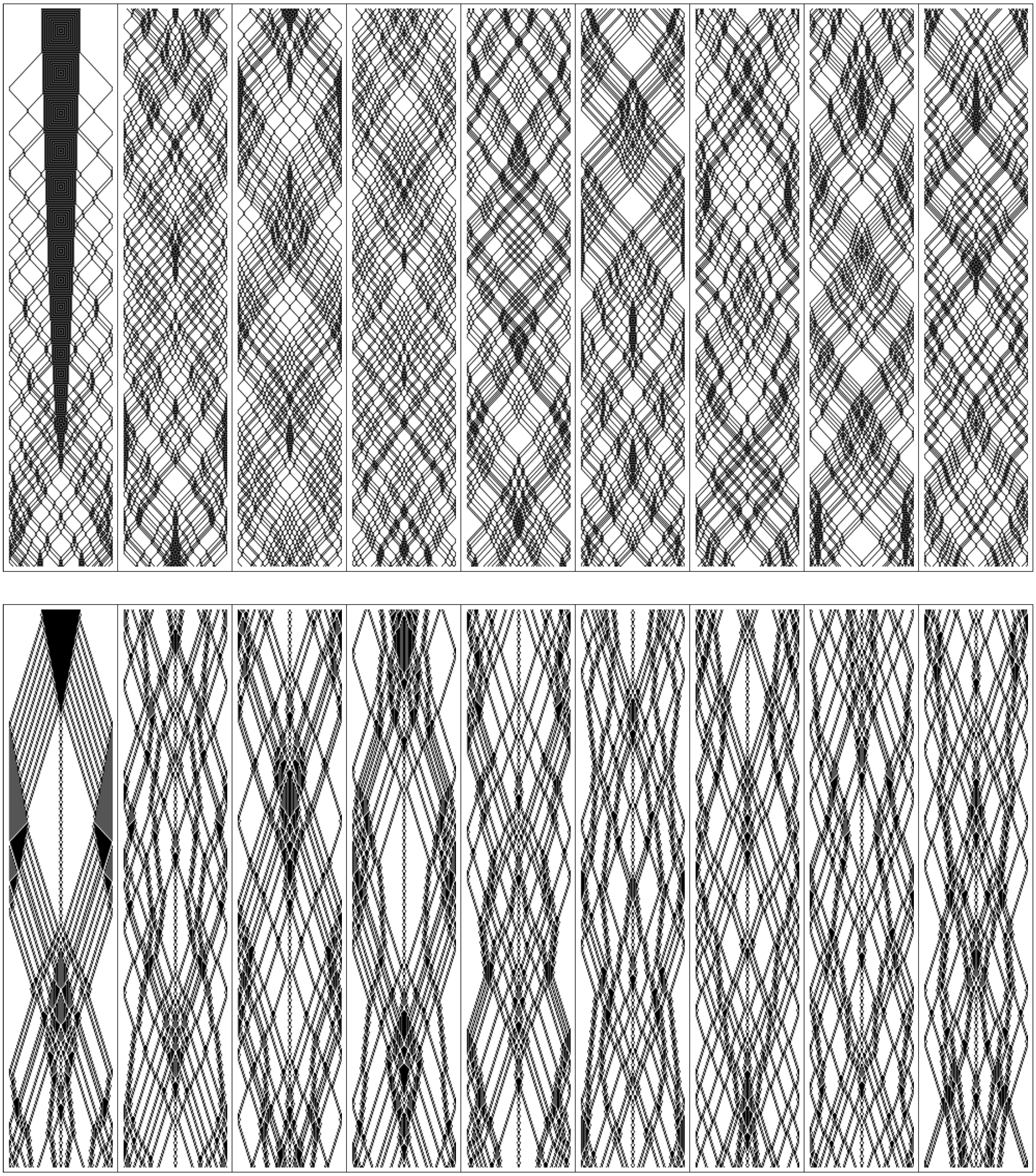
Block cellular automata with two possible colors and blocks of size two that conserve the total number of black cells (the last example has this property only on alternate steps). It so happens that all but the second of the rules shown here not only conserve the total number of black cells but also turn out to be reversible.

But if one allows three possible colors, and requires, say, that the total number of black and gray cells together be conserved, then more complicated behavior can occur, as in the pictures below.



Block cellular automata with three possible colors which conserve the combined number of black and gray cells. In rule (a), black and gray cells remain in localized regions. In rule (b), they move in fairly simple ways, and in rules (c) and (d), they move in a seemingly somewhat random way. The rules shown here are reversible, although their behavior is similar to that of non-reversible rules, at least after a few steps.

Indeed, as the pictures on the next page demonstrate, such systems can produce considerable randomness even when starting from very simple initial conditions.

