EXCERPTED FROM

STEPHEN WOLFRAM
A NEW KIND OF SCIENCE

SECTION 7.6

The Phenomenon of Continuity

The Phenomenon of Continuity

Many systems that we encounter in nature have behavior that seems in some way smooth or continuous. Yet cellular automata and most of the other programs that we have discussed involve only discrete elements. So how can such systems ever reproduce what we see in nature?

The crucial point is that even though the individual components in a system may be discrete, the average behavior that is obtained by looking at a large number of these components may still appear to be smooth and continuous. And indeed, there are many familiar systems in nature where exactly this happens.

Thus, for example, air and water seem like continuous fluids, even though we know that at a microscopic level they are both in fact made up of discrete molecules. And in a similar way, sand flows much like a continuous fluid, even though we can easily see that it is actually made up of discrete grains. So what is the basic mechanism that allows systems with discrete components to produce behavior that seems smooth and continuous?

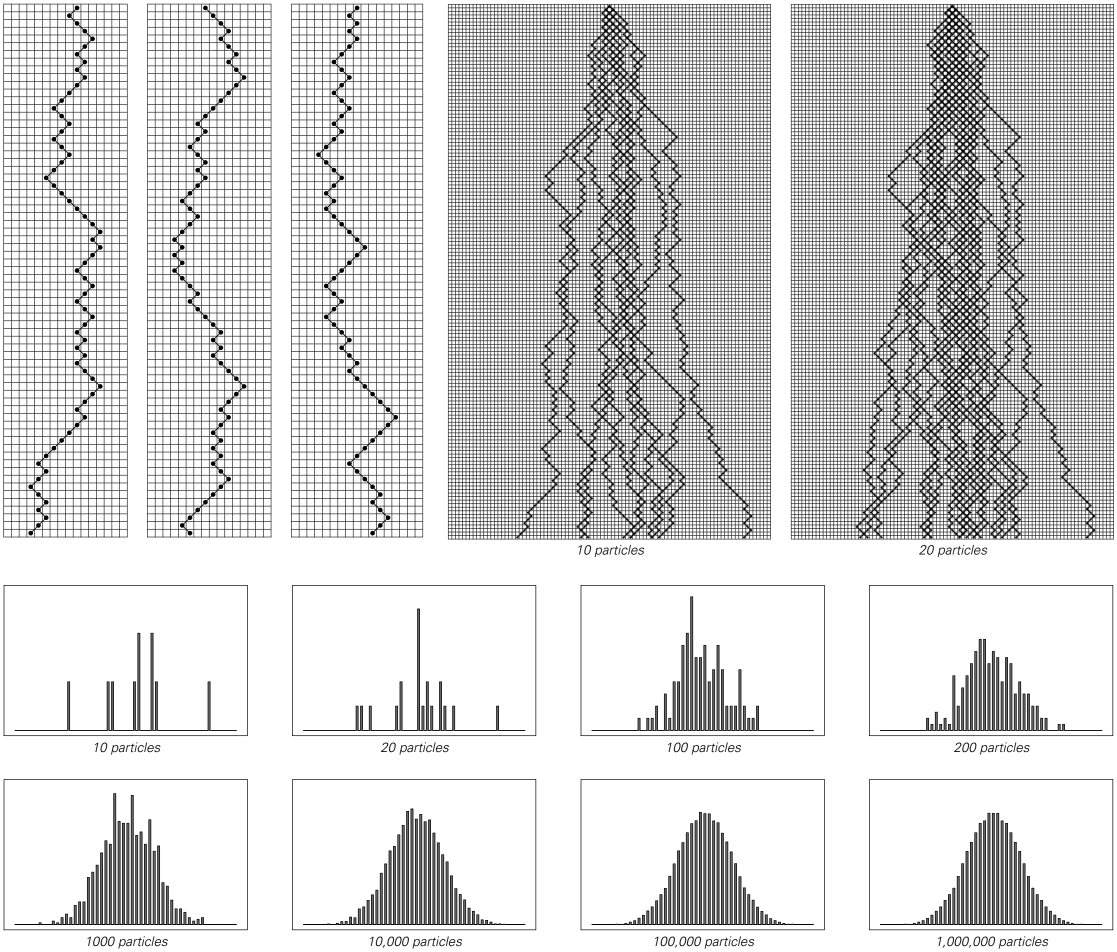
Most often, the key ingredient is randomness.

