EXCERPTED FROM

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A NEW KIND OF SCIENCE

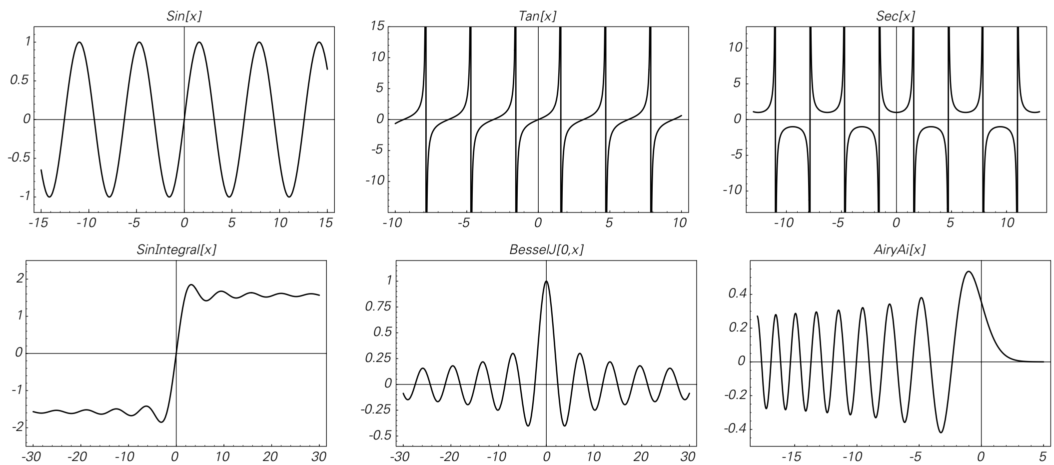
SECTION 4.6

Mathematical Functions

Mathematical Functions

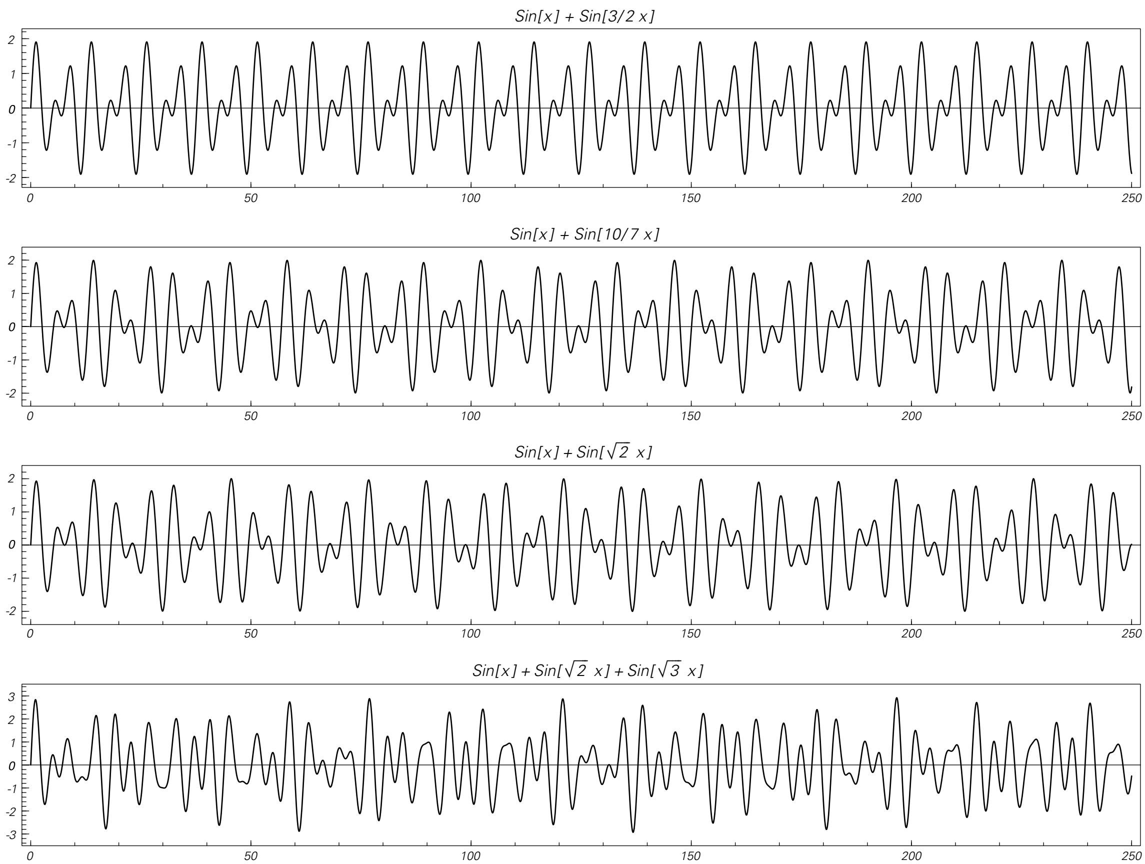
The last section showed that individual numbers obtained by applying various simple mathematical functions can have features that are quite complex. But what about the functions themselves?

