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Authors
Billey, Sara C.
Swanson, Joshua P.

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The metric space of limit laws for q-hook formulas

Sara C. Billey*[1] and Joshua P. Swanson[2]

[1]*Department of Mathematics, University of Washington, Seattle, U.S.A.*
billey@uw.edu
[2]*Department of Mathematics, University of Southern California, Los Angeles, U.S.A.*
swansonj@usc.edu

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Abstract. Billey–Konvalinka–Swanson studied the asymptotic distribution of the coefficients of Stanley's q-hook length formula, or equivalently the major index on standard tableaux of straight shape and certain skew shapes. We extend those investigations to Stanley's q-hook-content formula related to semistandard tableaux and q-hook length formulas of Björner–Wachs related to linear extensions of labeled forests. We show that, while their coefficients are "generically" asymptotically normal, there are uncountably many non-normal limit laws. More precisely, we introduce and completely describe the compact closure of the metric space of distributions of these statistics in several regimes. The additional limit distributions involve generalized uniform sum distributions which are topologically parameterized by certain decreasing sequence spaces with bounded 2-norm. The closure of these distributions in the Lévy metric gives rise to the space of DUSTPAN distributions. As an application, we completely classify the limiting distributions of the size statistic on plane partitions fitting in a box.

Keywords. Hook length, q-analogues, major index, semistandard tableaux, plane partitions, forests, asymptotic normality, limit laws, Irwin–Hall distribution

Mathematics Subject Classifications. 05A16 (Primary), 60C05, 60F05 (Secondary)

1. Introduction

The famed Frame–Robinson–Thrall *hook length formula* is a rational product formula for counting the number of *standard Young tableaux* of a given partition shape λ [FRT54], denoted $\mathrm{SYT}(\lambda)$. Stanley's q-analogue of the hook length formula [Sta99, Cor. 7.21.5] is a remarkably simple generalization for the polynomial generating function of the *major index* statistic on

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$\mathrm{SYT}(\lambda)$. His *q-hook length formula* replaces each integer n with the corresponding q-integer $[n]_q := 1 + q + \cdots + q^{n-1}$, times an overall shift of $q^{r(\lambda)}$ where $r(\lambda) := \sum_{i\geqslant 1}(i-1)\lambda_i$:

$$\sum_{T\in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)} = q^{r(\lambda)} \frac{[n]_q!}{\prod_{u\in\lambda}[h_u]_q}. \tag{1.1}$$

Consequently, (1.1) encodes probabilistic information concerning the distribution of the major index statistic when sampling from $\mathrm{SYT}(\lambda)$ uniformly at random.

In [BKS20], the present authors together with Konvalinka considered the distribution of maj on $\mathrm{SYT}(\lambda)$. Given a sequence of partitions, we were able to completely determine when the corresponding sequence of standardized random variables converges in distribution. Equivalently, we determined the asymptotic distribution of the coefficients of Stanley's q-hook length formula. For these random variables, countably many continuous limit laws are possible: one gets the normal distribution "generically" and, in certain degenerate regimes, the *Irwin–Hall distributions*. A key technical tool in [BKS20] is an exact formula for the *cumulants* of the underlying random variables, which follows easily from work of Chen–Wang–Wang [CWW08] and Hwang–Zacharovas [HZ15] together with Stanley's q-hook length formula (1.1).

The present work generalizes the study in [BKS20] to the next most famous q-analogues of the hook length formula: Stanley's q-hook-content formula for semistandard tableaux, and formulae of Björner–Wachs for linear extensions of labeled forests. See Table 1.1 for a summary of the q-hook-type formulas we use. The limit laws in these cases turn out to be much more intricate than in [BKS20], with uncountably many rather than countably many possible limits.

Typical central limit theorems are based on an integer sequence so they "let $n \to \infty$," even when the limit laws are complicated such as in the work of Chatterjee–Diaconis [CD14]. By contrast, the combinatorial statistics considered here and in [BKS20] have much more complex indexing sets involving objects like integer partitions and forests. We address this complication by considering sets of standardized distributions as metric spaces under the Lévy metric on all distributions, together with a corresponding space of parameters. Our overarching goal is to describe the closure of these metric spaces and to completely classify which sequences tend to which limit points in terms of the relevant parameter spaces.

A key step in our approach is the introduction of a new family of continuous univariate distributions which we call *DUSTPAN distributions*[1]. These distributions involve convolutions of the normal law with a countable family of uniform measures supported on some intervals. More precisely, we have the following abstract description. See Definition 3.24 for the concrete version.

Theorem 1.1. *The family of DUSTPAN distributions with variance* 1 *is uniquely characterized as the smallest family* $\mathcal{F}$ *of standardized real-valued distributions such that:*

(i) $\mathcal{U}[0,1]^* \in \mathcal{F}$

(ii) If $X, Y \in \mathcal{F}$*, then the standardized independent sum random variable* $\frac{\alpha X + \beta Y}{\sqrt{\alpha^2+\beta^2}}$ *belongs to* $\mathcal{F}$ *for any* $\alpha, \beta \in \mathbb{R}$ *not both* 0.

(iii) $\mathcal{F}$ *is closed under convergence in distribution.*

[1] A "*distribution associated to a uniform sum for* t *plus an independent normal distribution.*"

Statistic(s)	set	q-hook formula(s)	cumulant expression(s)
maj	$\mathrm{SYT}(\lambda)$	$q^{r(\lambda)} \frac{[n]_q!}{\prod_{u\in\lambda}[h_u]_q}$	$\sum_{i=1}^n j^d - \sum_{u\in\lambda} h_u^d$
rank	$\mathrm{SSYT}_{\leqslant m}(\lambda)$	$q^{r(\lambda)} \prod_{u\in\lambda} \frac{[m+c_u]_q}{[h_u]_q}$ $q^{r(\lambda)} \prod_{1\leqslant i<j\leqslant m} \frac{[\lambda_i-\lambda_j+j-i]_q}{[j-i]_q}$	$\sum_{u\in\lambda}(m+c_u)^d - h_u^d$ $\sum_{1\leqslant i<j\leqslant m}(\lambda_i-\lambda_j+j-i)^d-(j-i)^d$
size	$\mathrm{PP}(a\times b\times c)$	$\prod_{i=1}^a \prod_{j=1}^b \prod_{k=1}^c \frac{[i+j+k-1]_q}{[i+j+k-2]_q}$	$\sum_{i,j,k}(i+j+k-1)^d-(i+j+k-2)^d$
maj inv	$\mathcal{L}(P,w)$	$q^{\mathrm{maj}(P,w)} \frac{[n]_q!}{\prod_{u\in P}[h_u]_q}$ $q^{\mathrm{inv}(P,w)} \frac{[n]_q!}{\prod_{u\in P}[h_u]_q}$	$\sum_{i=1}^n j^d - \sum_{u\in\lambda} h_u^d$

Table 1.1: Summary of combinatorial objects, statistics, q-hook formulas, and cumulant expressions used in this paper. Cumulants are obtained from cumulant expressions by multiplying by $\frac{B_d}{d}$ for $d > 1$. See Section 2 for details.