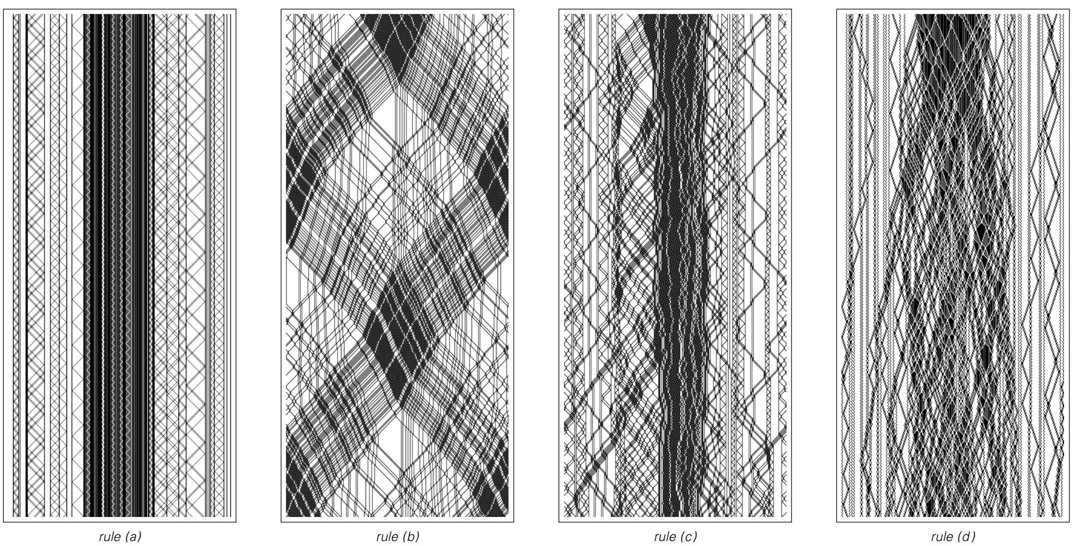


The behavior of rules (c) and (d) from the previous page, starting with very simple initial conditions. Each panel shows 500 steps of evolution, and rapid randomization is evident. The black and gray cells behave much like physical particles: their total number is conserved, and with the particular rules used here, their interactions are reversible. Note that the presence of boundaries is crucial; for without them there would in a sense be no collisions between particles, and the behavior of both systems would be rather trivial.

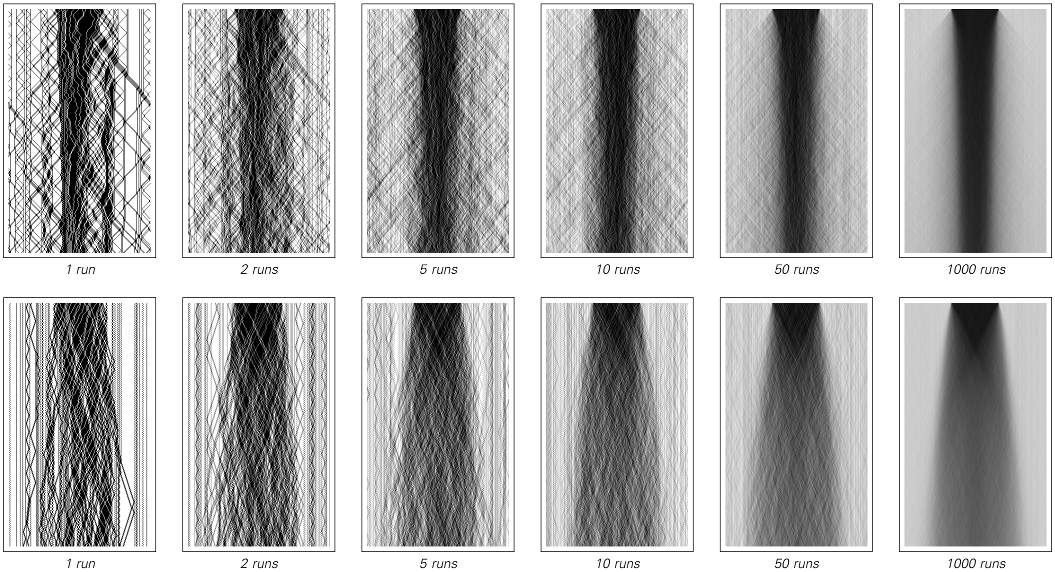
But there is still an important constraint on the behavior: even though black and gray cells may in effect move around randomly, their total number must always be conserved. And this means that if one looks at the total average density of colored cells throughout the system, it must always remain the same. But local densities in different parts of the system need not—and in general they will change as colored cells flow in and out.

The pictures below show what happens with four different rules, starting with higher density in the middle and lower density on the sides. With rules (a) and (b), each different region effectively remains separated forever. But with rules (c) and (d) the regions gradually mix.

As in many kinds of systems, the details of the initial arrangement of cells will normally have an effect on the details of the behavior that occurs. But what the pictures below suggest is that if one looks only at the overall distribution of density, then these details will become largely irrelevant—so that a given initial distribution of density will always tend to evolve in the same overall way, regardless of what particular arrangement of cells happened to make up that distribution.



The block cellular automata from previous pages started from initial conditions containing regions of different density. In rules (a) and (b) the regions remain separated forever, but in rules (c) and (d) they gradually diffuse into each other.



The evolution of overall density for block cellular automata (c) and (d) from the previous page. Even though at an underlying level these systems consist of discrete cells, their overall behavior seems smooth and continuous. The results shown here are obtained by averaging over progressively larger numbers of runs with initial conditions that differ in detail, but have the same overall density distribution. In the limit of an infinite number of runs (or infinite number of cells), the behavior in the second case approaches the form implied by the continuum diffusion equation. (In the first case correlations in effect last too long to yield exactly such behavior.)

The pictures above then show how the average density evolves in systems (c) and (d). And what is striking is that even though at the lowest level both of these systems consist of discrete cells, the overall distribution of density that emerges in both cases shows smooth continuous behavior.

And much as in physical systems like fluids, what ultimately leads to this is the presence of small-scale apparent randomness that washes out details of individual cells or molecules—as well as of conserved quantities that force certain overall features not to change too quickly. And in fact, given just these properties it turns out that essentially the same overall continuum behavior always tends to be obtained.

One might have thought that continuum behavior would somehow rely on special features of actual systems in physics. But in fact what we have seen here is that once again the fundamental mechanisms responsible already occur in a much more minimal way in programs that have some remarkably simple underlying rules.