COHOMOLOGICAL FIELD THEORY CALCULATIONS

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Abstract

Cohomological field theories (CohFTs) were defined in the mid 1990s by Kontsevich and Manin to capture the formal properties of the virtual fundamental class in Gromov–Witten theory. A beautiful classification result for semisimple CohFTs (via the action of the Givental group) was proven by Teleman in 2012. The Givental–Teleman classification can be used to explicitly calculate the full CohFT in many interesting cases not approachable by earlier methods.

My goal here is to present an introduction to these ideas together with a survey of the calculations of the CohFTs obtained from

- Witten's classes on the moduli spaces of r-spin curves,
- Chern characters of the Verlinde bundles on the moduli of curves,
- Gromov–Witten classes of Hilbert schemes of points of $\mathbb{C}^2$.

The subject is full of basic open questions.

Introduction

0.1 Moduli of curves. The moduli space $\mathfrak{M}_g$ of complete, nonsingular, irreducible, algebraic curves over $\mathbb{C}$ of genus g has been a central object in mathematics since Riemann's work in the middle of the 19th century. The Deligne–Mumford compactification

$$\mathfrak{M}_g \subset \overline{\mathfrak{M}}_g$$

by nodal curves was defined almost 50 years ago Deligne and Mumford [1969].

We will be concerned here with the moduli space of curves with marked points,

$$\mathfrak{M}_{g,n} \subset \overline{\mathfrak{M}}_{g,n},$$

in the stable range $2g - 2 + n > 0$. As a Deligne–Mumford stack (or orbifold), $\overline{\mathfrak{M}}_{g,n}$ is nonsingular, irreducible, and of (complex) dimension $3g - 3 + n$. There are natural forgetful morphisms

$$p : \overline{\mathfrak{M}}_{g,n+1} \to \overline{\mathfrak{M}}_{g,n}$$

dropping the last marking.

The *boundary*[1] of the Deligne–Mumford compactification is the closed locus parameterizing curves with a least one node,

$$\partial \overline{\mathcal{M}}_{g,n} = \overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n} .$$

By identifying the last two markings of a single $(n+2)$-pointed curve of genus $g-1$, we obtain a morphism

$$q : \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n} .$$

Similarly, by identifying the last markings of separate pointed curves, we obtain

$$r : \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g,n} ,$$

where $n = n_1 + n_2$ and $g = g_1 + g_2$. The images of both q and r lie in the boundary $\partial \overline{\mathcal{M}}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$.

The cohomology and Chow groups of the moduli space of curves are

$$H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \quad \text{and} \quad A^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) .$$

While there has been considerable progress in recent years, many basic questions about the cohomology and algebraic cycle theory remain open.[2]

0.2   Gromov–Witten classes. Let X be a nonsingular projective variety over $\mathbb{C}$, and let

$$\overline{\mathcal{M}}_{g,n}(X, \beta)$$

be the moduli space of genus g, n-pointed stable maps to X representing the class $\beta \in H_2(X, \mathbb{Z})$. The basic structures carried by $\overline{\mathcal{M}}_{g,n}(X, \beta)$ are forgetful maps,

$$\pi : \overline{\mathcal{M}}_{g,n}(X, \beta) \to \overline{\mathcal{M}}_{g,n} ,$$

to the moduli space of curves via the domain (in case $2g - 2 + n > 0$) and evaluation maps,

$$\mathrm{ev}_i : \overline{\mathcal{M}}_{g,n}(X, \beta) \to X ,$$

for each marking $1 \leq i \leq n$.

Given cohomology classes $v_1, \ldots, v_n \in H^*(X, \mathbb{Q})$, the associated *Gromov–Witten class* is defined by

$$\Omega^X_{g,n,\beta}(v_1, \ldots, v_n) = \pi_* \left(\prod_{i=1}^n \mathrm{ev}_i^*(v_i) \cap [\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\mathrm{vir}} \right) \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) .$$

Central to the construction is the *virtual fundamental class* of the moduli space of stable maps,

$$[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\mathrm{vir}} \in H_{2 \cdot \mathrm{virdim}}(\overline{\mathcal{M}}_{g,n}(X, \beta), \mathbb{Q}) ,$$

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[1]Since $\overline{\mathcal{M}}_{g,n}$ is a closed nonsingular orbifold, the boundary here is *not* in the sense of orbifold with boundary. If $g = 0$, there is no boundary map q.

[2]See Pandharipande [2018] for a survey of results and open questions.

of *virtual dimension*

$$\text{virdim} = \int_\beta c_1(X) + (1-g) \cdot (\dim_\mathbb{C}(X) - 3) + n \,.$$

Gromov–Witten classes contain much more information than the *Gromov–Witten invariants* defined by integration,

$$\langle v_1, \ldots, v_n \rangle^X_{g,n,\beta} = \int_{\overline{\mathcal{M}}_{g,n}} \Omega^X_{g,n,\beta}(v_1, \ldots, v_n) \,.$$

We refer the reader to Behrend [1997], Behrend and Fantechi [1997], Cox and Katz [1999], and Fulton and Pandharipande [1997] for a detailed treatment of stable maps, virtual fundamental classes, and Gromov–Witten invariants in algebraic geometry.

The Gromov–Witten classes satisfy formal properties with respect to the natural forgetful and boundary maps p, q, and r discussed in Section 0.1. The idea of a cohomological field theory was introduced by Kontsevich and Manin Kontsevich and Manin [1994] to fully capture these formal properties.

0.3 Cohomological field theories. The starting point for defining a cohomological field theory is a triple of data $(V, \eta, \mathbf{1})$ where

- V is a finite dimensional $\mathbb{Q}$-vector space,

- η is a non-degenerate symmetric 2-form on V,

- $\mathbf{1} \in V$ is a distinguished element.

Given a $\mathbb{Q}$-basis $\{e_i\}$ of V, the symmetric form η can be written as a matrix

$$\eta_{jk} = \eta(e_j, e_k) \,.$$

The inverse matrix is denoted, as usual, by η^{jk}.

A *cohomological field theory* consists of a system $\Omega = (\Omega_{g,n})_{2g-2+n>0}$ of tensors

$$\Omega_{g,n} \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \otimes (V^*)^{\otimes n}.$$

The tensor $\Omega_{g,n}$ associates a cohomology class in $H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ to vectors

$$v_1, \ldots, v_n \in V$$

assigned to the n markings. We will use both

$$\Omega_{g,n}(v_1 \otimes \cdots \otimes v_n) \quad \text{and} \quad \Omega_{g,n}(v_1, \ldots, v_n)$$

to denote the associated cohomology class in $H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$.

In order to define a cohomological field theory, the system $\Omega = (\Omega_{g,n})_{2g-2+n>0}$ must satisfy the CohFT axioms:

(i) Each tensor $\Omega_{g,n}$ is Σ_n-invariant for the natural action of the symmetric group Σ_n on

$$H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \otimes (V^*)^{\otimes n}$$

obtained by simultaneously permuting the n marked points of $\overline{\mathcal{M}}_{g,n}$ and the n factors of V^*.

(ii) The tensor $q^*(\Omega_{g,n}) \in H^*(\overline{\mathcal{M}}_{g-1,n+2}, \mathbb{Q}) \otimes (V^*)^{\otimes n}$, obtained via pull-back by the boundary morphism

$$q : \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n} ,$$

is required to equal the contraction of $\Omega_{g-1,n+2}$ by the bi-vector

$$\sum_{j,k} \eta^{jk} e_j \otimes e_k$$

inserted at the two identified points:

$$q^*(\Omega_{g,n}(v_1, \ldots, v_n)) = \sum_{j,k} \eta^{jk} \Omega_{g-1,n+2}(v_1, \ldots, v_n, e_j, e_k)$$

in $H^*(\overline{\mathcal{M}}_{g-1,n+2}, \mathbb{Q})$ for all $v_i \in V$.

The tensor $r^*(\Omega_{g,n})$, obtained via pull-back by the boundary morphism

$$r : \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g,n} ,$$

is similarly required to equal the contraction of $\Omega_{g_1,n_1+1} \otimes \Omega_{g_2,n_2+1}$ by the same bi-vector:

$$r^*(\Omega_{g,n}(v_1, \ldots, v_n)) =$$
$$\sum_{j,k} \eta^{jk} \Omega_{g_1,n_1+1}(v_1, \ldots, v_{n_1}, e_j) \otimes \Omega_{g_2,n_2+1}(v_{n_1+1}, \ldots, v_n, e_k)$$

in $H^*(\overline{\mathcal{M}}_{g_1,n_1+1}, \mathbb{Q}) \otimes H^*(\overline{\mathcal{M}}_{g_2,n_2+1}, \mathbb{Q})$ for all $v_i \in V$.

(iii) The tensor $p^*(\Omega_{g,n})$, obtained via pull-back by the forgetful map

$$p : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n} ,$$

is required to satisfy

$$\Omega_{g,n+1}(v_1, \ldots, v_n, \mathbf{1}) = p^*\Omega_{g,n}(v_1, \ldots, v_n)$$

for all $v_i \in V$. In addition, the equality

$$\Omega_{0,3}(v_1, v_2, \mathbf{1}) = \eta(v_1, v_2)$$

is required for all $v_i \in V$.