The general strategy of our arguments is as follows. First, we convert formulas involving ratios of q-integers into explicit expressions for the cumulants. In most cases these expressions involve significant cancellation. Next comes the difficult step where we find an asymptotically cancellation-free approximation to the cumulants in a suitable regime; see for instance Lemma 5.8. Finally, in all cases considered in this paper, we use the approximate cumulants to identify the limiting standardized distributions pertaining to SSYT's and linear extensions of trees as some particular DUSTPAN distribution. While the first step is quite generic, the combinatorial arguments and inequalities underlying the second step are highly domain-specific and expand on the corresponding approach from [BKS20].

In Section 1.1, we summarize the results of [BKS20] and reframe them in terms of metric spaces as a prelude to our new, more technical results on semi-standard tableaux and forests. To keep this introduction to a manageable length and avoid frequent digressions, we assume familiarity with tableaux combinatorics and cumulants. Detailed background on these topics is provided in [BKS20, §2] or [Sta99, Ch.7]. The main new results in this paper are outlined in Section 1.2 and Section 1.3. See Section 2 for background necessary for the new material.

1.1. Standard tableaux

Let $\mathcal{X}_\lambda[\mathrm{maj}]$ denote the random variable associated with maj on $\mathrm{SYT}(\lambda)$, sampled uniformly at random. Then the probability $\mathbb{P}(\mathcal{X}_\lambda[\mathrm{maj}] = k) = a_k^\lambda / f^\lambda$ where $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) = \sum a_k^\lambda q^k$ and $f^\lambda = \mathrm{SYT}(\lambda)^{\mathrm{maj}}(1)$ is the number of standard Young tableaux of shape λ. Hence, studying the distribution of the random variable $\mathcal{X}_\lambda[\mathrm{maj}]$ and the sequence of coefficients $\{a_k^\lambda : k \geqslant 0\}$ for $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q)$ are essentially equivalent. Furthermore, any polynomial in q with nonnegative integer coefficients can be associated to a random variable in a similar way.

For the sake of understanding limiting distributions, we typically standardize the random variables involved so they have mean 0 and variance 1. In general, given any random variable $\mathcal{X}$ with mean μ and standard deviation $\sigma > 0$, let $\mathcal{X}^* := (\mathcal{X} - \mu)/\sigma$ denote the corresponding *standardized random variable* with mean 0 and variance 1. To avoid overemphasizing trivialities, we implicitly ignore degenerate distributions with $\sigma = 0$ throughout the paper without further comment, so every distribution we consider does have a standardization. Write $\mathcal{X}_n \Rightarrow \mathcal{X}$ to mean that the sequence $\mathcal{X}_n$ converges in distribution to $\mathcal{X}$. Let $\mathcal{N}(\mu, \sigma^2)$ denote a normal distribution, and let $\mathcal{IH}_M$ denote the Mth *Irwin–Hall distribution*, obtained by summing M independent continuous uniform $[0, 1]$ random variables. These distributions are also referred to as *uniform sum distributions* in the literature. Note that the normal and Irwin–Hall distributions are continuous, while each of the random variables coming from q-hook formulas below determine discrete distributions.

We may completely describe the possible limit distributions of $\mathcal{X}_\lambda[\mathrm{maj}]^*$ using a simple auxiliary statistic on partitions, aft. In particular, let $\mathrm{aft}(\lambda) := |\lambda| - \max\{\lambda_1, \lambda'_1\}$.

Theorem 1.2. *[BKS20, Thm. 1.7] Let $\lambda^{(1)}, \lambda^{(2)}, \ldots$ be a sequence of partitions where $|\lambda^{(N)}| \to \infty$ as $N \to \infty$.*

(i) $\mathcal{X}_{\lambda^{(N)}}[\mathrm{maj}]^* \Rightarrow \mathcal{N}(0, 1)$ *if and only if* $\mathrm{aft}(\lambda^{(N)}) \to \infty$.

(ii) $\mathcal{X}_{\lambda^{(N)}}[\mathrm{maj}]^* \Rightarrow \mathcal{IH}_M^*$ *if and only if* $\mathrm{aft}(\lambda^{(N)}) \to M < \infty$.

Theorem 1.2 shows that the set $\mathbb{Z}_{\geqslant 1} \cup \{\infty\}$ parameterizes the set of all possible limit distributions associated to the q-hook length formulas and the standardized random variables $\mathcal{X}_\lambda[\mathrm{maj}]^*$. If we instead parameterize the limit distributions by $\left\{\frac{1}{n} : n \in \mathbb{Z}_{\geqslant 1}\right\} \cup \{0\}$, we get a parameter space and a distribution space which are homeomorphic as topological spaces. Hence, we introduce the notion of a metric space of standardized distributions.

Definition 1.3. The *metric space of Irwin–Hall distributions* is

$$\mathbf{M}_{\mathcal{IH}} := \{\mathcal{IH}_M^* : M \in \mathbb{Z}_{\geqslant 1}\},$$

and the *metric space of* SYT *distributions* is

$$\mathbf{M}_{\mathrm{SYT}} := \{\mathcal{X}_\lambda[\mathrm{maj}]^* : \lambda \in \mathrm{Par}, f^\lambda > 1\}.$$

Endow $\mathbf{M}_{\mathcal{IH}}$ and $\mathbf{M}_{\mathrm{SYT}}$ with the topology inherited from the topology of distributions of real-valued random variables under the Lévy metric, which is characterized by convergence in distribution [Bil95, Ex. 14.5].