If there is no randomness, then the overall forms that one sees tend to reflect the discreteness of the underlying components. Thus, for example, the faceted shape of a crystal reflects the regular microscopic arrangement of discrete atoms in the crystal.

But when randomness is present, such microscopic details often get averaged out, so that in the end no trace of discreteness is left, and the results appear to be smooth and continuous. The next page shows a classic example of this phenomenon, based on so-called random walks.

Each random walk is made by taking a discrete particle, and then at each step randomly moving the particle one position to the left or right. If one starts off with several particles, then at any particular time, each particle will be at a definite discrete position. But what happens if one looks not at the position of each individual particle, but rather at the overall distribution of all particles?

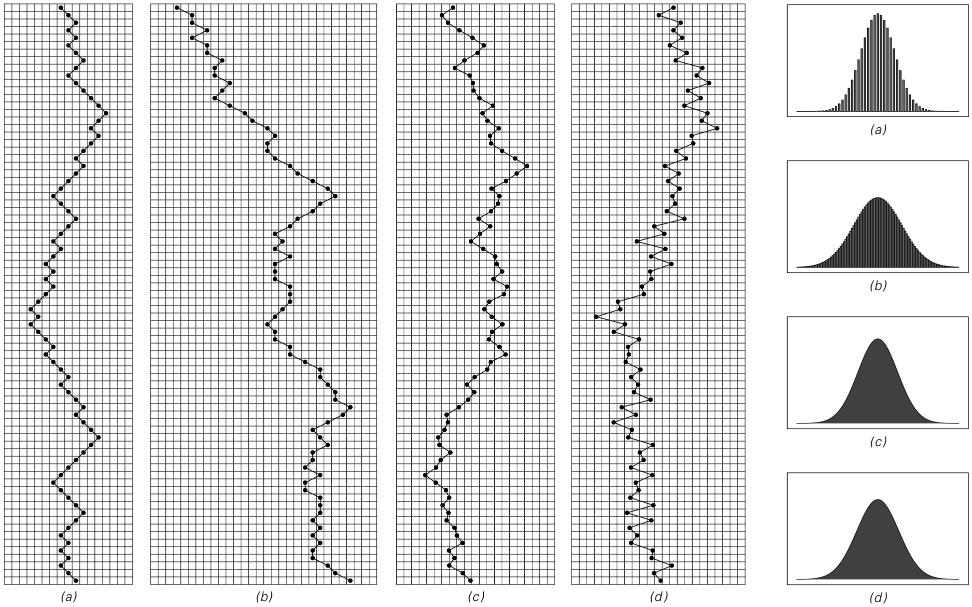
The answer, as illustrated on the next page, is that if there are enough particles, then the distribution one sees takes on a smooth and



The distribution of positions by reached particles that follow random walks. The top left shows three individual examples of random walks, in which each particle randomly moves one position to the left or right. Even though the individual particles are discrete, the pictures show that when a large number of particles are considered, the overall behavior obtained seems smooth and continuous.

continuous form, and shows no trace of the underlying discreteness of the system; the randomness has in a sense successfully washed out essentially all the microscopic details of the system.

The pictures at the top of the facing page show what happens if one uses several different underlying rules for the motion of each particle. And what one sees is that despite differences at a microscopic level, the overall distribution obtained in each case has exactly the same continuous form.