The pictures below show curves obtained by plotting standard mathematical functions. All of these curves have fairly simple, essentially repetitive forms. And indeed it turns out that almost all the standard mathematical functions that are defined, for example, in *Mathematica*, yield similarly simple curves.



Plots of some standard mathematical functions. The top row shows three trigonometric functions. The bottom row shows three so-called special functions that are commonly encountered in mathematical physics and other areas of traditional science. In all cases the curves shown have fairly simple repetitive forms.

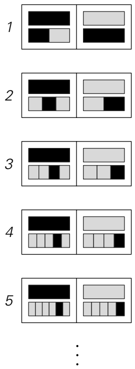
But if one looks at combinations of these standard functions, it is fairly easy to get more complicated results. The pictures on the next page show what happens, for example, if one adds together various sine functions. In the first picture, the curve one gets has a fairly simple repetitive structure. In the second picture, the curve is more complicated, but still has an overall repetitive structure. But in the third and fourth pictures, there is no such repetitive structure, and indeed the curves look in many respects random.

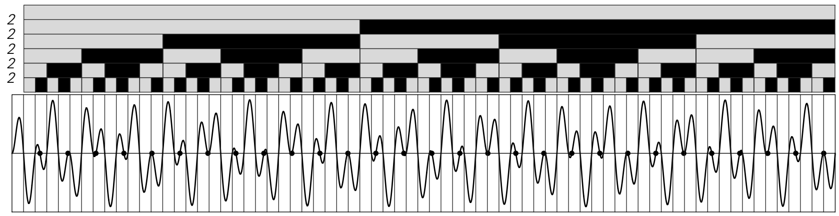


Curves obtained by adding together various sine functions. In the first two cases, the curves are ultimately repetitive; in the second two cases they are not. If viewed as waveforms for sounds, then these curves correspond to chords. The first curve yields a perfect fifth, while the third curve yields a diminished fifth (or tritone) in an equal temperament scale.

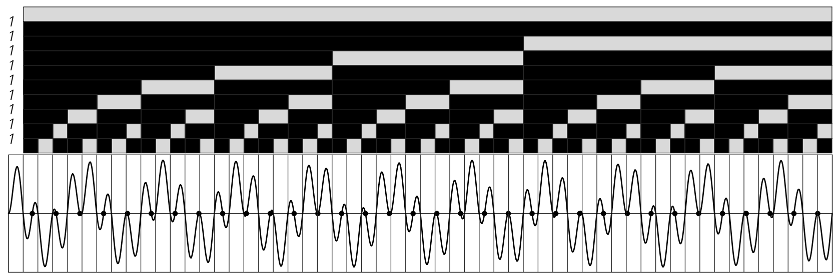
In the third picture, however, the points where the curve crosses the axis come in two regularly spaced families. And as the pictures on the facing page indicate, for any curve like Sin[x] + Sin[αx] the relative arrangements of these crossing points turn out to be related to the output of a generalized substitution system in which the rule at each step is obtained from a term in the continued fraction representation of $(\alpha - 1)/(\alpha + 1)$.

When α is a square root, then as discussed in the previous section, the continued fraction representation is purely repetitive,