By the Central Limit Theorem, $\overline{\mathbf{M}_{\mathcal{IH}}} = \mathbf{M}_{\mathcal{IH}} \cup \{\mathcal{N}(0, 1)\}$. In light of Theorem 1.2, we have the following very precise description of the minimal compactification of the metric space of SYT distributions.

Corollary 1.4. *In the Lévy metric,*

$$\overline{\mathbf{M}_{\mathrm{SYT}}} = \mathbf{M}_{\mathrm{SYT}} \sqcup \overline{\mathbf{M}_{\mathcal{IH}}}, \tag{1.2}$$

which is compact. Moreover, the set of limit points of $\mathbf{M}_{\mathrm{SYT}}$ is exactly $\overline{\mathbf{M}_{\mathcal{IH}}}$.

When the set parametrizing our combinatorial statistics has a natural topology, one might hope that it is homeomorphic to the space of distributions. For example, let

$$\mathbf{P}_{\mathcal{IH}} \coloneqq \left\{ \frac{1}{n} : n \in \mathbb{Z}_{\geqslant 1} \right\}$$

be the *Irwin–Hall parameter space*. We endow $\mathbf{P}_{\mathcal{IH}} \subset [0, 1]$ with the topology of pointwise convergence, so $\overline{\mathbf{P}_{\mathcal{IH}}} = \mathbf{P}_{\mathcal{IH}} \sqcup \{0\}$. Since $\mathcal{IH}_M^* \Rightarrow \mathcal{N}(0, 1)$ as $M \to \infty$, the bijection $\overline{\mathbf{P}_{\mathcal{IH}}} \to \overline{\mathbf{M}_{\mathcal{IH}}}$ given by $\frac{1}{M} \mapsto \mathcal{IH}_M^*$ and $0 \mapsto \mathcal{N}(0, 1)$ is a homeomorphism. It is less clear how to impose a topology on standard Young tableaux, but a characterization of the multiset of hook lengths would be a key consideration. See [BKS20, Thm. 7.1].

Remark 1.5. Recent work of Kim–Lee identified certain normal [KL20] and bivariate normal [KL21] distributions as limits of normalizations of des and $(\mathrm{des}, \mathrm{maj})$ over conjugacy classes in the symmetric group. In their context, the space of limit distributions is parameterized by real numbers in $[0, 1]$.

1.2. Semistandard tableaux and plane partitions

Stanley's *hook-content formula* is a rational product formula for counting the set $\mathrm{SSYT}_{\leqslant m}(\lambda)$ of *semistandard tableaux* of shape λ with entries at most m. He gave a natural q-analogue of this formula, which is in fact the polynomial generating function for the *rank* statistic on $\mathrm{SSYT}_{\leqslant m}(\lambda)$. A second rational product formula for rank on $\mathrm{SSYT}_{\leqslant m}(\lambda)$ with important representation-theoretic meaning is given by the type A case of the q-Weyl dimension formula. Explicitly,

$$\sum_{T \in \mathrm{SSYT}_{\leqslant m}(\lambda)} q^{\mathrm{rank}(T)} = q^{r(\lambda)} \prod_{u \in \lambda} \frac{[m + c_u]_q}{[h_u]_q} = q^{r(\lambda)} \prod_{1 \leqslant i < j \leqslant m} \frac{[\lambda_i - \lambda_j + j - i]_q}{[j - i]_q}. \tag{1.3}$$

See Section 2.2 for more details.

Let $\mathcal{X}_{\lambda;m}[\mathrm{rank}]$ denote the random variable associated with the rank statistic on $\mathrm{SSYT}_{\leqslant m}(\lambda)$, sampled uniformly at random. In Section 2, we derive simple explicit cumulant formulas from these rational expressions which allow us to study the possible limiting distributions for the $\mathcal{X}_{\lambda;m}[\mathrm{rank}]^*$. While the closures $\mathbf{M}_{\mathrm{SYT}}$ and $\mathbf{M}_{\mathcal{IH}}$ are completely characterized above, the closure of the *metric space of* SSYT *distributions*,

$$\mathbf{M}_{\mathrm{SSYT}} \coloneqq \{\mathcal{X}_{\lambda;m}[\mathrm{rank}]^* : \lambda \in \mathrm{Par}, \ell(\lambda) \leqslant m\},$$

is much more complicated. In particular, we show that the following generalization of the Irwin–Hall distributions are related to limit laws for $\mathcal{X}_{\lambda;m}[\mathrm{rank}]^*$.

Definition 1.6. Given a finite multiset $\mathbf{t}$ of non-negative real numbers, let

$$\mathcal{S}_{\mathbf{t}} \coloneqq \sum_{t \in \mathbf{t}} \mathcal{U}\left[-\frac{t}{2}, \frac{t}{2}\right], \tag{1.4}$$

where we assume the summands are independent and $\mathcal{U}[a,b]$ denotes the continuous uniform distribution supported on $[a,b]$. If $\mathbf{t}$ consists of M copies of 1, then $\mathcal{S}_{\mathbf{t}} + \frac{M}{2} = \mathcal{IH}_M$. By convention, we consider the multiset $\mathbf{t}$ as a weakly decreasing sequence of real numbers $\mathbf{t} = \{t_1 \geqslant t_2 \geqslant \cdots \geqslant t_m\}$ where $t_m \geqslant 0$. We call the distribution associated to $\mathcal{S}_{\mathbf{t}}$ a *finite generalized uniform sum distribution*.

Certain sequences of random variables $\mathcal{X}_{\lambda;m}[\mathrm{rank}]^*$ which converge to a finite generalized uniform sum distribution are completely characterized by an auxiliary multiset called the distance multiset. This auxiliary set also comes up in the *Turnpike Reconstruction Problem*, which is essentially the problem of identifying all possible sequences $\mathbf{t}$ from the following multiset $\Delta\mathbf{t}$, which has applications in DNA sequencing and X-ray crystallography [Wei95, Sect. 10.5.1]. The Turnpike Reconstruction Problem is a potential candidate for being in **NP-Intermediate**. See [LSS03] for further computational complexity considerations.