Definition 1. *A system* $\Omega = (\Omega_{g,n})_{2g-2+n>0}$ *of tensors*

$$\Omega_{g,n} \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \otimes (V^*)^{\otimes n}$$

satisfying (i) and (ii) is a cohomological field theory *or a* CohFT. *If (iii) is also satisfied,* Ω *is a CohFT with unit.*

The simplest example of a cohomological field theory with unit is given by the *trivial CohFT,*

$$V = \mathbb{Q}, \quad \eta(1,1) = 1, \quad \mathbf{1} = 1, \quad \Omega_{g,n}(1, \ldots, 1) = 1 \in H^0(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}).$$

A more interesting example is given by the total Chern class

$$c(\mathbb{E}) = 1 + \lambda_1 + \ldots + \lambda_g \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$$

of the rank g Hodge bundle $\mathbb{E} \to \overline{\mathcal{M}}_{g,n}$,

$$V = \mathbb{Q}, \quad \eta(1,1) = 1, \quad \mathbf{1} = 1, \quad \Omega_{g,n}(1, \ldots, 1) = c(\mathbb{E}) \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}).$$

Definition 2. *For a CohFT* $\Omega = (\Omega_{g,n})_{2g-2+n>0}$*, the* topological part ω *of* Ω *is defined by*

$$\omega_{g,n} = [\Omega_{g,n}]^0 \in H^0(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \otimes (V^*)^{\otimes n}.$$

The degree 0 part $[\;]^0$ of Ω is simply obtained from the canonical summand projection

$$[\;]^0 : H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \to H^0(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}).$$

If Ω is a CohFT with unit, then ω is also a CohFT with unit. The topological part of the CohFT obtained from the total Chern class of the Hodge bundle is the trivial CohFT.

The motivating example of a CohFT with unit is obtained from the Gromov–Witten theory of a nonsingular projective variety X. Here,

$$V = H^*(X, \mathbb{Q}), \quad \eta(v_1, v_2) = \int_X v_1 \cup v_2, \quad \mathbf{1} = 1.$$

Of course, the Poincaré pairing on $H^*(X, \mathbb{Q})$ is symmetric only if X has no odd cohomology.[3] The tensor $\Omega_{g,n}$ is defined using the Gromov–Witten classes $\Omega^X_{g,n,\beta}$ of Section 0.2 (together with a Novikov[4] parameter q),

$$\Omega_{g,n}(v_1, \ldots, v_n) = \sum_{\beta \in H_2(X, \mathbb{Q})} \Omega^X_{g,n,\beta} \, q^\beta.$$

The CohFT axioms here coincide exactly with the axioms[5] of Gromov–Witten theory related to the morphisms p, q, and r. For example, axiom (ii) of a CohFT here is the *splitting axiom* of Gromov–Witten theory, see Kontsevich and Manin [1994].

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[3]To accommodate the case of arbitrary X, the definition of a CohFT can be formulated with signs and $\mathbb{Z}/2\mathbb{Z}$-gradings. We do not take the super vector space path here.

[4]Formally, we must extend scalars in the definition of a CohFT from $\mathbb{Q}$ to the Novikov ring to capture the Gromov–Witten theory of X.

[5]The divisor axiom of Gromov–Witten theory (which concerns divisor and curve classes on X) is not part of the CohFT axioms.

0.4 Semisimplicity. A CohFT with unit Ω defines a *quantum product* $\bullet$ on V by[6]

$$\eta(v_1 \bullet v_2, v_3) = \Omega_{0,3}(v_1 \otimes v_2 \otimes v_3).$$

The quantum product $\bullet$ is commutative by CohFT axiom (i). The associativity of $\bullet$ follows from CohFT axiom (ii). The element $\mathbf{1} \in V$ is the identity for $\bullet$ by the second clause of CohFT axiom (iii). Hence,

$$(V, \bullet, \mathbf{1})$$

is a commutative $\mathbb{Q}$-algebra.

Lemma 3. *The topological part ω of Ω is uniquely and effectively determined by the coefficients*

$$\Omega_{0,3}(v_1, v_2, v_3) \in H^*(\overline{\mathcal{M}}_{0,3}, \mathbb{Q})$$

of the quantum product $\bullet$.

Proof.   Let the moduli point $[C, p_1, \dots, p_n] \in \overline{\mathcal{M}}_{g,n}$ correspond to a maximally degenerate curve (with every component isomorphic to $\mathbb{P}^1$ with exactly 3 special points). Since

$$\omega_{g,n}(v_1, \dots, v_n) \in H^0(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}),$$

is a multiple of the identity class, $\omega_{g,n}(v_1, \dots, v_n)$ is determined by the pull-back to the point $[C, p_1, \dots, p_n]$. The equality

$$\omega_{g,n}(v_1, \dots, v_n)\big|_{[C, p_1, \dots, p_n]} = \Omega_{g,n}(v_1, \dots, v_n)\big|_{[C, p_1, \dots, p_n]}$$

holds, and the latter restriction is determined by 3-point values $\Omega_{0,3}(w_1, w_2, w_3)$ from repeated application of CohFT axiom (ii). $\diamond$

A finite dimensional $\mathbb{Q}$-algebra is *semisimple* if there exists a basis $\{e_i\}$ of idempotents,

$$e_i e_j = \delta_{ij} e_i,$$

after an extension of scalars to $\mathbb{C}$.

Definition 4. *A CohFT with unit $\Omega = (\Omega_{g,n})_{2g-2+n>0}$ is* semisimple *if $(V, \bullet, \mathbf{1})$ is a semisimple algebra.*

0.5 Classification and calculation. The Givental–Teleman classification concerns semisimple CohFTs with unit.[7] The form of the classification result is as follows: *a semisimple CohFT with unit Ω is uniquely determined by the following two structures:*

- *the topological part ω of Ω,*

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[6]Since $\overline{\mathcal{M}}_{0,3}$ is a point, we canonically identify $H^*(\overline{\mathcal{M}}_{0,3}, \mathbb{Q}) \cong \mathbb{Q}$, so $\Omega_{0,3}(v_1, v_2, v_3) \in \mathbb{Q}$.
[7]Semisimple CohFTs without unit are also covered, but we are interested here in the unital case. Semisimplicity is an essential condition.

- *an R-matrix*

$$R(z) = \mathsf{Id} + R_1 z + R_2 z^2 + R_3 z^3 + \dots, \quad R_k \in \mathrm{End}(V)$$

   satisfying the symplectic property

$$R(z) \cdot R^\star(-z) = \mathsf{Id},$$

   where $\star$ denotes the adjoint with respect to the metric η.

The precise statement of the Givental–Teleman classification will be discussed in Section 1.

Via the Givental–Teleman classification, a semisimple CohFT with unit Ω can be calculated in three steps:

(i) determine the ring $(V, \bullet, \mathbf{1})$ as explicitly as possible,

(ii) find a closed formula for the topological part ω of Ω via Lemma 3,

(iii) calculate the R-matrix of the theory.

In the language of Gromov–Witten theory, step (i) is the determination of the *small quantum cohomology* ring $QH^*(X, \mathbb{Q})$ via the 3-pointed genus 0 Gromov–Witten invariants. Step (ii) is then to calculate the Gromov–Witten invariants where the domain has a *fixed* complex structure of higher genus. New ideas are often required for the leap to higher genus moduli in step (iii). Finding a closed formula for the R-matrix requires a certain amount of luck.

Explaining how the above path to calculation plays out in three important CohFTs is my goal here. The three theories are:

- Witten's class on the moduli of r-spin curves,

- the Chern character of the Verlinde bundle on the moduli of curves,

- the Gromov–Witten theory of the Hilbert scheme of points of $\mathbb{C}^2$.

While each theory has geometric interest and the calculations have consequences in several directions, the focus of the paper will be on the CohFT determination. The paths to calculation pursued here are applicable in many other cases.

0.6 Past and future directions. The roots of the classification of semisimple CohFTs can be found in Givental's analysis Givental [2001a,b] and Lee and Pandharipande [2004] of the torus localization formula Graber and Pandharipande [1999] for the higher genus Gromov–Witten theory of toric varieties. The three CohFTs treated here are not directly accessible via the older torus localization methods. Givental's approach to the R-matrix via oscillating integrals (used often in the study of toric geometries) is not covered in the paper.

Many interesting CohFTs are *not* semisimple. For example, the Gromov–Witten theory of the famous Calabi–Yau quintic 3-fold,

$$X_5 \subset \mathbb{P}^4,$$

does *not* define a semisimple CohFT. However, in the past year, an approach to the quintic via the semisimple *formal quintic* theory Guo, Janda, and Ruan [2017] and Lho and Pandharipande [2017] appears possible. These developments are not surveyed here.

1 Givental–Teleman classification

1.1 Stable graphs. The boundary strata of the moduli space of curves correspond to *stable graphs*

$$\Gamma = (V, H, L, \ g : V \to \mathbb{Z}_{\geq 0}, \ v : H \to V, \ i : H \to H)$$

satisfying the following properties:

(i) V is a vertex set with a genus function $g : V \to \mathbb{Z}_{\geq 0}$,

(ii) H is a half-edge set equipped with a vertex assignment $v : H \to V$ and an involution i,

(iii) E, the edge set, is defined by the 2-cycles of i in H (self-edges at vertices are permitted),

(iv) L, the set of legs, is defined by the fixed points of i and endowed with a bijective correspondence with a set of markings,

(v) the pair (V, E) defines a *connected* graph,

(vi) for each vertex v, the stability condition holds:

$$2g(v) - 2 + n(v) > 0,$$

where $n(v)$ is the valence of Γ at v including both half-edges and legs.

An automorphism of Γ consists of automorphisms of the sets V and H which leave invariant the structures g, v, and i (and hence respect E and L). Let Aut(Γ) denote the automorphism group of Γ.

