Catalan Numbers

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Abstract

How likely is it that a vote with 3030 participants ends up in a tie, with 1515 YESes and 1515 NOs? The answer is very interesting, especially when the independence of Catalonia is at stake.

1. A cup of coffee. A few days ago I returned home after a week of being disconnected from the world, and went to the corner café. Luis, the owner, treated me to a coffee and said he wanted to tell me some news and ask me a math question.

2. ¿Catalunya Lliure? Catalonia has spent decades debating whether to separate from Spain, and the independentist movement has gained great momentum in recent years. In an effort to stay in power in these times of change, conservative president Artur Mas formed the *Junts pel Sí* (Together for Yes) alliance with sectors of the Catalonian left in favor of independence. The numbers fell short, though; they did not have enough seats to gain control of the parliament. To obtain these seats, they needed a very unlikely alliance with the CUP, a small, independentist, anti-capitalist party.

Suddenly holding unexpected power, the CUP found itself at a great ideological crossroads: Should they ally with Mas, a symbol of the traditional politics that they oppose, in order to obtain the independence of Catalonia?

On December 27, 2015, each one of the 3030 members present at the CUP Assembly cast her vote. (In accordance with their anti-patriarchal principles, CUP members use the feminine to include all possible genders.) The result:

1515 in favor of Mas, 1515 against him.

Shocked, Catalonians started asking themselves, not without some suspicion:

What were the odds of a tie?!

Some skeptics computed the probability to be 1 in 3031, and said the outcome was virtually impossible. Others said a tie was as likely as a tossed coin landing on its edge. The question is very interesting, and the answer is even better.

3. Catalan numbers 1. Let's assume that in a vote with $2m$ voters, each person can vote YES or NO.[1]

The probability that the final result is a tie
of m YESes and m NOs is approximately $1/\sqrt{m\pi}$.

Yes, that's a pi: $\pi = 3.14159\ldots$. In the specific case that interests us:

The odds that the CUP vote was tied with
1515 YESes and 1515 NOs was approximately 1 in 69.

3.1. Why $2m$ people? Luis told me, laughing: "Imagine a member of the CUP who woke up too late to cast her vote. She could have decided the fate of Catalonia all by herself!" For a tie to be possible, the number of voters must be an even number, say $2m$.

3.2. Why $1/\sqrt{m\pi}$? Imagine we registered the $2m$ votes, in the order they came in: YES, YES, NO, YES, NO, etc. Each voter had two options (YES or NO), for a total of $2 \times 2 \times \cdots \times 2 = 2^{2m}$ possible voting choices.

Out of those 2^{2m} possibilities, the ties are those with m YESes and m NOs. They correspond to the choices of m (out of the $2m$) people who will vote YES; the remaining m will vote NO. The number of choices is denoted $\binom{2m}{m}$ and called "$2m$ *choose* m". There is a formula which I could prove to you if we had a bit more time: the number of ties is $\binom{2m}{m} = \frac{(2m)!}{m!m!}$, where the symbol $n!$ denotes the number "n *factorial*", which is $n! = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n$. Therefore:

$$\text{probability of a tie} = \frac{(\#\text{ of possible ties})}{(\#\text{ of possible voting choices})} = \frac{(2m)!}{m!m!\,2^{2m}}. \tag{1}$$

In the case of Catalonia, if one had a computer within reach, one could ask it to compute this number for $2m = 3030$, and obtain $1.44938\ldots\%$. But I had just returned from a week away from technology, and I had left my phone at home, so I had to use a napkin. (I had to earn my free coffee, right?)

Fortunately, Stirling discovered a very useful approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2n\pi} \tag{2}$$

where $\approx$ means *is almost equal to*. Proving this formula requires a bit of calculus. Although I have shown it to my students many times, it never ceases to amaze me. What are $e = 2.71828\ldots$ and $\pi = 3.14159\ldots$, mathematicians' two favorite numbers, doing there?!

[1] As we will explain in Section 3.3, we are making more assumptions about the vote.