Definition 1.7. The *distance multiset* of $\mathbf{t} = \{t_1 \geqslant t_2 \geqslant \cdots \geqslant t_m\}$ is the multiset

$$\Delta\mathbf{t} \coloneqq \{t_i - t_j : 1 \leqslant i < j \leqslant m\}.$$

To avoid highly cluttered notation coming from the terms in a sequence indexed by a parameter $N = 1, 2, \ldots$, we will often drop the explicit dependence on N. For example, let λ and m denote a sequence of partitions $\lambda^{(1)}, \lambda^{(2)}, \ldots$ and a sequence of values $m^{(1)}, m^{(2)}, \ldots$ respectively. If we assume $\ell(\lambda^{(N)}) < m^{(N)}$ for each N, we will simply write $\ell(\lambda) < m$. Also, $|\lambda| = n$ means there is another sequence $n^{(1)}, n^{(2)}, \ldots$ such that the size of the partition $|\lambda^{(N)}| = n^{(N)}$, thus $|\lambda| \to \infty$ and $n \to \infty$ both imply $|\lambda^{(N)}| \to \infty$ as $N \to \infty$. Similarly, let $\mathcal{X}_{\lambda;m}[\mathrm{rank}]$ denote the sequence of uniform random variables associated with $\mathrm{SSYT}_{\leqslant m^{(N)}}(\lambda^{(N)})^{\mathrm{rank}}(q)$.

Theorem 1.8. *Let λ be an infinite sequence of partitions with $\ell(\lambda) < m$ where $\lambda_1/m^3 \to \infty$. Let $\mathbf{t}(\lambda) = (t_1, \ldots, t_m) \in [0,1]^m$ be the finite multiset with $t_k \coloneqq \frac{\lambda_k}{\lambda_1}$ for $1 \leqslant k \leqslant m$. Then $\mathcal{X}_{\lambda;m}[\mathrm{rank}]^*$ converges in distribution if and only if the multisets $\Delta\mathbf{t}(\lambda)$ converge pointwise. In that case, the limit distribution is $\mathcal{N}(0,1)$ if $m \to \infty$ and $\mathcal{S}^*_{\mathbf{d}}$ where $\Delta\mathbf{t}(\lambda) \to \mathbf{d}$ if m is bounded.*

Theorem 1.8 suggests we consider the *metric space of distance distributions*

$$\mathbf{M}_{\mathrm{DIST}} \coloneqq \bigcup_{m \geqslant 2} \{\mathcal{S}^*_{\Delta\mathbf{t}} : \mathbf{t} = \{1 = t_1 \geqslant \cdots \geqslant t_m = 0\}\} \tag{1.5}$$

and its associated parameter space $\mathbf{P}_{\mathrm{DIST}}$ defined in Section 3.4. By padding with 0's, we consider $\mathbf{P}_{\mathrm{DIST}} \subset \mathbb{R}^{\mathbb{N}}$ as a sequence space with the topology of pointwise convergence. The metric space of distance distributions is significantly more complex than the metric space of Irwin–Hall distributions. Nonetheless, a careful analysis involving the topology of the parameter space of distance multisets done in Section 3.4 yields the following results. We will show that both $\mathbf{P}_{\mathrm{DIST}}$ and $\mathbf{M}_{\mathrm{DIST}}$ have natural one point compactifications,

$$\overline{\mathbf{P}_{\mathrm{DIST}}} = \mathbf{P}_{\mathrm{DIST}} \sqcup \{\mathbf{0}\} \text{ and } \overline{\mathbf{M}_{\mathrm{DIST}}} = \mathbf{M}_{\mathrm{DIST}} \sqcup \{\mathcal{N}(0,1)\},$$

where $\mathbf{0}$ is the infinite sequence of 0's. Furthermore, in analogy with Corollary 1.4, we will show that the map $\overline{\mathbf{P}_{\mathrm{DIST}}} \to \overline{\mathbf{M}_{\mathrm{DIST}}}$ given by $\mathbf{d} \mapsto \mathcal{S}^*_{\mathbf{d}}$ and $\mathbf{0} \mapsto \mathcal{N}(0,1)$ is a homeomorphism between sequentially compact spaces. See Theorem 3.32. Therefore, Theorem 1.8 and Theorem 3.32 combine to give the following complete characterization of the possible limit laws for a particular family of semistandard tableaux in analogy with Corollary 1.4.

Corollary 1.9. *For any fixed* $\epsilon > 0$, *let*

$$\mathbf{M}_{\epsilon\,\mathrm{SSYT}} := \{\mathcal{X}_{\lambda;m}[\mathrm{rank}]^* : \ell(\lambda) < m \text{ and } \lambda_1/m^3 > (|\lambda| + m)^\epsilon\} \subset \mathbf{M}_{\mathrm{SSYT}}.$$

Then

$$\overline{\mathbf{M}_{\epsilon\,\mathrm{SSYT}}} = \mathbf{M}_{\epsilon\,\mathrm{SSYT}} \sqcup \overline{\mathbf{M}_{\mathrm{DIST}}}, \tag{1.6}$$

which is compact. Moreover, the set of limit points of $\mathbf{M}_{\epsilon\,\mathrm{SSYT}}$ *is* $\overline{\mathbf{M}_{\mathrm{DIST}}}$.

Corollary 1.9 already indicates that the limiting distributions associated to semistandard tableaux are much more varied than the case of standard Young tableaux. See Summary 4.20 for a synopsis of all of the asymptotic limits we have identified for $\mathcal{X}_{\lambda,m}[\mathrm{rank}]^*$. This includes several "generic" asymptotic normality criteria and a partial analogue of aft, called weft, which controls asymptotic normality in many cases of interest. A complete description of the closure of $\mathbf{M}_{\mathrm{SSYT}}$ akin to Theorem 1.2 and Corollary 1.4 remains open.