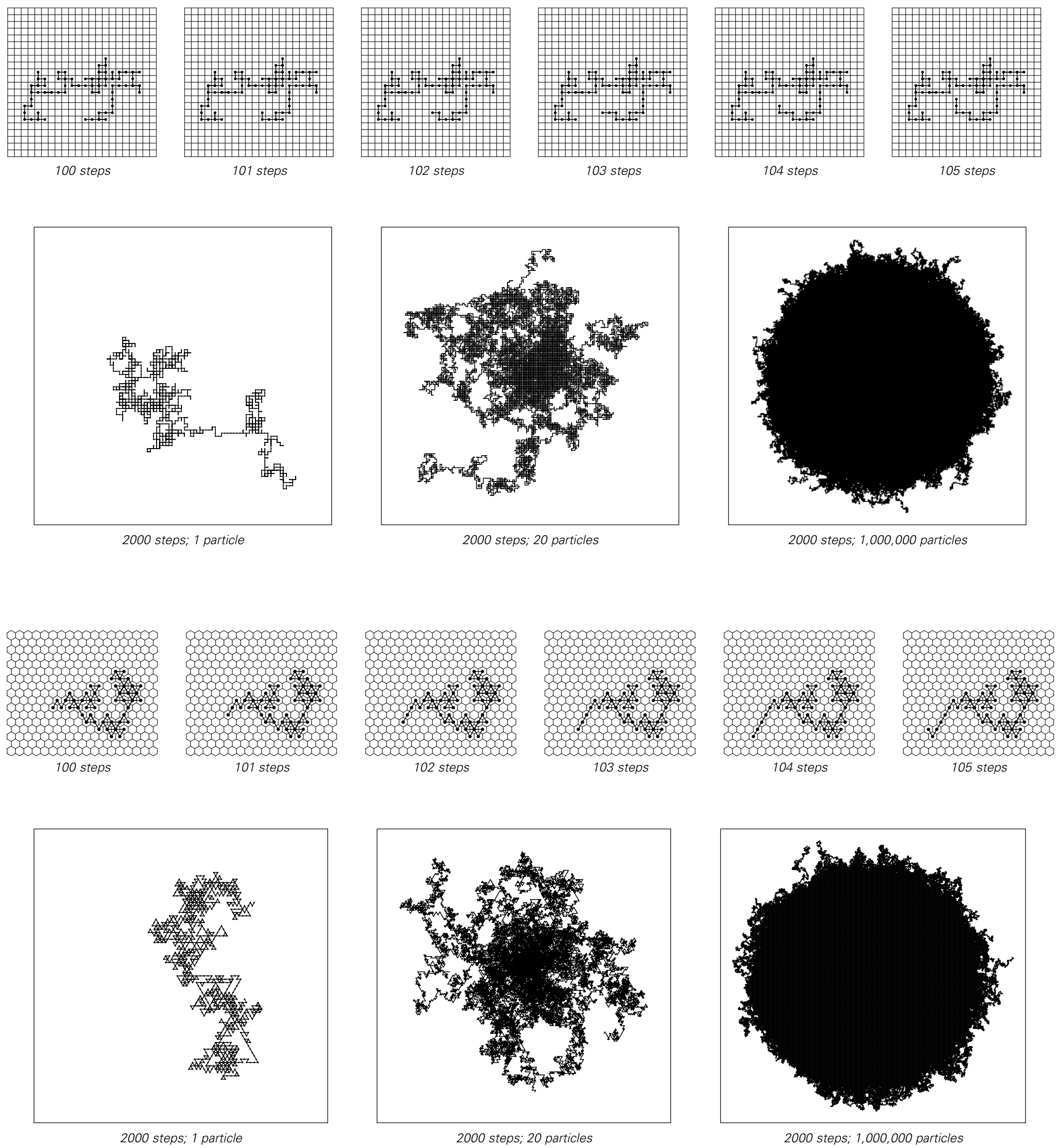


A demonstration of the fact that for a wide range of underlying rules for each step in a random walk, the overall distribution obtained always has the same continuous form. In case (a), each particle moves just one position to the left or right at each step. In case (b), it can move between 0, 1 or 2 positions, while in case (c) it can move any distance between 0 and 1 at each step. Finally, in case (d), on alternate steps the particle moves either always to the right or always to the left.

Indeed, in the particular case of systems such as random walks, the Central Limit Theorem suggested over two centuries ago ensures that for a very wide range of underlying microscopic rules, the same continuous so-called Gaussian distribution will always be obtained.