Using Stirling's formula (2) we can simplify the probability (1). My wife was at the café with me, and she took the opportunity to dust off her algebra. You should do it too, especially if you are one of those people who enjoy tidying up your desk when it's messy: throwing out everything you don't need, and putting everything else in its right place. The final result is:

$$\text{probability of a tie} = \frac{(2m)!}{m!m!\,2^{2m}} \approx \frac{1}{\sqrt{m\pi}}.$$

and then we were able to estimate this quantity by hand for $2m = 3030$:

$$\sqrt{1515\pi} \approx \sqrt{(\text{more than 1500})(\text{more than 3})} \approx \sqrt{\text{about 4900}} \approx \text{about 70}.$$

So the probability of a tie was about 1 in 70. Indeed, now that I am in front of a computer, I see that

$$\tfrac{1}{\sqrt{1515\pi}} \approx \tfrac{1}{68.989...} \approx 1.44950\ldots\%.$$

It is important to ask ourselves: how good is Stirling's approximation? It is very good! For $2m = 3030$:

$$\text{real probability of a tie} = \tbinom{3030}{1515}/2^{3030} \approx 1.44938\ldots\%$$

$$\text{Stirling's aproximation} = 1/\sqrt{1515\pi} \approx 1.44950\ldots\%.$$

3.3. The fine print. What does it mean to say that the probability of a tie is 1 in 69? We can't very well ask the CUP to vote 69 times, and see if they tie exactly once.

The only way of making sense of this statement is to choose a model of how people vote. Our model to obtain odds of 1/69 is that each person decides independently, and votes YES or NO with equal probability of 1/2. To obtain odds of 1/3031, it appears that the skeptics assumed that the number of YESes is equally likely to be $0, 1, 2, \ldots, 3029$, or 3030. This assumption does not seem to have any real basis.

Clearly our model is not exactly correct either, even if the final result of 1515 YESes and 1515 NOs does not make it seem entirely absurd. No model of human behaviour will be exactly correct. Our practice as mathematicians is to choose a model which is close to reality and which we are able to analyze. This practice comes with a huge responsibility: to question every mathematical model critically, especially when its results affect our lives. [1] In that sense, this note is just the beginning of a rigorous analysis.

4. Catalan numbers 2. As a combinatorialist, I can't tell this story without talking about *our* Catalan numbers. Allow me some mathematical-poetic license.

Let's imagine that an unwavering independentist oversaw the CUP vote. As she counted the votes one by one she noticed that the YES vote was **never** behind. When she saw that the final result was a tie; her surprise and deception were enormous. What are the odds that the 1515 YESes and 1515 NOs lined up like this, always feeding her optimism?

If $2m$ people vote YES or NO in an election which ends in a tie,
there is a 1 in $m + 1$ probability that
YES is never behind as the votes are counted in order.

In the CUP vote, the odds are 1 in 1516.

In fact, the number of voting choices which end in a tie, where YES is never losing as the votes are counted, is known to mathematicians as the "*Catalan number*" (no relation to Catalunya), and is equal to $\frac{1}{m+1}\binom{2m}{m}$.

The Catalan numbers were discovered by Ming'antu in Mongolia and Euler in Prussia in the 18th century; they are named after Belgian mathematician Eugène Catalan. These numbers appear all over the place. While poets consult the dictionary, combinatorialists consult the Online Encyclopedia of Integer Sequences (`www.oeis.org`); the entry for the Catalan sequence $1, 2, 5, 14, 42, 132, 429, \ldots$ occupies 20 pages and is still growing. To this sequence, Richard Stanley devoted a collection of over 200 appearances, a book [3] (including work of the Catalans Anna de Mier, Sergi Elizalde and Marc Noy), and a joke:

> [2, Problem 6.24] Explain the significance of the following sequence:
> un, dos, tres, quatre, cinc, sis, set, vuit, nou, deu, . . .

5. The outcome. In the tie-breaker on January 3, the CUP directives decided not to support Mas, with 36 NOs, 30 YESes, and one abstention. (Apparently someone did not want the tie to be broken.) On January 9, 2016, after long negotiations, Mas agreed not to run for reelection as president, and in return the CUP agreed to join the *Junts pel Sí* coalition, which thus gains control of the parliament; a great victory for the independentist project. In the end, this is more than a curious mathematical anecdote. It is likely that this alliance, born out of and shaped by the arithmetic of the parliamentary system, will end up playing an important role in the political future of Catalonia.

References

[1] D. Huff. How to lie with statistics. W. W. Norton & Co., 1993.

[2] R. Stanley. Enumerative Combinatorics Vol. 2. Cambridge University Press, 2001.

[3] R. Stanley. Catalan Numbers. Cambridge University Press, 2015.