Open Problem 1.10. *Describe* $\overline{\mathbf{M}_{\mathrm{SSYT}}}$ *in the Lévy metric. What are all possible limit points?*

By studying one more special family of semistandard tableaux, we will show that the Irwin–Hall distributions are also among the limit points. Thus, the strongest statement we have shown for the metric space of limit laws for Stanley's q-hook-content formula is

$$\mathbf{M}_{\mathrm{SSYT}} \cup \mathbf{M}_{\mathrm{DIST}} \cup \mathbf{M}_{\mathcal{IH}} \cup \{\mathcal{N}(0,1)\} \subset \overline{\mathbf{M}_{\mathrm{SSYT}}}.$$

Using a well-known bijection, the two product formulas in (1.3) imply product formulas for the generating function of the *size* statistic on the set $\mathrm{PP}(a \times b \times c)$ of *plane partitions* fitting in a box. See the second and third rows of Table 1.1. Let $\mathcal{X}_{a\times b\times c}[\mathrm{size}]$ similarly denote the random variable associated with the size statistic on $\mathrm{PP}(a \times b \times c)$. In the theorem below, we give a complete characterization of the limit laws for plane partitions and $\{\mathcal{X}_{a\times b\times c}[\mathrm{size}]^*\}$. This leads to an analog of Corollary 1.4 for the *metric space of plane partition distributions*, denoted $\mathbf{M}_{\mathrm{PP}} := \{\mathcal{X}_{a\times b\times c}[\mathrm{size}]^*\}$.

Theorem 1.11. *Let* a, b, c *each be a sequence of positive integers.*

(i) $\mathcal{X}_{a\times b\times c}[\mathrm{size}]^* \Rightarrow \mathcal{N}(0,1)$ *if and only if* $\mathrm{median}\{a,b,c\} \to \infty$.

(ii) $\mathcal{X}_{a\times b\times c}[\mathrm{size}]^* \Rightarrow \mathcal{IH}_M$ *if* $ab \to M < \infty$ *and* $c \to \infty$.

Corollary 1.12. *In the Lévy metric,*

$$\overline{\mathbf{M}_{\mathrm{PP}}} = \mathbf{M}_{\mathrm{PP}} \sqcup \overline{\mathbf{M}_{\mathcal{IH}}}, \tag{1.7}$$

which is compact. Moreover, the set of limit points of $\mathbf{M}_{\mathrm{PP}}$ *is exactly* $\overline{\mathbf{M}_{\mathcal{IH}}}$.

1.3. Linear extensions of forests

Knuth [Knu73, p. 70] gave a rational product formula for counting the set $\mathcal{L}(P)$ of linear extensions of a forest P, analogous to the Frame–Robinson–Thrall hook length formula. Using a fixed bijection $w\colon P \to [n]$, one may interpret $\mathcal{L}(P)$ as a set of permutations $\mathcal{L}(P, w) \subset S_n$ and consider the distribution of the major index or inversion number statistics on these permutations. Stanley [Sta72] and Björner–Wachs [BW89] gave q-analogues of Knuth's formula for major index and number of inversions using certain labelings w. All of these statistics agree up to an overall shift. See the fourth row of Table 1.1 and Section 2.3 for details.

Let $\mathcal{X}_P$ denote the random variable associated with the maj or inv statistic on $\mathcal{L}(P, w)$ where w is order-preserving. The distribution of $\mathcal{X}_P^*$ is independent of the choice of statistic and the choice of w. Let

$$\mathrm{M}_{\text{Forest}} := \{\mathcal{X}_P^* : P \text{ is a forest}\}$$

be the *metric space of forest distributions*. We show that the behavior of the possible limiting distributions for $\mathcal{X}_P^*$ breaks into two distinct regimes. The first "generic" regime exhibits classic asymptotic normality, while the second "degenerate" regime allows even more continuous limit laws than have appeared in the theory for standard or semistandard tableaux.

Let $\operatorname{rank}(P)$ denote the length of a maximal chain in P. Let $|P|$ denote the number of vertices. For example, the rank of a complete binary tree with $2^n - 1$ vertices is n, so $\operatorname{rank}(P) \approx \log_2 |P|$. Typically, $\operatorname{rank}(P)$ is much smaller than $|P|$, so the following theorem covers the "generic" regime.

Theorem 1.13. *Given a sequence of forests P, the corresponding sequence of random variables $\mathcal{X}_P^*$ is asymptotically normal if*

$$|P| \to \infty \qquad \textit{and} \qquad \limsup \frac{\operatorname{rank}(P)}{|P|} < 1.$$

In the "degenerate" regime, $\operatorname{rank}(P) \sim |P|$, so the number of vertices not in a chosen maximal chain is relatively small. We completely describe the possible limit distributions when $|P| - \operatorname{rank}(P) = o(|P|^{1/2})$. To do so, we generalize both the distance distributions and the Irwin–Hall distributions to the distributions associated to countable sums of independent, continuous, uniform random variables with finite mean and variance. We call these *generalized uniform sum distributions*. Again we can reduce to sums of independent centralized random variables $\mathcal{S}_{\mathbf{t}}$ exactly as in (1.4), except now we consider countably infinite multisets $\mathbf{t} = \{t_1 \geqslant t_2 \geqslant \dots\}$ of nonnegative real numbers. See Section 3.1 for details such as cumulants, the density function, and the relation to pointwise convergence in $\mathbb{R}^{\mathbb{N}}$.

The variance of a uniform sum random variable $\mathcal{S}_{\mathbf{t}}$ is closely related to the 2-*norm* of $\mathbf{t}$,

$$|\mathbf{t}|_2 := \left(\sum_{t \in \mathbf{t}} t^2\right)^{1/2}.$$

In this notation, $\operatorname{Var}[\mathcal{S}_{\mathbf{t}}] = \frac{B_2}{2}|\mathbf{t}|_2^2$, where $B_2 = \frac{1}{6}$ is a Bernoulli number. Thus, in order for $\mathcal{S}_{\mathbf{t}}$ to be well defined, it must have finite variance, so $|\mathbf{t}|_2 < \infty$ is required. Let

$\widetilde{\ell}_2 := \{\mathbf{t} = (t_1, t_2, \ldots) : t_1 \geqslant t_2 \geqslant \cdots \geqslant 0, |\mathbf{t}|_2 < \infty\}$. The standardized general uniform sum distributions are indexed by the decreasing sequences $\mathbf{t} \in \widetilde{\ell}_2$ such that $1 = \mathrm{Var}[\mathcal{S}_{\mathbf{t}}] = \frac{B_2}{2}|\mathbf{t}|_2^2$, so $|\mathbf{t}|_2^2 = \frac{2}{B_2} = 12$. Thus, we will see the number 12 coming up in several places. In particular, define the *hat-operation* on $\mathbf{t} \in \widetilde{\ell}_2$ with positive 2-norm by

$$\widehat{\mathbf{t}} := \frac{\sqrt{12} \cdot \mathbf{t}}{|\mathbf{t}|_2}, \tag{1.8}$$

so that $\mathrm{Var}[\mathcal{S}_{\widehat{\mathbf{t}}}] = 1$ and $\mathcal{S}_{\widehat{\mathbf{t}}} = \mathcal{S}^*_{\widehat{\mathbf{t}}}$.