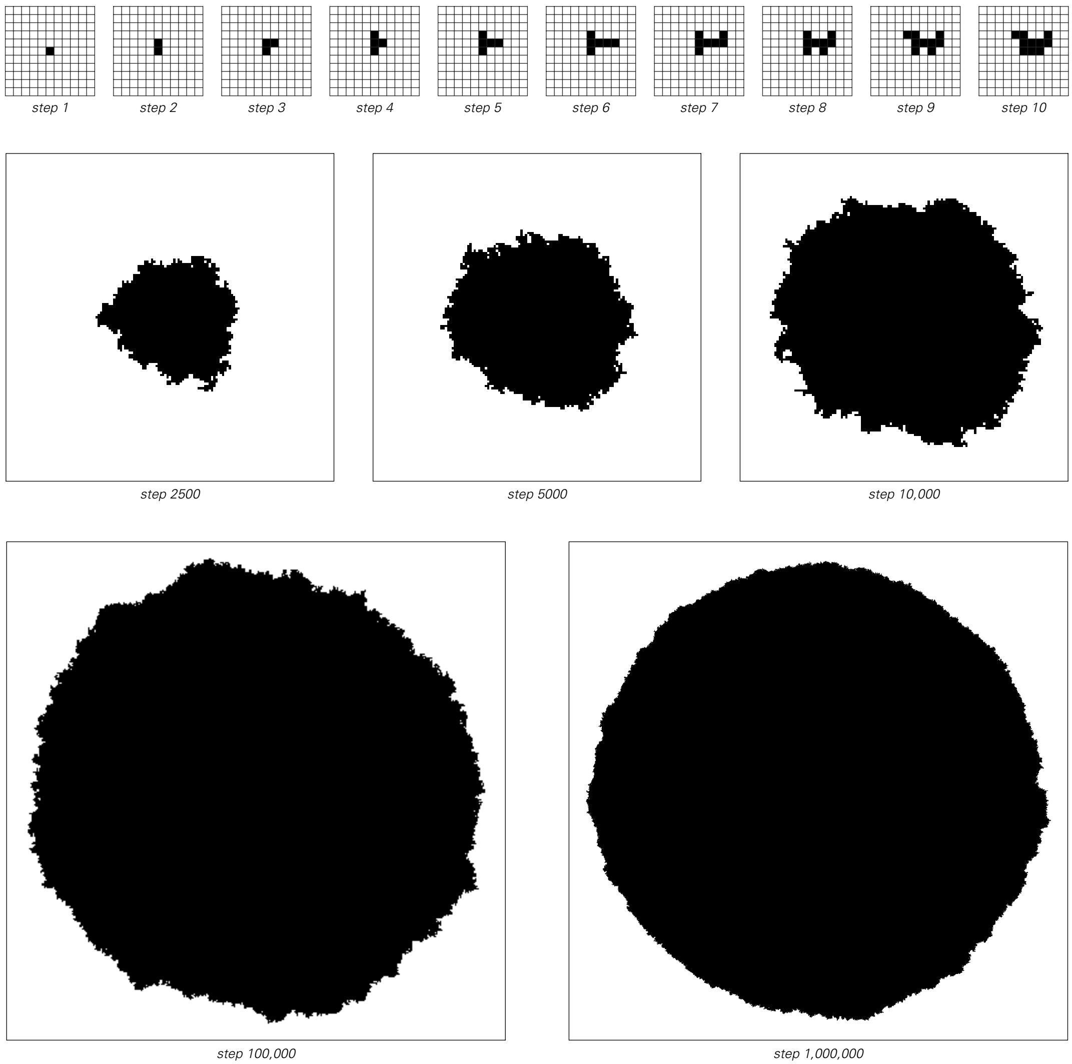
This kind of independence of microscopic details has many important consequences. The pictures on the next page show, for example, what happens if one looks at two-dimensional random walks on square and hexagonal lattices.

One might expect that the different underlying forms of these lattices would lead to different shapes in overall distributions. But the remarkable fact illustrated on the next page is that when enough particles are considered, one gets in the end distributions that have a purely circular shape that shows no trace of the different discrete structures of the underlying lattices.



Examples of random walks on square and hexagonal lattices. Despite the different underlying lattices the average of sufficiently many particles yields ultimately circular behavior in both cases—as implied by the Central Limit Theorem.

Beyond random walks, there are many other systems based on discrete components in which randomness at a microscopic level also leads to continuous behavior on a large scale. The picture below shows as one example what happens in a simple aggregation model.