The genus of a stable graph Γ is defined by:

$$g(\Gamma) = \sum_{v \in V} g(v) + h^1(\Gamma).$$

Let $\mathsf{G}_{g,n}$ denote the set of all stable graphs (up to isomorphism) of genus g with n legs. The strata[8] of the moduli space $\overline{\mathcal{M}}_{g,n}$ of Deligne–Mumford stable curves are in bijective correspondence to $\mathsf{G}_{g,n}$ by considering the dual graph of a generic pointed curve parameterized by the stratum.

To each stable graph Γ, we associate the moduli space

$$\overline{\mathcal{M}}_{\Gamma} = \prod_{v \in V} \overline{\mathcal{M}}_{g(v),n(v)}.$$

Let π_v denote the projection from $\overline{\mathcal{M}}_{\Gamma}$ to $\overline{\mathcal{M}}_{g(v),n(v)}$ associated to the vertex v. There is a canonical morphism

(1) $\iota_{\Gamma} : \overline{\mathcal{M}}_{\Gamma} \to \overline{\mathcal{M}}_{g,n}$

with image[9] equal to the boundary stratum associated to the graph Γ.

1.2 *R*-matrix action.

1.2.1 First action. Let $\Omega = (\Omega_{g,n})_{2g-2+n>0}$ be a CohFT[10] on the vector space (V, η). Let R be a matrix series

$$R(z) = \sum_{k=0}^{\infty} R_k z^k \in \mathsf{Id} + z \cdot \mathrm{End}(V)[[z]]$$

which satisfies the symplectic condition

$$R(z) \cdot R^{\star}(-z) = \mathsf{Id}.$$

We define a new CohFT $R\Omega$ on the vector space (V, η) by summing over stable graphs Γ with summands given by a product of vertex, edge, and leg contributions,

(2) $(R\Omega)_{g,n} = \sum_{\Gamma \in \mathsf{G}_{g,n}} \dfrac{1}{|\mathrm{Aut}\,(\Gamma)|} \iota_{\Gamma\star} \left(\prod_{v \in V} \mathrm{Cont}(v) \prod_{e \in E} \mathrm{Cont}(e) \prod_{l \in L} \mathrm{Cont}(l) \right),$

where

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[8] We consider here the standard stratification by topological type of the pointed curve
[9] The degree of ι_{Γ} is $|\mathrm{Aut}(\Gamma)|$.
[10] Ω is not assumed here to be unital – only CohFT axioms (i) and (ii) are imposed.

(i) the vertex contribution is

$$\mathsf{Cont}(v) = \Omega_{g(v),n(v)},$$

where $g(v)$ and $n(v)$ denote the genus and number of half-edges and legs of the vertex,

(ii) the leg contribution is the $\mathrm{End}(V)$-valued cohomology class

$$\mathsf{Cont}(l) = R(\psi_l),$$

where $\psi_l \in H^2(\overline{\mathcal{M}}_{g(v),n(v)}, \mathbb{Q})$ is the cotangent class at the marking corresponding to the leg,

(iii) the edge contribution is

$$\mathsf{Cont}(e) = \frac{\eta^{-1} - R(\psi'_e)\eta^{-1}R(\psi''_e)^{\top}}{\psi'_e + \psi''_e},$$

where ψ'_e and ψ''_e are the cotangent classes at the node which represents the edge e. The symplectic condition guarantees that the edge contribution is well-defined.

We clarify the meaning of the edge contribution (iii),

$$\mathsf{Cont}(e) \in V^{\otimes 2} \otimes H^{\star}(\overline{\mathcal{M}}_{g',n'}) \otimes H^{\star}(\overline{\mathcal{M}}_{g'',n''}),$$

where (g', n') and (g'', n'') are the labels of the vertices adjacent to the edge e by writing the formula explicitly in coordinates.

Let $\{e_\mu\}$ be a $\mathbb{Q}$-basis of V. The components of the R-matrix in the basis are $R^\nu_\mu(z)$,

$$R(z)(e_\mu) = \sum_\nu R^\nu_\mu(z) \cdot e_\nu.$$

The components of $\mathsf{Cont}(e)$ are

$$\mathsf{Cont}(e)^{\mu\nu} = \frac{\eta^{\mu\nu} - \sum_{\rho,\sigma} R^\mu_\rho(\psi'_e) \cdot \eta^{\rho\sigma} \cdot R^\nu_\sigma(\psi''_e)}{\psi'_e + \psi''_e} \in H^{\star}(\overline{\mathcal{M}}_{g',n'}) \otimes H^{\star}(\overline{\mathcal{M}}_{g'',n''}).$$

The fraction

$$\frac{\eta^{\mu\nu} - \sum_{\rho,\sigma} R^\mu_\rho(z) \cdot \eta^{\rho\sigma} \cdot R^\nu_\sigma(w)}{z + w}$$

is a power series in z and w since the numerator vanishes when $z = -w$ as a consequence of the symplectic condition which, in coordinates, takes the form

$$\sum_{\rho,\sigma} R^\mu_\rho(z) \cdot \eta^{\rho\sigma} \cdot R^\nu_\sigma(-z) = \eta^{\mu\nu}.$$

The substitution $z = \psi'_e$ and $w = \psi''_e$ is therefore unambiguously defined.

Definition 5. *Let $R\Omega$ be the CohFT obtained from Ω by the R-action* (2).

The above R-action was first defined[11] on Gromov–Witten potentials by Givental [2001a]. An abbreviated treatment of the lift to CohFTs appears in papers by Teleman [2012] and Shadrin [2009]. A careful proof that $R\Omega$ satisfies CohFT axioms (i) and (ii) can be found in Pandharipande, Pixton, and Zvonkine [2015, Section 2].

If Ω is a CohFT with unit on $(V, \eta, \mathbf{1})$, then $R\Omega$ may not respect the unit $\mathbf{1}$. To handle the unit, a second action is required.

1.2.2   Second action. A second action on the CohFT[12] Ω on (V, η) is given by translations. Let $T \in V[[z]]$ be a series with *no terms of degree* 0 *or* 1,

$$T(z) = T_2 z^2 + T_3 z^3 + \dots, \quad T_k \in V.$$

Definition 6. *Let $T\Omega$ be the CohFT obtained from Ω by the formula*

$$(T\Omega)_{g,n}(v_1, \dots, v_n) = \sum_{m=0}^{\infty} \frac{1}{m!} p_{m\star}\Big(\Omega_{g,n+m}(v_1, \dots, v_n, T(\psi_{n+1}), \dots, T(\psi_{n+m}))\Big),$$

where $p_m : \overline{\mathcal{M}}_{g,n+m} \to \overline{\mathcal{M}}_{g,n}$ is the morphism forgetting the last m markings.

The right side of the formula in Definition 6 is a formal expansion by distributing the powers of the ψ classes as follows:

$$\Omega_{g,n+m}(\cdots, T(\psi_{\bullet}), \cdots) = \sum_{k=2}^{\infty} \psi_{\bullet}^k \cdot \Omega_{g,n+m}(\cdots, T_k, \cdots).$$

The summation is finite because T has no terms of degree 0 or 1.

1.3   Reconstruction. We can now state the Givental–Teleman classification result Teleman [2012]. Let Ω be a semisimple CohFT with unit on $(V, \eta, \mathbf{1})$, and let ω be the topological part of Ω. For a symplectic matrix R, define

$$R.\omega = R(T(\omega)) \quad \text{with} \quad T(z) = z((\mathsf{Id} - R(z)) \cdot \mathbf{1}) \in V[[z]].$$

By Pandharipande, Pixton, and Zvonkine [2015, Proposition 2.12], $R.\omega$ is a CohFT with unit on $(V, \eta, \mathbf{1})$. The Givental–Teleman classification asserts the *existence* of a unique R-matrix which exactly recovers Ω.

Theorem 7. *There exists a unique symplectic matrix*

$$R \in \mathsf{Id} + z \cdot \mathrm{End}(V)[[z]]$$

which reconstructs Ω from ω,

$$\Omega = R.\omega,$$

as a CohFT with unit.

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[11]To simplify our formulas, we have changed Givental's and Teleman's conventions by replacing R with R^{-1}. Equation (2) above then determines a right group action on CohFTs rather than a left group action as in Givental's and Teleman's papers.

[12]To define the translation action, Ω is required only to be CohFT and not necessarily a CohFT with unit.

The first example concerns the total Chern class CohFT of Section 0.3,

$$V = \mathbb{Q}, \quad \eta(1,1) = 1, \quad \mathbf{1} = 1, \quad \Omega_{g,n}(1,\ldots,1) = c(\mathbb{E}) \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}).$$

The topological part is the trivial CohFT, and the R-matrix is

$$R(z) = \exp\left(-\sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)(2k-1)} z^{2k-1}\right).$$

That the above R-matrix reconstructs the total Chern class CohFT is a consequence of Mumford's calculation Mumford [1983] of the Chern character of the Hodge bundle by Grothendieck–Riemann–Roch.

1.4   Chow field theories. Let $(V, \eta, \mathbf{1})$ be a $\mathbb{Q}$-vector space with a non-degenerate symmetric 2-form and a distinguished element. Let $\Omega = (\Omega_{g,n})_{2g-2+n>0}$ be a system of tensors

$$\Omega_{g,n} \in A^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \otimes (V^*)^{\otimes n}$$

where A^* is the Chow group of algebraic cycles modulo rational equivalence. In order to define a Chow field theory, the system Ω must satisfy the CohFT axioms of Section 0.3 with cohomology H^* replaced everywhere by Chow A^*.