Now, we can return to the limiting distributions of forests in the "degenerate" regime. We show in Remark 2.22 that it suffices to consider only standardized trees in order to characterize all of $\mathbf{M}_{\mathrm{Forest}}$. In Definition 5.7, we associate to each tree P an *elevation multiset* $\mathbf{e}$ depending on a maximal chain in P. These multisets determine a new type of limiting distribution related to the generalized uniform sum distributions, but with another normal summand.

Theorem 1.14. *Let P be an infinite sequence of standardized trees with $|P| - \mathrm{rank}(P) = o(|P|^{1/2})$. Then $\mathcal{X}^*_P$ converges in distribution if and only if the multisets $\widehat{\mathbf{e}}$ converge pointwise to some element $\mathbf{t} \in \widetilde{\ell}_2$. In that case, the limit distribution is $\mathcal{S}_{\mathbf{t}} + \mathcal{N}(0, \sigma^2)$ where $|\mathbf{t}|_2^2/12 + \sigma^2 = 1$.*

Inspired by Theorem 1.14, we begin the study of *DUSTPAN distributions* associated to random variables of the form $\mathcal{S}_{\mathbf{t}} + \mathcal{N}(0, \sigma^2)$, assuming the two random variables are independent, $\mathbf{t} \in \widetilde{\ell}_2$, and $\sigma \in \mathbb{R}_{\geqslant 0}$. The nomenclature DUSTPAN refers to a *distribution associated to a uniform sum for* t *plus an independent normal distribution*. The generalized uniform sum distributions with variance 1 are the special case when $\sigma = 0$. Let

$$\mathbf{P}_{\mathrm{DUST}} := \left\{\mathbf{t} \in \widetilde{\ell}_2 : |\mathbf{t}|_2^2 \leqslant 12\right\} \tag{1.9}$$

be the *standardized DUSTPAN parameter space*, considered as a sequence space with the topology of pointwise convergence. Define the *metric space of standardized DUSTPAN distributions* to be

$$\mathbf{M}_{\mathrm{DUST}} := \{\mathcal{S}_{\mathbf{t}} + \mathcal{N}(0, \sigma^2) : |\mathbf{t}|_2^2/12 + \sigma^2 = 1\}. \tag{1.10}$$

The standardized DUSTPAN parameter space $\mathbf{P}_{\mathrm{DUST}}$ is a closed subset of the sequence space $\widetilde{\ell}_2 \subset \mathbb{R}^{\mathbb{N}}$ considered as a *Fréchet space* (rather than a Banach space). See e.g. [MV97, Ex. 5.18(1)] for more details on this structure. In fact, $\mathbf{M}_{\mathrm{DUST}}$ is closed as well, and we will show we have the following homeomorphism of compact spaces.

Theorem 1.15. *The map $\Phi \colon \mathbf{P}_{\mathrm{DUST}} \to \mathbf{M}_{\mathrm{DUST}}$ given by $\mathbf{t} \mapsto \mathcal{S}_{\mathbf{t}} + \mathcal{N}(0, \sigma^2)$ where $\sigma := \sqrt{1 - |\mathbf{t}|_2^2/12}$ is a homeomorphism between compact spaces.*

Corollary 1.16. *The limit laws for all possible standardized general uniform sum distributions $\mathbf{M}_{\mathrm{SUMS}} := \{\mathcal{S}^*_{\mathbf{t}} : \mathbf{t} \in \widetilde{\ell}_2\}$ is exactly the metric space of DUSTPAN distributions,*

$$\overline{\mathbf{M}_{\mathrm{SUMS}}} = \mathbf{M}_{\mathrm{DUST}}.$$

Corollary 1.17. *For any fixed* $\epsilon > 0$*, let* $\epsilon\,\mathrm{TREE}$ *be the set of standardized trees* P *for which* $|P| - \mathrm{rank}(P) < |P|^{\frac{1}{2}-\epsilon}$*. Let* $\mathbf{M}_{\epsilon\,\mathrm{TREE}} \coloneqq \{\mathcal{X}_P^* : P \in \epsilon\,\mathrm{TREE}\} \subset \mathbf{M}_{\mathrm{Forest}}$ *be the corresponding metric space of distributions. Then*

$$\overline{\mathbf{M}_{\epsilon\,\mathrm{TREE}}} = \mathbf{M}_{\epsilon\,\mathrm{TREE}} \sqcup \mathbf{M}_{\mathrm{DUST}}, \tag{1.11}$$

which is compact. Moreover, the set of limit points of $\mathbf{M}_{\epsilon\,\mathrm{TREE}}$ *is* $\mathbf{M}_{\mathrm{DUST}}$*.*

Remark 1.18. The foundational idea of information geometry is to endow spaces of distributions with the structure of Riemannian manifolds. Consequently, one may be tempted to recast Theorem 1.15 in the context of manifold theory. However, the infinite-dimensional case is generally "not mathematically easy" [Ama16, §2.5, p.39]. Here, $\widetilde{\ell}_2$ is a Hilbert manifold and a Banach manifold under the $\widetilde{\ell}_2$-norm, as well as a Fréchet manifold under pointwise convergence. There does not appear to be a generally agreed-upon Hilbert, Banach, or Fréchet manifold structure which the closed subset $\mathbf{P}_{\mathrm{DUST}}$ inherits from $\widetilde{\ell}_2$, though it could perhaps be thought of as a manifold with corners. In any case, the inherited Hilbert and Banach topology on $\mathbf{P}_{\mathrm{DUST}}$ disagrees with the Fréchet topology, so for our purposes, Theorem 1.15 requires us to use the Fréchet structure of pointwise convergence. It is consequently unclear if a useful differentiable structure exists for $\mathbf{P}_{\mathrm{DUST}}$.