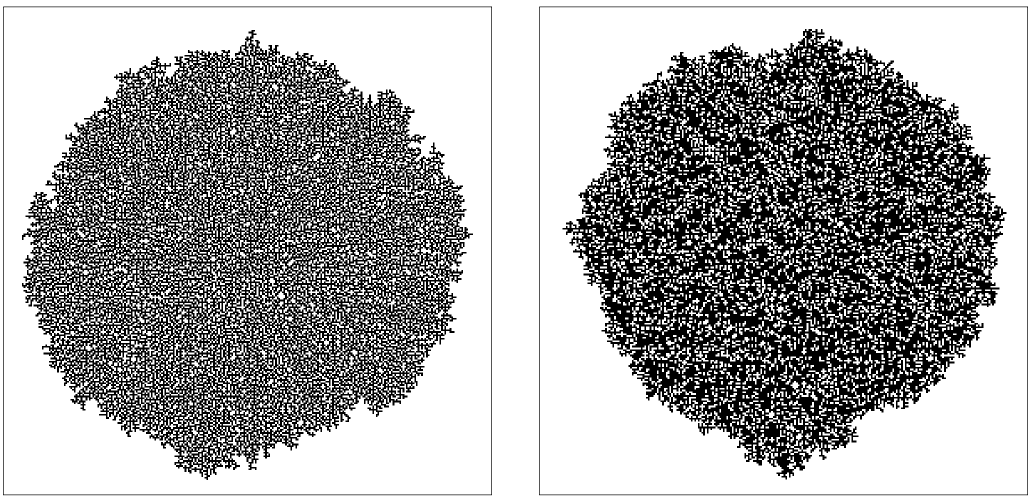


Behavior of a simple aggregation model, in which a single new black cell is added at each step at a randomly chosen position adjacent to the existing cluster of black cells. The system is a version of the so-called Eden model. The shape obtained is ultimately an almost perfect circle.

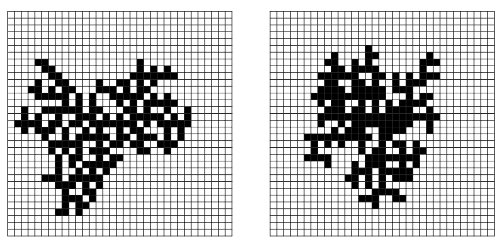
The idea of this model is to build up a cluster of black cells by adding just one new cell at each step. The position of this cell is chosen entirely at random, with the only constraint being that it should be adjacent to an existing cell in the cluster.

At early stages, clusters that are grown in this way look quite irregular. But after a few thousand steps, a smooth overall roughly circular shape begins to emerge. Unlike for the case of random walks, there is as yet no known way to make a rigorous mathematical analysis of this process. But just as for random walks, it appears once again that the details of the underlying rules for the system do not have much effect on the main features of the behavior that is seen.

The pictures below, for example, show generalizations of the aggregation model in which new cells are added only at positions that have certain numbers of existing neighbors. And despite such changes



(a) (b)



(a) (b)

Patterns produced by generalized aggregation models in which a new cell is added only if (a) it would have only one immediate neighbor (out of four), or (b) it would have either one or four neighbors. The pictures above show step 30,000, while those on the right show step 200. Despite the difference in underlying rules, the same basic overall shape of pattern is eventually produced.

in underlying rules, the overall shapes of the clusters produced remain very much the same.

In all these examples, however, the randomness that is involved comes from the same basic mechanism: it is explicitly inserted from outside at each step in the evolution of the system.