Definition 8. *A system* $\Omega = (\Omega_{g,n})_{2g-2+n>0}$ *of elements*

$$\Omega_{g,n} \in A^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \otimes (V^*)^{\otimes n}$$

satisfying (i) and (ii) is a Chow field theory *or a* ChowFT. *If (iii) is also satisfied,* Ω *is a ChowFT with unit.*

For ChowFTs, the quantum product $(V, \bullet, \mathbf{1})$ and semisimplicity are defined just as for CohFTs. The R- and T-actions of Sections 1.2 also lift immediately to ChowFTs. However, the classification of semisimple ChowFTs is an open question.

Question 9. *Does the Givental–Teleman classification of Theorem 7 hold for a semisimple Chow field theory* Ω *with unit?*

2   Witten's r-spin class

2.1 r-spin CohFT. Let $r \geq 2$ be an integer. Let $(V_r, \eta, \mathbf{1})$ be the following triple:

- V_r is an $(r-1)$-dimensional $\mathbb{Q}$-vector space with basis $e_0, \ldots, e_{r-2}$,

- η is the non-degenerate symmetric 2-form

$$\eta_{ab} = \langle e_a, e_b \rangle = \delta_{a+b, r-2},$$

- $\mathbf{1} = e_0.$

Witten's r-spin theory provides a family of classes

$$\mathcal{W}^r_{g,n}(a_1, \ldots, a_n) \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$$

for $a_1, \ldots, a_n \in \{0, \ldots, r-2\}$ which define a CohFT $\mathsf{W}^r = (\mathsf{W}^r_{g,n})_{2g-2+n>0}$ by

$$\mathsf{W}^r_{g,n} : V^{\otimes n}_r \to H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}), \quad \mathsf{W}^r_{g,n}(e_{a_1} \otimes \cdots \otimes e_{a_n}) = \mathcal{W}^r_{g,n}(a_1, \ldots, a_n).$$

The class $\mathcal{W}^r_{g,n}(a_1, \ldots, a_n)$ has (complex) degree[13]

(3)
$$\deg_{\mathbb{C}} \mathcal{W}^r_{g,n}(a_1, \ldots, a_n) = \mathsf{D}^r_{g,n}(a_1, \ldots, a_n)$$
$$= \frac{(r-2)(g-1) + \sum_{i=1}^n a_i}{r}.$$

If $\mathsf{D}^r_{g,n}(a_1, \ldots, a_n)$ is not an integer, the corresponding Witten's class vanishes.

The construction of $\mathcal{W}^r_{0,n}(a_1, \ldots, a_n)$ in genus 0 was carried out by Witten [1993] using r-spin structures. Let $\overline{\mathcal{M}}^r_{0,n}(a_1, \ldots, a_n)$ be the Deligne–Mumford moduli space parameterizing r^{th} roots,

$$\mathcal{L}^{\otimes r} \cong \omega_C\left(-\sum_{i=1}^n a_i p_i\right) \quad \text{where} \quad [C, p_1, \ldots, p_n] \in \overline{\mathcal{M}}_{0,n}.$$

The class $\frac{1}{r}\mathcal{W}^r_{0,n}(a_1, \ldots, a_n)$ is defined to be the push-forward to $\overline{\mathcal{M}}_{0,n}$ of the top Chern class of the bundle on $\overline{\mathcal{M}}^r_{0,n}(a_1, \ldots, a_n)$ with fiber $H^1(C, \mathcal{L})^*$.

The existence of Witten's class in higher genus is both remarkable and highly non-trivial. Polishchuk [2004] and Polishchuk and Vaintrob [2001] constructed

$$\mathcal{W}^r_{g,n}(a_1, \ldots, a_n) \in A^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$$

as an algebraic cycle class and proved and the CohFT axioms (i-iii) for a Chow field theory hold. The algebraic approach was later simplified in Chang, J. Li, and W.-P. Li [2015] and Chiodo [2006]. Analytic constructions appear in Fan, Jarvis, and Ruan [2013] and Mochizuki [2006].

2.2 Genus 0.

2.2.1 3 and 4 markings. Witten [1993] determined the following initial conditions in genus 0 with $n = 3, 4$:

(4)
$$\int_{\overline{\mathcal{M}}_{0,3}} \mathcal{W}^r_{0,3}(a_1, a_2, a_3) = \begin{vmatrix} 1 & \text{if } a_1 + a_2 + a_3 = r - 2, \\ 0 & \text{otherwise}, \end{vmatrix}$$

$$\int_{\overline{\mathcal{M}}_{0,4}} \mathcal{W}^r_{0,4}(1, 1, r-2, r-2) = \frac{1}{r}.$$

---

[13]So $\mathcal{W}^r_{g,n}(a_1, \ldots, a_n) \in H^{2 \cdot \mathsf{D}^r_{g,n}(a_1, \ldots, a_n)}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$.

Uniqueness of the r-spin CohFT in genus 0 follows easily from the initial conditions (4) and the axioms of a CohFT with unit.

The genus 0 sector of the CohFT W^r defines a quantum product[14] $\bullet$ on V_r. The resulting algebra $(V_r, \bullet, \mathbf{1})$, even after extension to $\mathbb{C}$, is *not* semisimple. Therefore, the Givental–Teleman classification can not be directly applied.

2.2.2 Witten's r-spin class and representations of $\mathsf{sl}_2(\mathbb{C})$.

Consider the Lie algebra $\mathsf{sl}_2 = \mathsf{sl}_2(\mathbb{C})$. Denote by ρ_k the kth symmetric power of the standard 2-dimensional representation of sl_2,

$$\rho_k = \mathrm{Sym}^k(\rho_1), \quad \dim \rho_k = k + 1.$$

The following complete solution of the genus 0 part of the CohFT W^r (after integration) was found by Pixton, see Pandharipande, Pixton, and Zvonkine [2016] for a proof.

Theorem 10. *Let $\mathbf{a} = (a_1, \ldots, a_{n \geq 3})$ with $a_i \in \{0, \ldots, r-2\}$ satisfy the degree constraint $\mathsf{D}_{0,n}^r(\mathbf{a}) = n - 3$. Then,*

$$\int_{\overline{\mathcal{M}}_{0,n}} \mathsf{W}_{0,n}^r(\mathbf{a}) = \frac{(n-3)!}{r^{n-3}} \dim \left[\rho_{r-2-a_1} \otimes \cdots \otimes \rho_{r-2-a_n} \right]^{\mathsf{sl}_2},$$

where the superscript sl_2 denotes the sl_2-invariant subspace.

2.2.3 Shifted Witten class.

Definition 11. *For $\gamma \in V_r$, the* shifted r-spin CohFT $\mathsf{W}^{r,\gamma}$ *is defined by*

$$\mathsf{W}_{g,n}^{r,\gamma}(v_1 \otimes \cdots \otimes v_n) = \sum_{m \geq 0} \frac{1}{m!} \, p_{m\star} \, \mathsf{W}_{g,n+m}^r(v_1 \otimes \cdots \otimes v_n \otimes \gamma^{\otimes m}),$$

where $p_m \colon \overline{\mathcal{M}}_{g,n+m} \to \overline{\mathcal{M}}_{g,n}$ is the map forgetting the last m markings.

Using degree formula $\mathsf{D}_{g,n}^r$, the summation in the definition of the shift is easily seen to be finite. The shifted Witten class $\mathsf{W}^{r,\gamma}$ determines a CohFT with unit, see Pandharipande, Pixton, and Zvonkine [2015, Section 1.1].

Definition 12. *Define the CohFT $\widehat{\mathsf{W}}^r$ with unit on $(V_r, \eta, \mathbf{1})$ by the shift*

$$\widehat{\mathsf{W}}^r = \mathsf{W}^{r,(0,\ldots,0,r)}$$

along the special vector $r e_{r-2} \in V_r$. Let $(V_r, \widehat{\bullet}, \mathbf{1})$ be the $\mathbb{Q}$-algebra determined by the quantum product defined by $\widehat{\mathsf{W}}^r$.

The *Verlinde algebra* of level r for sl_2 is spanned by the weights of sl_2 from 0 to $r-2$. The coefficient of c in the product $a \bullet b$ is equal to the dimension of the sl_2-invariant subspace of the representation $\rho_a \otimes \rho_b \otimes \rho_c$ provided the inequality

$$a + b + c \leq 2r - 4$$

is satisfied. Using Theorem 10 for the integral r-spin theory in genus 0, the following basic result is proven in Pandharipande, Pixton, and Zvonkine [2016].

---

[14]See Section 0.4.

Theorem 13. *The algebra* $(V_r, \widehat{\bullet}, \mathbf{1})$ *is isomorphic to the Verlinde algebra of level* r *for* $\mathfrak{sl}_2$.

Since the Verlinde algebra is well-known to be semisimple[15], the Givental–Teleman classification of Theorem 7 can be applied to the CohFT $\widehat{W}^r$. Using the degree formula (3) and Definition 12, we see the (complex) degree $D_{g,n}^r(a_1, \ldots, a_n)$ part of $\widehat{W}^r$ equals W^r,

$$\left[\widehat{W}_{g,n}^r(a_1, \ldots, a_n)\right]^{D_{g,n}^r(a_1, \ldots, a_n)} = W_{g,n}^r(a_1, \ldots, a_n).$$

Hence, a complete computation of $\widehat{W}^r$ also provides a computation of W^r.

2.3 The topological field theory. After the studying genus 0 theory, we turn our attention to the topological part $\widehat{\omega}^r$ of $\widehat{W}^r$. The following two results of Pandharipande, Pixton, and Zvonkine [ibid.] provide a complete calculation.