As with $\mathbf{M}_{\mathrm{SSYT}}$, it remains an open problem to completely classify all possible limit points of $\mathbf{M}_{\mathrm{Forest}}$. The strongest results we have proven for q-hook length formulas for forests show $\mathbf{M}_{\mathrm{Forest}} \cup \mathbf{M}_{\mathrm{DUST}} \subset \overline{\mathbf{M}_{\mathrm{Forest}}}$, implying there are an uncountable number of possible limit laws for distributions associated to forests. In the case of forests, the underlying distributions are always symmetric and unimodal, in contrast to $\mathbf{M}_{\mathrm{SYT}}$ which are not always unimodal, see [BKS20, Conj. 8.1]. So, $\overline{\mathbf{M}_{\mathrm{Forest}}}$ does not contain $\mathbf{M}_{\mathrm{SYT}}$.

More generally, it is natural to ask which limit laws are possible for the coefficients of arbitrary q-hook-type formulas, namely polynomials with nonnegative integer coefficients of the form $\prod_{i=1}^n [a_i]_q/[b_i]_q$. In [BS22], we call such q-integer quotients *cyclotomic generating functions* (CGF's) and study their properties from a variety of algebraic and probabilistic perspectives. Let $\mathbf{M}_{\mathrm{CGF}}$ denote the corresponding metric space of standardized distributions. By Prohkorov's Theorem, $\overline{\mathbf{M}_{\mathrm{CGF}}}$ is compact.

Open Problem 1.19. *Describe* $\overline{\mathbf{M}_{\mathrm{CGF}}}$ *in the Lévy metric. What are all possible limit points? Is* $\mathbf{M}_{\mathrm{CGF}} \cup \mathbf{M}_{\mathrm{DUST}}$ *__the__ metric space of limit laws for* q*-hook formulas, referring back to the title of this article?*

1.4. Paper organization

The rest of the paper is organized as follows. In Section 2, we provide background for the hook and cumulant formulas summarized in Table 1.1. In Section 3, we analyze the metric space of generalized uniform sum distributions and its variations in order to prove Theorem 1.15 and its analog for the distance distributions. The analysis of $\mathbf{M}_{\mathrm{SSYT}}$ and $\mathbf{M}_{\mathrm{PP}}$ is in Section 4. The analysis of $\mathbf{M}_{\mathrm{Forest}}$ is in Section 5. Some additional open questions and avenues for future work are listed in Section 6.

2. Background

In this section, we briefly recall statements from the literature we will need related to asymptotic distributions, semistandard tableaux and forests. All of our arguments for determining asymptotic distributions use the Method of Moments/Cumulants. Using work of Hwang–Zacharovas, we explain a key insight for this paper, namely that rational product formulas such as appear in Table 1.1 give rise to explicit formulas for cumulants of the corresponding distributions. See [BKS20, §2-3] for a more extensive exposition aimed at an audience familiar with enumerative combinatorics. See [Bil95] for background in probability.

2.1. Asymptotic distributions

Let $\mathcal{X}$ be a real-valued random variable. For $d \in \mathbb{Z}_{\geqslant 0}$, the *dth moment* $\mathcal{X}$ is

$$\mu_d := \mathbb{E}[\mathcal{X}^d].$$

The *moment-generating function* of $\mathcal{X}$ is

$$M_{\mathcal{X}}(t) := \mathbb{E}[e^{t\mathcal{X}}] = \sum_{d=0}^{\infty} \mu_d \frac{t^d}{d!},$$

which for us will always have a positive radius of convergence. The *characteristic function* of $\mathcal{X}$ is

$$\phi_{\mathcal{X}}(t) := \mathbb{E}[e^{it\mathcal{X}}],$$

which exists for all $t \in \mathbb{R}$ and which is the Fourier transform of the density or mass function associated to $\mathcal{X}$. We will need the following technical details for the proofs in future sections.

Remark 2.1. The characteristic function $\phi_{\mathcal{X}}(s) := \mathbb{E}[e^{is\mathcal{X}}]$ in general converges only for $s \in \mathbb{R}$. However, if there is a complex analytic function $\psi(s)$ defined in an open ball $|s| < \rho$ such that $\phi_{\mathcal{X}}(s) = \psi(s)$ for $-\rho < s < \rho$, then $\phi_{\mathcal{X}}(s)$ exists and is analytic in some strip $-\beta < \mathrm{Im}(s) < \alpha$ where $\alpha, \beta \geqslant \rho$. Moreover, for $|s| < \rho$, $\phi_{\mathcal{X}}(s) = \psi(s)$. In particular, the moment-generating function $\mathbb{E}[e^{t\mathcal{X}}]$ converges for $-\rho < t < \rho$, so $\mathcal{X}$ has moments of all orders and is determined by its moments. See e.g. [Luk70, Thm. 7.1.1, pp.191-193] and [Bil95, Thm. 30.1] for details.

The *cumulants* $\kappa_1, \kappa_2, \ldots$ of $\mathcal{X}$ are defined to be the coefficients of the exponential generating function

$$K_{\mathcal{X}}(t) := \sum_{d=1}^{\infty} \kappa_d \frac{t^d}{d!} := \log M_{\mathcal{X}}(t) = \log \mathbb{E}[e^{t\mathcal{X}}].$$

Hence, they satisfy the recurrence

$$\mu_d = \kappa_d + \sum_{m=1}^{d-1} \binom{d-1}{m-1} \kappa_m \mu_{d-m}, \tag{2.1}$$

so the moments can similarly be recovered from the cumulants and vice versa. In particular, (2.1) implies $\kappa_1 = \mu_1 = \mu = \mathbb{E}[\mathcal{X}]$ and $\kappa_2 = \mathrm{Var}[\mathcal{X}] = \sigma^2$. The cumulants also satisfy

1. *(Homogeneity)*: the dth cumulant of $c\mathcal{X}$ is $c^d\kappa_d$ for $c \in \mathbb{R}$, and
2. *(Additivity)* the cumulants of the sum of *independent* random variables are the sums of the cumulants.