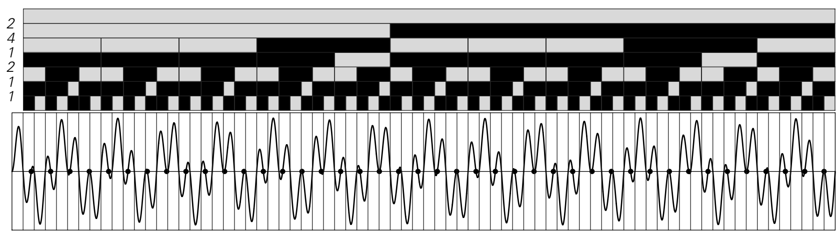




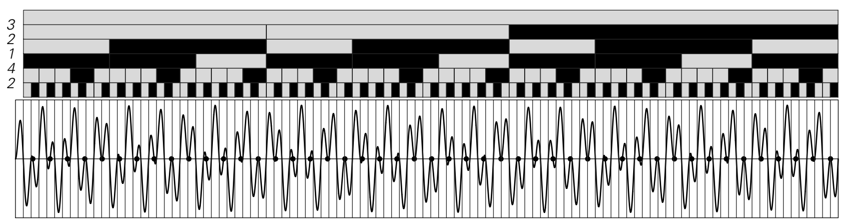
$Cos[x] - Cos[(1 + \sqrt{2}\,)x]$



$Cos[x] - Cos[(2 + \sqrt{5}\,)x]$



$Cos[x] - Cos[(2 + \sqrt[3]{5}\,)x]$



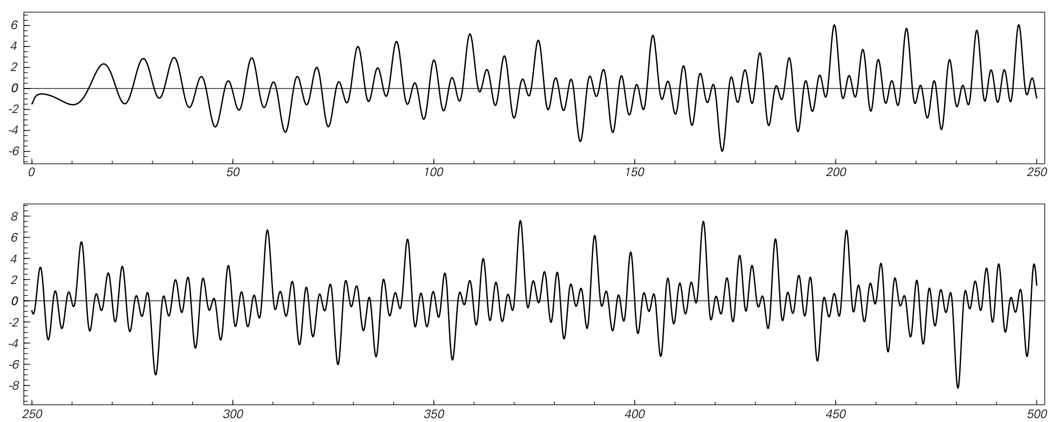
$Cos[x] - Cos[(1 + \sqrt{e}\,)x]$

Curves obtained by adding or subtracting exactly two sine or cosine functions turn out to have a pattern of axis crossings that can be reproduced by a generalized substitution system. In general there is an axis crossing within an interval when the corresponding element in the generalized substitution system is black, and there is not when the element is white. In the case of $Cos[x] - Cos[\alpha x]$ each step in the generalized substitution system has a rule determined as shown on the left from a term in the continued fraction representation of $(\alpha - 1)/(\alpha + 1)$. In the first two examples shown α is a quadratic irrational, so that the continued fraction is repetitive, and the pattern obtained is purely nested. (The second example is analogous to the Fibonacci substitution system on page 83.) In the last two examples, however, there is no such regularity. Note that successive terms in each continued fraction are shown alongside successive steps in the substitution system going up the page.

making the generated pattern nested. But when α is not a square root the pattern can be more complicated. And if more than two sine functions are involved there no longer seems to be any particular connection to generalized substitution systems or continued fractions.

Among all the various mathematical functions defined, say, in *Mathematica* it turns out that there are also a few—not traditionally common in natural science—which yield complex curves but which do not appear to have any explicit dependence on representations of individual numbers. Many of these are related to the so-called Riemann zeta function, a version of which is shown in the picture below.

The basic definition of this function is fairly simple. But in the end the function turns out to be related to the distribution of primes—and the curve it generates is quite complicated. Indeed, despite immense mathematical effort for over a century, it has so far been impossible even to establish for example the so-called Riemann Hypothesis, which in effect just states that all the peaks in the curve lie above the axis, and all the valleys below.



A curve associated with the so-called Riemann zeta function. The zeta function *Zeta[s]* is defined as *Sum[1/k^s, {k, ∞}]*. The curve shown here is the so-called Riemann-Siegel Z function, which is essentially *Zeta[1/2 + ⅈ t]*. The celebrated Riemann Hypothesis in effect states that all peaks after the first one in this curve must lie above the axis.