Proposition 14. *The basis of normalized idempotents of* $(V_r, \widehat{\bullet}, \mathbf{1})$ *is given by*

$$v_k = \sqrt{\frac{2}{r}} \sum_{a=0}^{r-2} \sin\left(\frac{(a+1)k\pi}{r}\right) e_a, \qquad k \in \{1, \ldots, r-1\}.$$

More precisely, we have

$$\eta(v_k, v_l) = (-1)^{k-1}\delta_{k,l}, \qquad v_k \widehat{\bullet} v_l = \frac{\sqrt{r/2}}{\sin(\frac{k\pi}{r})} v_k \delta_{k,l}.$$

Once the normalized idempotents are found, the computation of $\widehat{\omega}^r$ is straightforward by Lemma 3 and elementary trigonometric identities.

Proposition 15. *For* $a_1, \ldots, a_n \in \{0, \ldots, r-2\}$, *we have*

$$\widehat{\omega}_{g,n}^r(e_{a_1}, \ldots, e_{a_n}) = \left(\frac{r}{2}\right)^{g-1} \sum_{k=1}^{r-1} \frac{(-1)^{(k-1)(g-1)} \prod_{i=1}^n \sin\left(\frac{(a_i+1)k\pi}{r}\right)}{\left(\sin(\frac{k\pi}{r})\right)^{2g-2+n}}.$$

In Proposition 15, the CohFT $\widehat{\omega}^r$ is viewed as taking values in $\mathbb{Q}$ via the canonical identification

$$H^0(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \cong \mathbb{Q}.$$

2.4 The R-matrix. The last (and often hardest) step in the computation of a semisimple CohFT via the Givental–Teleman classification is to find the unique R-matrix. Remarkably, there exists a closed formula in hypergeometric series for the R-matrix of the CohFT $\widehat{W}^r$. The precise shift in Definition 12 of the CohFT $\widehat{W}^r$ is crucial: the shift re_{r-2} is (up to scale) the *only* shift of W^r for which closed formulas for the R-matrix are known.

---

[15]An explicit normalized idempotent basis is given in Proposition 14 below.

The method of finding the unique R-matrix for W^r uses the *Euler field* e_{r-2} at the shift re_{r-2}. The operator of quantum multiplication $\widehat{\bullet}$ by the Euler field in the basis $e_0, \ldots, e_{r-2}$ is

$$\xi = \begin{pmatrix} 0 & \cdots & \cdots & 0 & 2 \\ 0 & & 0 & 2 & 0 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ 0 & 2 & 0 & & 0 \\ 2 & 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

In the same frame, the *shifted degree operator* is

$$\mu = \frac{1}{2r} \begin{pmatrix} -(r-2) & 0 & \cdots & \cdots & 0 \\ 0 & -(r-4) & 0 & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & & & 0 & r-4 & 0 \\ 0 & \cdots & & \cdots & 0 & r-2 \end{pmatrix}.$$

Since $\widehat{\mathsf{W}}^r$ has an Euler field with an associated degree operator, the unique R-matrix for the classification is given by the solution of

(5) $$[R_{m+1}, \xi] = (m - \mu) R_m$$

with the initial condition $R_0 = \mathsf{Id}$, see Teleman [2012].

Definition 16. *For each integer $a \in \{0, \ldots, r-2\}$, define the hypergeometric series*

$$\boldsymbol{B}_{r,a}(z) = \sum_{m=0}^{\infty} \left[\prod_{i=1}^{m} \frac{\big((2i-1)r - 2(a+1)\big)\big((2i-1)r + 2(a+1)\big)}{i} \right] \left(-\frac{z}{16r^2} \right)^m.$$

Let $\boldsymbol{B}_{r,a}^{\mathrm{even}}$ and $\boldsymbol{B}_{r,a}^{\mathrm{odd}}$ the even and odd summands[16] of the series $\boldsymbol{B}_{r,a}$.

The unique solution to (5) is computed in Pandharipande, Pixton, and Zvonkine [2016]. The R-matrix of $\widehat{\mathsf{W}}^r$ has a surprisingly simple form.

Theorem 17. *The unique R-matrix classifying $\widehat{\mathsf{W}}^r$ has coefficients*

$$R_a^a = \boldsymbol{B}_{r,a}^{\mathrm{even}}(z), \quad a \in \{0, \ldots, r-2\}$$

on the main diagonal, and

$$R_a^{r-2-a} = \boldsymbol{B}_{r,a}^{\mathrm{odd}}(z), \quad a \in \{0, \ldots, r-2\}$$

---

[16] The even summand consists of all the even powers of z (and likewise for the odd summand).

on the antidiagonal (if r is even, the coefficient at the intersection of both diagonals is 1), and 0 everywhere else.

In case $r = 2$, the matrix is trivial $R(z) = \mathsf{Id}$. For $r = 3$ and 4 respectively, the R-matrices[17] are

$$R(z) = \begin{pmatrix} \boldsymbol{B}_{3,0}^{\mathrm{even}}(z) & \boldsymbol{B}_{3,1}^{\mathrm{odd}}(z) \\ \boldsymbol{B}_{3,0}^{\mathrm{odd}}(z) & \boldsymbol{B}_{3,1}^{\mathrm{even}}(z) \end{pmatrix},$$

$$R(z) = \begin{pmatrix} \boldsymbol{B}_{4,0}^{\mathrm{even}}(z) & 0 & \boldsymbol{B}_{4,2}^{\mathrm{odd}}(z) \\ 0 & 1 & 0 \\ \boldsymbol{B}_{4,0}^{\mathrm{odd}}(z) & 0 & \boldsymbol{B}_{4,2}^{\mathrm{even}}(z) \end{pmatrix}.$$

2.5 Calculation of W^r. The analysis of Sections 2.2-2.4 together complete the calculation of $\widehat{\mathsf{W}}^r$,

$$\widehat{\mathsf{W}}^r = R.\widehat{\omega}^r,$$

in exactly the steps (i)-(iii) proposed in Section 0.5 of the Introduction. Then, as we have seen,

$$\mathsf{W}_{g,n}^r(a_1, \ldots, a_n) = \left[\widehat{\mathsf{W}}_{g,n}^r(a_1, \ldots, a_n)\right]^{\mathsf{D}_{g,n}^r(a_1, \ldots, a_n)}.$$

The calculation has an immediate consequence Pandharipande, Pixton, and Zvonkine [ibid.].

Corollary 18. *Witten's r-spin class on $\overline{\mathfrak{M}}_{g,n}$ lies in the tautological ring (in cohomology),*

$$\mathsf{W}_{g,n}^r(a_1, \ldots, a_n) \in RH^*(\overline{\mathfrak{M}}_{g,n}, \mathbb{Q}).$$

We refer the reader to Pandharipande [2018] for a discussion of tautological classes on the moduli space of curves. In fact, the first proof of Pixton's relations in $RH^*(\overline{\mathfrak{M}}_{g,n}, \mathbb{Q})$ was obtained via the calculation of $\widehat{\mathsf{W}}^3$ in Pandharipande, Pixton, and Zvonkine [2015].

2.6 Questions. Whether Corollary 18 also holds in Chow is an interesting question: is

(6) $$\mathsf{W}_{g,n}^r(a_1, \ldots, a_n) \in R^*(\overline{\mathfrak{M}}_{g,n}, \mathbb{Q})?$$

A positive answer to Question 9 about the classification of Chow field theories would imply a positive answer here. The following question may be viewed as a refinement of (6).

Question 19. *Find a formula in algebraic cycles for Witten's r-spin class on $\overline{\mathfrak{M}}_{g,n}^r(a_1, \ldots, a_n)$ before push-forward to $\overline{\mathfrak{M}}_{g,n}$.*

---

[17] R here is R^{-1} in Pandharipande, Pixton, and Zvonkine [2015] and Pandharipande, Pixton, and Zvonkine [2016] because of a change of conventions.

Another open direction concerns the moduli spaces of holomorphic differentials Bainbridge, Chen, Gendron, Grushevsky, and Moeller [2016] and Farkas and Pandharipande [2018]. Let $(a_1, \ldots, a_n)$ be a partition of $2g - 2$ with non-negative parts. Let

$$\overline{\mathcal{H}}_g(a_1, \ldots, a_n) \subset \overline{\mathcal{M}}_{g,n}$$

be the closure of the locus of moduli points

$$[C, p_1, \ldots, p_n] \in \mathcal{M}_{g,n} \quad \text{where} \quad \omega_C \cong \mathcal{O}_C\Big(\sum_{i=1}^{n} a_i \, p_i\Big).$$

For $r - 2 \geq \text{Max}\{a_1, \ldots, a_n\}$, Witten's r-spin class $\mathbb{W}_{g,n}^r(a_1, \ldots, a_n)$ is well-defined and of degree *independent* of r,

$$\mathsf{D}_{g,n}^r(a_1, \ldots, a_n) = \frac{(r-2)(g-1) + \sum_{i=1}^{n} a_i}{r} = g - 1.$$

By Pandharipande, Pixton, and Zvonkine [2016, Theorem 7], after scaling by r^{g-1},

$$\mathbb{W}_{g,n}(a_1, \ldots, a_n)[r] = r^{g-1} \cdot \mathbb{W}_{g,n}^r(a_1, \ldots, a_n) \in RH^{g-1}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$$

is a *polynomial* in r for all sufficiently large r.