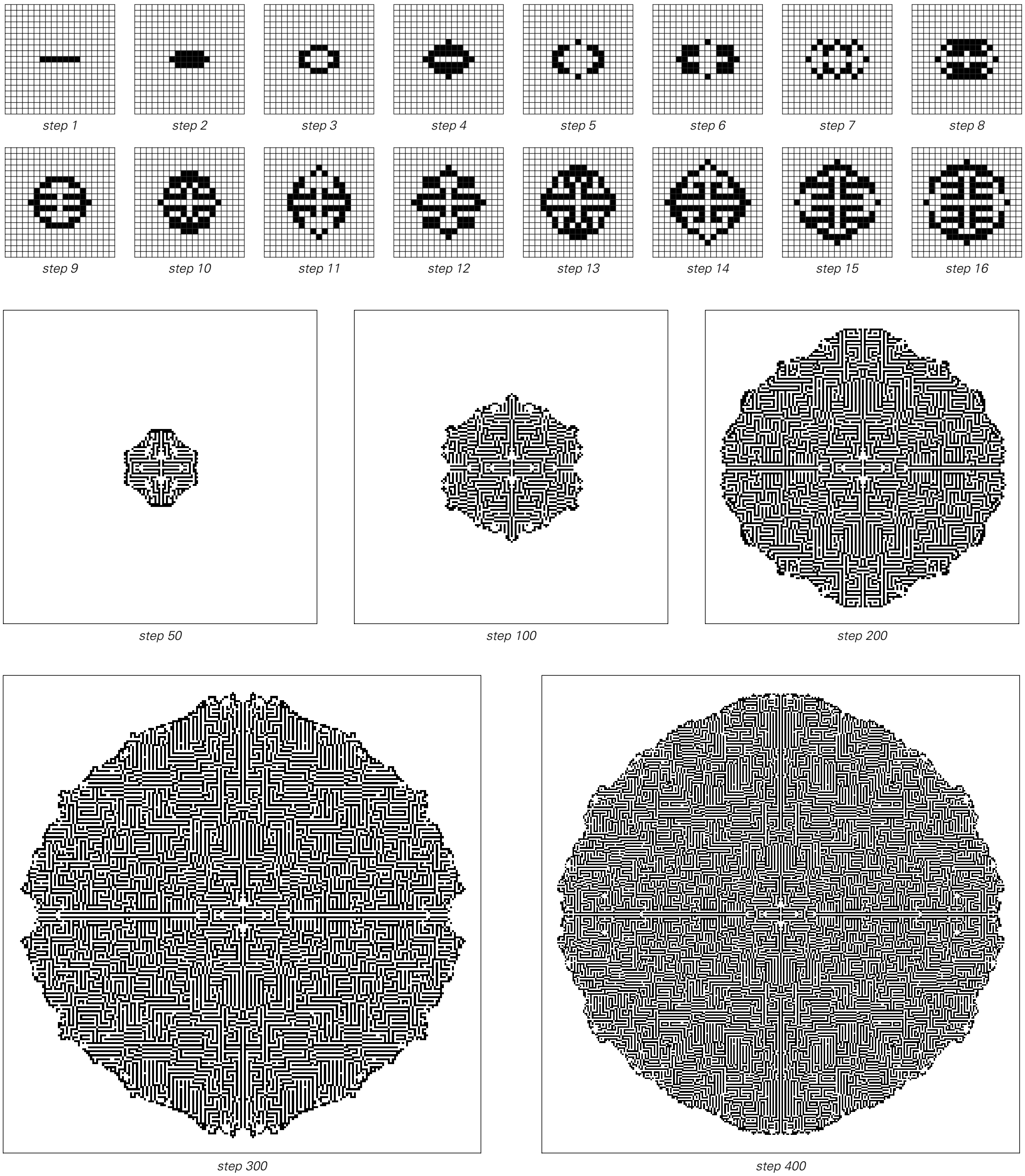
But it turns out that all that really seems to matter is that randomness is present: the mechanism through which it arises appears to be largely irrelevant. And in particular what this means is that randomness which comes from the mechanism of intrinsic randomness generation discussed in the previous section is able to make systems with discrete components behave in seemingly continuous ways.

The picture on the next page shows a two-dimensional cellular automaton where this happens. There is no randomness in the rules or the initial conditions for this system. But through the mechanism of intrinsic randomness generation, the behavior of the system exhibits considerable randomness. And this randomness turns out to lead to an overall pattern of growth that yields the same basic kind of smooth roughly circular form as in the aggregation model.

Having seen this, one might then wonder whether in fact any system that involves randomness will ultimately produce smooth overall patterns of growth. The answer is definitely no. In discussing two-dimensional cellular automata in Chapter 5, for example, we saw many examples where randomness occurs, but where the overall forms of growth that are produced have a complicated structure with no particular smoothness or continuity.

As a rough guide, it seems that continuous patterns of growth are possible only when the rate at which small-scale random changes occur is substantially greater than the overall rate of growth. For in a sense it is only then that there is enough time for randomness to average out the effects of the underlying discrete structure.