Question 20. *Prove the following conjecture of Pandharipande, Pixton, and Zvonkine [ibid., Appendix]:*

$$(-1)^g \mathbb{W}_{g,n}(a_1, \ldots, a_n)[0] = [\overline{\mathcal{H}}_g(a_1, \ldots, a_n)] \in H^{2(g-1)}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}).$$

3   Chern character of the Verlinde bundle

3.1   Verlinde CohFT.   Let G be a complex, simple, simply connected Lie group with Lie algebra $\mathfrak{g}$. Fix an integer *level* $\ell > 0$. Let $(V_\ell, \eta, \mathbf{1})$ be the following triple:

- V_ℓ is the $\mathbb{Q}$-vector space with basis indexed by the irreducible representations of $\mathfrak{g}$ at level ℓ,

- η is the non-degenerate symmetric 2-form

$$\eta(\mu, \nu) = \delta_{\mu, \nu^\star}$$

where $\nu^\star$ denotes the dual representation,

- $\mathbf{1}$ is the basis element corresponding to the trivial representation.

Let $\mu_1, \ldots, \mu_n$ be n irreducible representations of $\mathfrak{g}$ at level ℓ. A vector bundle

$$\mathbb{V}_g(\mu_1, \ldots, \mu_n) \to \overline{\mathcal{M}}_{g,n}$$

is constructed in Tsuchiya, Ueno, and Yamada [1989]. Over nonsingular curves, the fibers of $\mathbb{V}_g(\mu_1, \ldots, \mu_n)$ are the spaces of non-abelian theta functions – spaces of global

sections of the determinant line bundles over the moduli of parabolic G-bundles. To extend $\mathbb{V}_g(\mu_1, \ldots, \mu_n)$ over the boundary

$$\partial \mathfrak{M}_{g,n} \subset \overline{\mathfrak{M}}_{g,n},$$

the theory of *conformal blocks* is required Tsuchiya, Ueno, and Yamada [ibid.]. The vector bundle $\mathbb{V}_g(\mu_1, \ldots, \mu_n)$ has various names in the literature: the Verlinde bundle, the bundle of conformal blocks, and the bundle of vacua. A study in genus 0 and 1 can be found in Fakhruddin [2012].

A CohFT Ω^ℓ is defined via the Chern character[18] of the Verlinde bundle:

$$\Omega^\ell_{g,n}(\mu_1, \ldots, \mu_n) = \mathsf{ch}_t(\mathbb{V}_g(\mu_1, \ldots, \mu_n)) \in H^\star(\overline{\mathfrak{M}}_{g,n}, \mathbb{Q}).$$

CohFT axiom (i) for Ω^ℓ is trivial. Axiom (ii) follows from the fusion rules Tsuchiya, Ueno, and Yamada [1989]. Axiom (iii) for the unit $\mathbf{1}$ is the *propagation of vacua* Fakhruddin [2012, Proposition 2.4(i)].

3.2   Genus 0 and the topological part.   Since the variable t carries the degree grading, the topological part ω^ℓ of Ω^ℓ is obtained by setting $t = 0$,

$$\omega^\ell_{g,n} = \Omega^\ell_{g,n}\Big|_{t=0}.$$

The result is the just the rank of the Verlinde bundle,

$$\omega^\ell_{g,n}(\mu_1, \ldots, \mu_n) = \mathrm{rk}\, \mathbb{V}_g(\mu_1, \ldots, \mu_n) = d_g(\mu_1, \ldots, \mu_n).$$

With the quantum product obtained[19] from ω^ℓ, $(V_\ell, \bullet, \mathbf{1})$ is the *fusion algebra*.

Since the fusion algebra is well-known[20] to be semisimple, the CohFT with unit Ω^ℓ is also semisimple, The subject has a history starting in the mid 80s with the discovery and in 90s with several proofs of the *Verlinde formula* for the rank $d_g(\mu_1, \ldots, \mu_n)$, see Beauville [1996] for an overview.

Hence, steps (i) and (ii) of the computational strategy of Section 0.5 for Ω^ℓ are complete (and have been for many years). Step (iii) is the jump to moduli.

3.3   Path to the R-matrix.   The shifted r-spin CohFT $\widehat{W}^r$ has an Euler field obtained from the pure dimensionality of Witten's r-spin class which was used to find the unique R-matrix in Section 2. The CohFT Ω^ℓ is not of pure dimension and has no Euler field. A different path to the R-matrix is required here.

---

[18]For a vector bundle $\mathbb{V}$ with Chern roots $r_1, \ldots, r_k$,

$$\mathsf{ch}_t(\mathbb{V}) = \sum_{j=1}^{k} e^{t r_j}.$$

The parameter t may be treated either as a formal variable, in which case the CohFT is defined over the ring $\mathbb{Q}[[t]]$ instead of $\mathbb{Q}$, or as a rational number $t \in \mathbb{Q}$.

[19]Since the quantum product depends only upon the tensors of genus 0 with 3 markings, the quantum products of Ω^ℓ and ω^ℓ are equal.

[20]See, for example, Beauville [1996, Proposition 6.1].

The restriction of the tensor $\Omega^\ell_{g,n}$ to the open set of nonsingular curves

$$\mathfrak{M}_{g,n} \subset \overline{\mathfrak{M}}_{g,n}$$

forgets a lot of the data of the CohFT. However, by Marian, Oprea, Pandharipande, Pixton, and Zvonkine [2017, Lemma 2.2], the restriction is enough to uniquely determine the R-matrix of Ω^ℓ. Fortunately, the restriction is calculable in closed form:

- the first Chern class of the Verlinde bundle over $\mathfrak{M}_{g,n}$ is found in Tsuchimoto [1993],

- the existence of a projectively flat connection[21] Tsuchiya, Ueno, and Yamada [1989] on the Verlinde bundle over $\mathfrak{M}_{g,n}$ then determines the full Chern character over $\mathfrak{M}_{g,n}$.

As should be expected, the computation of Ω^ℓ relies significantly upon the past study of the Verlinde bundles.

3.4 The R-matrix. For a simple Lie algebra $\mathfrak{g}$ and a level ℓ, the conformal anomaly is

$$c = c(\mathfrak{g}, \ell) = \frac{\ell \dim \mathfrak{g}}{\check{h} + \ell},$$

where $\check{h}$ is the dual Coxeter number. For each representation with highest weight μ of level ℓ, define

$$\mathsf{w}(\mu) = \frac{(\mu, \mu + 2\rho)}{2(\check{h} + \ell)}.$$

Here, ρ is half of the sum of the positive roots, and the Cartan–Killing form $(,)$ is normalized so that the longest root θ satisfies

$$(\theta, \theta) = 2.$$

Example 21. *For $\mathfrak{g} = \mathfrak{sl}(r, \mathbb{C})$, the highest weight of a representation of level ℓ is given by an r-tuple of integers*

$$\mu = (\mu^1, \ldots, \mu^r), \ \ell \geq \mu^1 \geq \cdots \geq \mu^r \geq 0,$$

defined up to shifting the vector components by the same integer. Furthermore, we have

$$c(\mathfrak{g}, \ell) = \frac{\ell(r^2 - 1)}{\ell + r},$$

$$\mathsf{w}(\mu) = \frac{1}{2(\ell + r)} \left(\sum_{i=1}^{r} (\mu^i)^2 - \frac{1}{r} \left(\sum_{i=1}^{r} \mu^i \right)^2 + \sum_{i=1}^{r} (r - 2i + 1)\mu^i \right).$$

Via the path to the R-matrix discussed in Section 3.3, a simple closed form for the R-matrix of Ω^ℓ is found in Marian, Oprea, Pandharipande, Pixton, and Zvonkine [2017] using the constants $c(\mathfrak{g}, \ell)$ and $\mathsf{w}(\mu)$ from representation theory.

---

[21]Often called the Hitchin connection.

Theorem 22. *The CohFT Ω^ℓ is reconstructed from the topological part ω^ℓ by the diagonal R-matrix*

$$R(z)_\mu^\mu = \exp\left(tz \cdot \left(-\mathsf{w}(\mu) + \frac{c(\mathfrak{g}, \ell)}{24}\right)\right).$$

For the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2$ at level $\ell = 1$, there are only two representations $\{\varnothing, \square\}$ to consider[22],

$$c(\mathfrak{sl}_2, 1) = 1, \quad \mathsf{w}(\varnothing) = 0, \quad \mathsf{w}(\square) = \frac{1}{4}.$$

As an example of the reconstruction result of Theorem 22,

$$\Omega^\ell = R.\omega^\ell,$$

the total Chern character $\mathrm{ch}\,\mathbb{V}_g(\square, \ldots, \square)$ at $t = 1$ is

$$\exp\left(-\frac{\lambda_1}{2}\right) \cdot \sum_{\Gamma \in \mathsf{G}_{g,n}^{\mathrm{even}}} \frac{2^{g-h^1(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} \cdot {}^\iota\Gamma_\star \left(\prod_{e \in \mathrm{E}} \frac{1 - \exp\left(-\frac{1}{4}(\psi_e' + \psi_e'')\right)}{\psi_e' + \psi_e''} \cdot \prod_{l \in \mathrm{L}} e^{-\psi_l/4}\right)$$

in $H^*(\overline{\mathfrak{M}}_{g,n}, \mathbb{Q})$. A few remarks about the above formula are required:

- The classes λ_1 and ψ are the first Chern classes of the Hodge bundle and the cotangent line bundle respectively.

- The sum is over the set of *even* stable graphs,

$$\mathsf{G}_{g,n}^{\mathrm{even}} \subset \mathsf{G}_{g,n},$$

  defined by requiring the valence $\mathsf{n}(v)$ to be even for every vertex v of the graph.