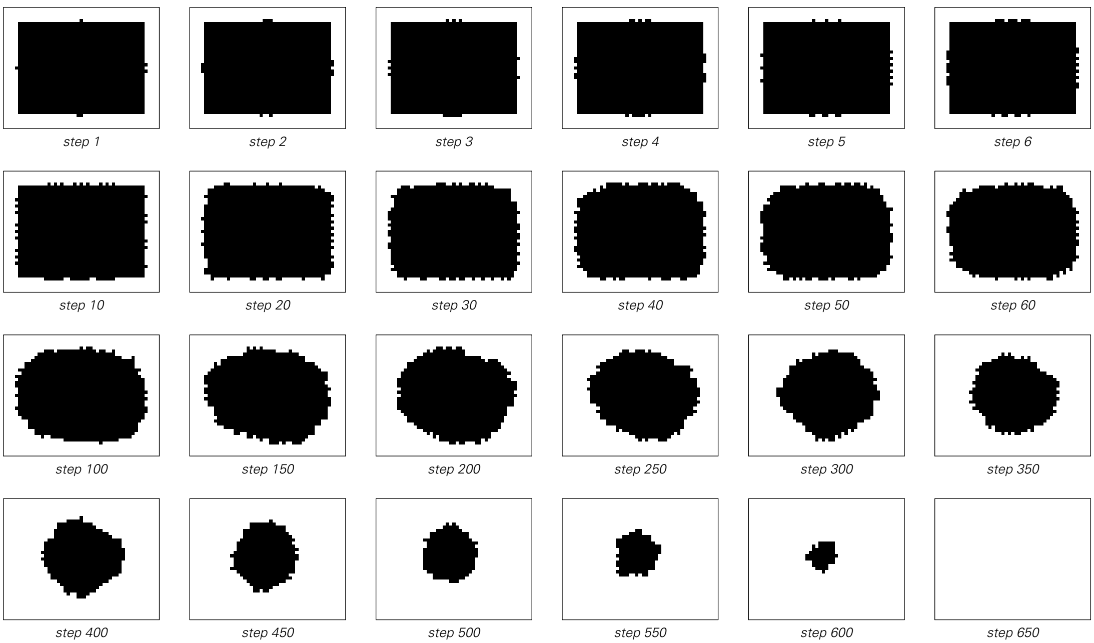
And indeed this same issue also exists for processes other than growth. In general the point is that continuous behavior can arise in systems with discrete components only when there are features that evolve slowly relative to the rate of small-scale random changes.



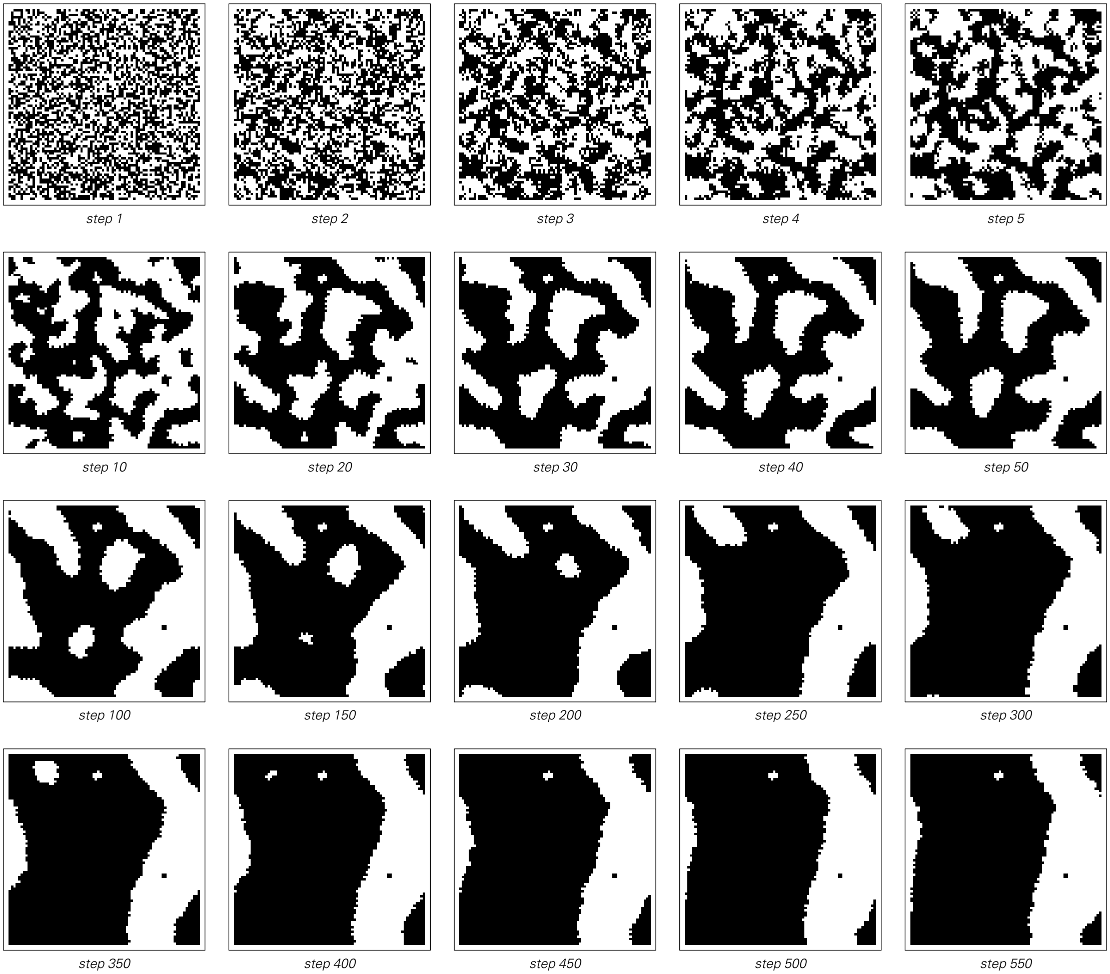
A two-dimensional cellular automaton first shown on page 178 with the rule that if out of the eight neighbors (including diagonals) around a given cell, there are exactly three black cells, then the cell itself becomes black on the next step. If the cell has 1, 2 or 4 black neighbors, then it stays the same color as before, and if it has 5 or more black neighbors, then it becomes white on the next step. (Outer totalistic code 746.) This simple rule produces randomness through the mechanism of intrinsic randomness generation, and this randomness in turn leads to a pattern of growth that takes on an increasingly smooth more-or-less circular form.

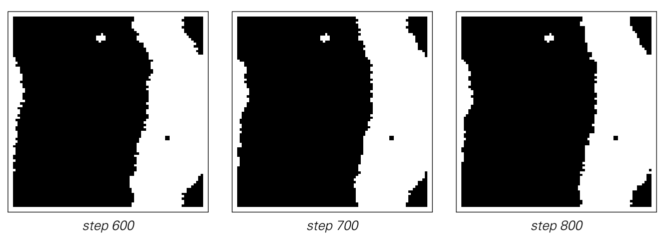
The pictures on the next page show an example where this happens. The detailed pattern of black and white cells in these pictures changes at every step. But the point is that the large domains of black and white that form have boundaries which move only rather slowly. And at an overall level these boundaries then behave in a way that looks quite smooth and continuous.

It is still true, however, that at a small scale the boundaries consist of discrete cells. But as the picture below shows, the detailed configuration of these cells changes rapidly in a seemingly random way. And just as in the other systems we have discussed, what then emerges on average from all these small-scale random changes is overall behavior that again seems in many ways smooth and continuous.



The behavior of an individual domain of black cells in the cellular automaton shown on the next page. The boundary of the domain exhibits seemingly random fluctuations. But at an overall level, the behavior that is produced seems in many respects quite smooth and continuous. The domain effectively behaves as if it has a surface tension, so that it first evolves to a roughly circular shape, then shrinks eventually to nothing. The main black rectangle is initially 39 × 29 cells in size.





Behavior of a two-dimensional cellular automaton starting from a random initial condition. At each step, each cell looks at the total number of black cells in the 9-cell neighborhood consisting of the cell itself and the 8 cells adjacent to it (including diagonals). If this total is less than 4, then the cell becomes white on the next step, while if the total is greater than 6, it becomes black. If the total is exactly 5, then the cell becomes white, and if the total is exactly 4, then it becomes black. (The rule has totalistic code 976.) The pictures show that on a large scale, the rule leads to regions of black and white whose boundaries behave in a seemingly smooth and continuous way. Note that each picture is 80 cells across, and is effectively wrapped around so that the left neighbor of the leftmost cell is the rightmost cell, and so on.