- The Verlinde rank $d_{\mathsf{g}(v)}(\square, \ldots, \square)$ with $\mathsf{n}(v)$ insertions equals 2^g in the even case, see Beauville [1996]. The product of $2^{\mathsf{g}(v)}$ over the vertices of Γ yields $2^{g-h^1(\Gamma)}$, where h^1 denotes the first Betti number.

3.5 Questions. A different approach to the calculation of the Chern character of the Verlinde bundle in the $\mathfrak{sl}_2$ case (for every level) was pursued in Faber, Marian, and Pandharipande [n.d.] using the geometry introduced by Thaddeus [1994] to prove the Verlinde formula. The outcome of Faber, Marian, and Pandharipande [n.d.] is a more difficult calculation (with a much more complex answer), but with one advantage: the projective flatness of the Hitchin connection over $\mathfrak{M}_{g,n}$ is *not* used. When the flatness is introduced, the method of Faber, Marian, and Pandharipande [ibid.] yields tautological relations. Unfortunately, no such relations are obtained by the above R-matrix calculation of Ω^ℓ since the projective flatness is an input.

Question 23. *Is there an alternative computation of Ω^ℓ which does* not *use the projective flatness of the Hitchin connection and which systematically produces tautological relations in $RH^*(\mathfrak{M}_{g,n}, \mathbb{Q})$?*

Of course, if the answer to Question 23 is yes, then the next question is whether *all* tautological relations are produced.[23]

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[22] Here, $\varnothing$ is the trivial representation (corresponding to **1**) and $\square$ is the standard representation.
[23] I first heard an early version of this question from R. Bott at Harvard in the 90s.

4 Gromov–Witten theory of $\mathsf{Hilb}^m(\mathbb{C}^2)$

4.1 T-equivariant cohomology of $\mathsf{Hilb}^m(\mathbb{C}^2)$. The *Hilbert scheme* $\mathsf{Hilb}^m(\mathbb{C}^2)$ of m points in the plane $\mathbb{C}^2$ parameterizes ideals $\mathcal{I} \subset \mathbb{C}[x, y]$ of colength m,

$$\dim_\mathbb{C} \mathbb{C}[x, y]/\mathcal{I} = m \,.$$

The Hilbert scheme $\mathsf{Hilb}^m(\mathbb{C}^2)$ is a nonsingular, irreducible, quasi-projective variety of dimension $2m$, see Nakajima [1999] for an introduction. An open dense set of $\mathsf{Hilb}^m(\mathbb{C}^2)$ parameterizes ideals associated to configurations of m distinct points.

The symmetries of $\mathbb{C}^2$ lift to the Hilbert scheme. The algebraic torus

$$\mathsf{T} = (\mathbb{C}^*)^2$$

acts diagonally on $\mathbb{C}^2$ by scaling coordinates,

$$(z_1, z_2) \cdot (x, y) = (z_1 x, z_2 y) \,.$$

We review the Fock space description of the T-equivariant cohomology of the Hilbert scheme of points of $\mathbb{C}^2$ following the notation of Okounkov and Pandharipande [2010a, Section 2.1].

By definition, the *Fock space* $\mathcal{F}$ is freely generated over $\mathbb{Q}$ by commuting creation operators α_{-k}, $k \in \mathbb{Z}_{>0}$, acting on the vacuum vector $v_\varnothing$. The annihilation operators α_k, $k \in \mathbb{Z}_{>0}$, kill the vacuum

$$\alpha_k \cdot v_\varnothing = 0, \quad k > 0 \,,$$

and satisfy the commutation relations $[\alpha_k, \alpha_l] = k \, \delta_{k+l}$.

A natural basis of $\mathcal{F}$ is given by the vectors

$$(7) \qquad\qquad |\mu\rangle = \frac{1}{\mathfrak{z}(\mu)} \prod_i \alpha_{-\mu_i} \, v_\varnothing$$

indexed by partitions μ. Here, $\mathfrak{z}(\mu) = |\mathrm{Aut}(\mu)| \prod_i \mu_i$ is the usual normalization factor. Let the length $\ell(\mu)$ denote the number of parts of the partition μ.

The *Nakajima basis* defines a canonical isomorphism,

$$(8) \qquad\qquad \mathcal{F} \otimes_\mathbb{Q} \mathbb{Q}[t_1, t_2] \cong \bigoplus_{n \geq 0} H_\mathsf{T}^*(\mathsf{Hilb}^m(\mathbb{C}^2), \mathbb{Q}).$$

The Nakajima basis element corresponding to $|\mu\rangle$ is

$$\frac{1}{\prod_i \mu_i} [\mathcal{V}_\mu]$$

where $[\mathcal{V}_\mu]$ is (the cohomological dual of) the class of the subvariety of $\mathsf{Hilb}^{|\mu|}(\mathbb{C}^2)$ with generic element given by a union of schemes of lengths

$$\mu_1, \dots, \mu_{\ell(\mu)}$$

supported at $\ell(\mu)$ distinct points[24] of $\mathbb{C}^2$. The vacuum vector $v_\varnothing$ corresponds to the unit in

$$1 \in H_{\mathsf{T}}^*(\mathsf{Hilb}^0(\mathbb{C}^2), \mathbb{Q}) \, .$$

The variables t_1 and t_2 are the equivariant parameters corresponding to the weights of the T-action on the tangent space $\mathsf{Tan}_0(\mathbb{C}^2)$ at the origin of $\mathbb{C}^2$.

The subspace $\mathcal{F}_m \subset \mathcal{F} \otimes_{\mathbb{Q}} \mathbb{Q}[t_1, t_2]$ corresponding to $H_{\mathsf{T}}^*(\mathsf{Hilb}^m(\mathbb{C}^2), \mathbb{Q})$ is spanned by the vectors (7) with $|\mu| = n$. The subspace can also be described as the n-eigenspace of the *energy operator*:

$$|\cdot| = \sum_{k>0} \alpha_{-k}\, \alpha_k \, .$$

The vector $|1^n\rangle$ corresponds to the unit

$$1 \in H_{\mathsf{T}}^*(\mathsf{Hilb}^m(\mathbb{C}^2), \mathbb{Q}) \, .$$

The standard inner product on the T-equivariant cohomology of $\mathsf{Hilb}^m(\mathbb{C}^2)$ induces the following *nonstandard* inner product on Fock space after an extension of scalars:

$$(9) \qquad\qquad \langle \mu | \nu \rangle = \frac{(-1)^{|\mu|-\ell(\mu)}}{(t_1 t_2)^{\ell(\mu)}} \frac{\delta_{\mu\nu}}{\mathfrak{z}(\mu)} \, .$$

4.2 Gromov–Witten CohFT of $\mathsf{Hilb}^m(\mathbb{C}^2)$.

Let $m > 0$ be a colength. Let $(V_m, \eta, \mathbf{1})$ be the following triple:

- V_m is the free $\mathbb{Q}(t_1, t_2)[[q]]$-module $\mathcal{F}_m \otimes_{\mathbb{Q}[t_1, t_2]} \mathbb{Q}(t_1, t_2)[[q]]$,

- η is the non-degenerate symmetric 2-form (9),

- $\mathbf{1}$ is the basis element $|1^n\rangle$.

Since $\mathsf{Hilb}^m(\mathbb{C}^2)$ is *not* proper, the Gromov–Witten theory is only defined after localization by T. The CohFT with unit

$$\Omega^{\mathsf{Hilb}^m(\mathbb{C}^2)} = (\Omega_{g,n}^{\mathsf{Hilb}^m(\mathbb{C}^2)})_{2g-2+n>0}$$

is defined via the localized T-equivariant Gromov–Witten classes of $\mathsf{Hilb}^m(\mathbb{C}^2)$,

$$\Omega_{g,n}^{\mathsf{Hilb}^m(\mathbb{C}^2)} \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}(t_1, t_2)[[q]]) \otimes (V_m^*)^n \, .$$

Here, q is the Novikov parameter. Curves of degree d are counted with weight q^d, where the curve degree is defined by the pairing with the divisor

$$D = -|2, 1^{m-2}\rangle \, , \quad d = \int_\beta D \, .$$

Formally, $\Omega^{\mathsf{Hilb}^m(\mathbb{C}^2)}$ is a CohFT *not* over the field $\mathbb{Q}$ as in the r-spin and Verlinde cases, but over the ring $\mathbb{Q}(t_1, t_2)[[q]]$. To simplify notation, let

$$\Omega^m = \Omega^{\mathsf{Hilb}^m(\mathbb{C}^2)} \, .$$

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[24]The points and parts of μ are considered here to be unordered.

4.3   Genus 0. Since the T-action on $\mathsf{Hilb}^m(\mathbb{C}^2)$ has finitely many T-fixed points, the localized T-equivariant cohomology

$$H^*_\mathsf{T}(\mathsf{Hilb}^m(\mathbb{C}^2), \mathbb{Q}) \otimes_{\mathbb{Q}[t_1, t_2]} \mathbb{Q}(t_1, t_2)$$

is semisimple. At $q = 0$, the quantum cohomology ring,

(10) $(V_m, \bullet, \mathbf{1})$,

defined by Ω^m specializes to the localized T-equivariant cohomology of $\mathsf{Hilb}^m(\mathbb{C}^2)$. Hence, the quantum cohomology (10) is semisimple over the ring $\mathbb{Q}(t_1, t_2)[[q]]$, see Lee and Pandharipande [2004].

Let M_D denote the operator of T-equivariant quantum multiplication by the divisor D. A central result of Okounkov and Pandharipande [2010a] is the following explicit formula for M_D an as operator on Fock space:

$$\mathsf{M}_D(q, t_1, t_2) = (t_1 + t_2) \sum_{k > 0} \frac{k}{2} \frac{(-q)^k + 1}{(-q)^k - 1} \alpha_{-k} \alpha_k \; - \; \frac{t_1 + t_2}{2} \frac{(-q) + 1}{(-q) - 1} |\cdot|$$

$$+ \frac{1}{2} \sum_{k,l > 0} \left[t_1 t_2 \, \alpha_{k+l} \, \alpha_{-k} \, \alpha_{-l} - \alpha_{-k-l} \, \alpha_k \, \alpha_l \right].$$

The q-dependence of M_D occurs only in the first two terms (which act diagonally in the basis (7)).

Let μ^1 and μ^2 be partitions of m. The T-equivariant Gromov–Witten invariants of $\mathsf{Hilb}^m(\mathbb{C}^2)$ in genus 0 with 3 cohomology insertions given (in the Nakajima basis) by μ^1, D, and μ^2 are determined by M_D:

$$\sum_{d=0}^\infty \Omega^m_{0,3,d}(\mu^1, D, \mu^2) \, q^d \; = \; \langle \mu^1 \, | \, \mathsf{M}_D \, | \, \mu^2 \rangle.$$

The following result is proven in Okounkov and Pandharipande [ibid.].

Theorem 24. *The restriction of Ω^m to genus 0 is uniquely and effectively determined from the calculation of* M_D.

While Theorem 24 in principle completes the genus 0 study of Ω^m, the result is not as strong as the genus 0 determinations in the r-spin and Verlinde cases. The proof of Theorem 24 provides an effective linear algebraic procedure, but not a formula, for calculating the genus 0 part of Ω^m from M_D.

4.4   The topological part. Let ω^m be the topological part of the CohFT with unit Ω^m. A closed formula for ω^m can not be expected since closed formulas are already missing in the genus 0 study.

The CohFT with unit ω^m has been considered earlier from another perspective. Using fundamental correspondences Maulik, Nekrasov, Okounkov, and Pandharipande [2006], ω^m is equivalent to the *local GW/DT theory* of 3-folds of the form

(11) $\mathbb{C}^2 \times C$,

where C is a curve or arbitrary genus. Such local theories have been studied extensively Bryan and Pandharipande [2008] in the investigation of the GW/DT theory of 3-folds[25].

4.5   The R-matrix. Since Ω^m is not of pure dimension (and does not carry an Euler field), the R-matrix is *not* determined by the T-equivariant genus 0 theory alone. As in the Verlinde case, a different method is required. Fortunately, together with the divisor equation, an evaluation of the T-equivariant higher genus theory in degree 0 is enough to uniquely determine the R-matrix.

Let $\text{Part}(m)$ be the set of of partitions of m corresponding to the T-fixed points of $\text{Hilb}^m(\mathbb{C}^2)$. For each $\eta \in \text{Part}(m)$, let $\text{Tan}_\eta(\text{Hilb}^m(\mathbb{C}^2))$ be the T-representation on the tangent space at the T-fixed point corresponding to η. As before, let

$$\mathbb{E} \to \overline{\mathcal{M}}_{g,n}$$

be the Hodge bundle. The follow result is proven in Pandharipande and Tseng [2017].

Theorem 25. *The R-matrix of Ω^m is uniquely determined by the divisor equation and the degree 0 invariants*

$$\langle \mu \rangle_{1,0}^{\text{Hilb}^m(\mathbb{C}^2)} = \sum_{\eta \in \text{Part}(m)} \mu|_\eta \int_{\overline{\mathcal{M}}_{1,1}} \frac{e\left(\mathbb{E}^* \otimes \text{Tan}_\eta(\text{Hilb}^m(\mathbb{C}^2))\right)}{e\left(\text{Tan}_\eta(\text{Hilb}^m(\mathbb{C}^2))\right)},$$

$$\langle \rangle_{g \geq 2,0}^{\text{Hilb}^m(\mathbb{C}^2)} = \sum_{\eta \in \text{Part}(m)} \int_{\overline{\mathcal{M}}_g} \frac{e\left(\mathbb{E}^* \otimes \text{Tan}_\eta(\text{Hilb}^m(\mathbb{C}^2))\right)}{e\left(\text{Tan}_\eta(\text{Hilb}^m(\mathbb{C}^2))\right)}.$$

While Theorem 25 is weaker than the explicit R-matrix solutions in the r-spin and Verlinde cases, the result nevertheless has several consequences. The first is a rationality result Pandharipande and Tseng [ibid.].

Theorem 26. *For all genera $g \geq 0$ and $\mu^1, \ldots, \mu^n \in \text{Part}(m)$, the series*[26]

$$\int_{\overline{\mathcal{M}}_{g,n}} \Omega_{g,n}^m(\mu^1, \ldots, \mu^n) \in \mathbb{Q}(t_1, t_2)[[q]]$$

is the Taylor expansion in q of a rational function in $\mathbb{Q}(t_1, t_1, q)$.

The statement of Theorem 26 can be strengthened (with an R-matrix argument using Theorem 25) to prove that the CohFT with unit Ω^m can be defined over the field $\mathbb{Q}(t_1, t_2, q)$.

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[25] A natural generalization of the geometry (11) is to consider the 3-fold total space,

$$L_1 \oplus L_2 \to C,$$

of a sum of line bundles $L_1, L_2 \to C$. For particular pairs L_1 and L_2, simple closed form solutions were found Bryan and Pandharipande [2008] and have later played a role in the study of the structure of the Gromov–Witten theory of Calabi–Yau 3-folds by Ionel and Parker [2018].

[26] As always, g and n are required to be in the stable range $2g - 2 + n > 0$.

4.6   Crepant resolution.   The Hilbert scheme of points of $\mathbb{C}^2$ is well-known to be a crepant resolution of the symmetric product,

$$\epsilon : \mathsf{Hilb}^m(\mathbb{C}^2) \ \rightarrow \ \mathsf{Sym}^m(\mathbb{C}^2) = (\mathbb{C}^2)^m/S_m \,.$$

Viewed as an *orbifold*, the symmetric product $\mathsf{Sym}^m(\mathbb{C}^2)$ has a T-equivariant Gromov–Witten theory with insertions indexed by partitions of m and an associated CohFT with unit $\Omega^{\mathsf{Sym}^m(\mathbb{C}^2)}$ determined by the Gromov–Witten classes. The CohFT $\Omega^{\mathsf{Sym}^m(\mathbb{C}^2)}$ is defined over the ring $\mathbb{Q}(t_1, t_2)[[u]]$, where u is variable associated to the free ramification points, see Pandharipande and Tseng [2017] for a detailed treatment.

In genus 0, the equivalence of the T-equivariant Gromov–Witten theories of $\mathsf{Hilb}^m(\mathbb{C}^2)$ and the orbifold $\mathsf{Sym}^m(\mathbb{C}^2)$ was proven[27] in Bryan and Graber [2009]. Another consequence of the R-matrix study of $\Omega^{\mathsf{Hilb}^m(\mathbb{C}^2)}$ is the proof in Pandharipande and Tseng [2017] of the *crepant resolution conjecture* here.

Theorem 27. *For all genera $g \geq 0$ and $\mu^1, \ldots, \mu^n \in \mathrm{Part}(m)$, we have*

$$\Omega^{\mathsf{Hilb}^m(\mathbb{C}^2)}_{g,n}(\mu^1, \ldots, \mu^n) = (-i)^{\sum_{i=1}^n \ell(\mu^i) - |\mu^i|} \, \Omega^{\mathsf{Sym}^m(\mathbb{C}^2)}_{g,n}(\mu^1, \ldots, \mu^n)$$

after the variable change $-q = e^{iu}$.

The variable change of Theorem 27 is well-defined by the rationality of Theorem 26. The analysis Okounkov and Pandharipande [2010b] of the quantum differential equation of $\mathsf{Hilb}^m(\mathbb{C}^2)$ plays an important role in the proof. Theorem 27 is closely related to the GW/DT correspondence for local curves (11) in families, see Pandharipande and Tseng [2017].

4.7   Questions.   The most basic open question is to find an expression for the R-matrix of Ω^m in terms of natural operators on Fock space.

Question 28. *Is there a representation theoretic formula for the R-matrix of the CohFT with unit Ω^m?*

The difficulty in attacking Question 28 starts with the lack of higher genus calculations in closed form. The first nontrivial example Pandharipande and Tseng [ibid.] occurs in genus 1 for the Hilbert scheme of 2 points:

$$(12) \qquad \int_{\overline{\mathfrak{M}}_{1,1}} \Omega^{\mathsf{Hilb}^2(\mathbb{C}^2)}_{1,1}((2)) = -\frac{1}{24} \frac{(t_1 + t_2)^2}{t_1 t_2} \cdot \frac{1+q}{1-q} \,.$$

While there are numerous calculations to do, the higher m analogue of (12) surely has a simple answer.

Question 29. *Calculate the series*

$$\int_{\overline{\mathfrak{M}}_{1,1}} \Omega^{\mathsf{Hilb}^m(\mathbb{C}^2)}_{1,1}((2, 1^{n-2})) \ \in \ \mathbb{Q}(t_1, t_2, q) \,,$$

in closed form for all m.

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[27]The prefactor $(-i)^{\sum_{i=1}^n \ell(\mu^i) - |\mu^i|}$ was treated incorrectly in Bryan and Graber [2009] because of an arithmetical error. The prefactor here is correct.

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