For $d \geqslant 4$, the moments of independent random variables are not necessarily sums of the moments, so cumulants work much better for our purposes. By homogeneity and additivity, the associated *standardized random variable* $\mathcal{X}^* := (\mathcal{X} - \mu)/\sigma$ has cumulants $\kappa_1^{\mathcal{X}^*} = 0$, $\kappa_2^{\mathcal{X}^*} = 1$, and

$$\kappa_d^{\mathcal{X}^*} = \frac{\kappa_d^{\mathcal{X}}}{\sigma^d} = \frac{\kappa_d^{\mathcal{X}}}{(\kappa_2^{\mathcal{X}})^{d/2}} \quad \text{for } d \geqslant 2. \tag{2.2}$$

Example 2.2. The *normal distribution* $\mathcal{N}(0,1)$ is the unique distribution with $\kappa_1 = 0$, $\kappa_2 = 1$, and $\kappa_d = 0$ for $d \geqslant 3$. Therefore, $\mathcal{N}(\mu, \sigma^2)$ is the unique distribution with cumulants $\kappa_1 = \mu$, $\kappa_2 = \sigma^2$, and $\kappa_d = 0$ for $d \geqslant 3$.

Example 2.3. Let $\mathcal{U} = \mathcal{U}[0,1]$ be the continuous uniform random variable whose density takes the value 1 on the interval $[0,1]$ and 0 otherwise. Then the moment generating function is $M_{\mathcal{U}}(t) = \int_0^1 e^{tx} dx = (e^t - 1)/t$, so the cumulant generating function $\log M_{\mathcal{U}}(t)$ coincides with the exponential generating function for the *divided Bernoulli numbers* $\frac{B_d}{d}$ for $d \geqslant 1$. Their exponential generating function $E_D(t)$ satisfies

$$E_D(t) := \sum_{d \geqslant 1} \frac{B_d}{d} \frac{t^d}{d!} = \log\left(\frac{e^t - 1}{t}\right).$$

Hence, the d^{th} cumulant for $\mathcal{U}$ is $\kappa_d^{\mathcal{U}} = B_d/d$ for $d \geqslant 1$. Recall from Section 1, $\mathcal{IH}_m$ is the *Irwin–Hall* distribution obtained by adding m independent $\mathcal{U}[0,1]$ random variables. By additivity, the dth cumulant of $\mathcal{IH}_m$ is mB_d/d. More generally, let $\mathcal{S} := \sum_{k=1}^m \mathcal{U}[\alpha_k, \beta_k]$ be the sum of m independent uniform continuous random variables. Then the dth cumulant of $\mathcal{S}$ for $d \geqslant 2$ is

$$\kappa_d^{\mathcal{S}} = \frac{B_d}{d} \sum_{k=1}^m (\beta_k - \alpha_k)^d \tag{2.3}$$

by the homogeneity and additivity properties of cumulants.

The Method of Moments/Cumulants is based on the following theorem. All random variables we encounter will have moments of all orders.

Theorem 2.4 (Frechét–Shohat Theorem, [Bil95, Theorem 30.2])**.** *Let $\mathcal{X}_1, \mathcal{X}_2, \ldots$ be a sequence of real-valued random variables, and let $\mathcal{X}$ be a real-valued random variable. Suppose the moments of $\mathcal{X}_n$ and $\mathcal{X}$ all exist and the moment generating functions all have positive radius of convergence. If*

$$\lim_{n \to \infty} \mu_d^{\mathcal{X}_n} = \mu_d^{\mathcal{X}} \quad \forall d \in \mathbb{Z}_{\geqslant 1}, \tag{2.4}$$

then $\mathcal{X}_1, \mathcal{X}_2, \ldots$ converges in distribution to $\mathcal{X}$. Similarly, if

$$\lim_{n \to \infty} \kappa_d^{\mathcal{X}_n} = \kappa_d^{\mathcal{X}} \quad \forall d \in \mathbb{Z}_{\geqslant 1}, \tag{2.5}$$

then $\mathcal{X}_1, \mathcal{X}_2, \ldots$ converges in distribution to $\mathcal{X}$.

Corollary 2.5. *A sequence $\mathcal{X}_1, \mathcal{X}_2, \ldots$ of real-valued random variables on finite sets is asymptotically normal if for all $d \geqslant 3$ we have*

$$\lim_{n\to\infty} \kappa_d^{\mathcal{X}_n^*} = \lim_{n\to\infty} \frac{\kappa_d^{\mathcal{X}_n}}{(\sigma^{\mathcal{X}_n})^d} = 0. \tag{2.6}$$

For a positive integer n, define the associated *q-integer* to be the polynomial

$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1} = (1-q^n)/(1-q).$$

The q-integers factor into cyclotomic polynomials over the integers. Therefore, the hook length formulas considered in this paper are all products of cyclotomic polynomials. Because these rational product formulas are polynomial, all cancellation can be done efficiently by taking the multiset difference between the numerator and denominator of the cyclotomic factors.

In a forthcoming paper [BS22], we investigate general properties of generating functions which are products of cyclotomic polynomials with nonnegative coefficients. For this paper, we just need two facts. The first theorem first appeared explicitly in the work of Hwang–Zacharovas [HZ15, §4.1] building on the work of Chen–Wang–Wang [CWW08, Thm. 3.1], who in turn used an argument going back at least to Sachkov [Sac97, §1.3.1].

Theorem 2.6. *[HZ15, §4.1] Suppose $\{a_1, \ldots, a_m\}$ and $\{b_1, \ldots, b_m\}$ are multisets of positive integers such that*

$$P(q) = \prod_{k=1}^{m} \frac{1-q^{a_k}}{1-q^{b_k}} = \prod_{k=1}^{m} \frac{[a_k]_q}{[b_k]_q} = \sum_k c_k q^k \in \mathbb{Z}_{\geqslant 0}[q]. \tag{2.7}$$

Let $\mathcal{X}$ be a discrete random variable with $\mathbb{P}[X = k] = c_k/P(1)$. Then the dth cumulant of $\mathcal{X}$ is

$$\kappa_d^{\mathcal{X}} = \frac{B_d}{d}\left(\sum_{k=1}^{m} a_k^d - b_k^d\right) \tag{2.8}$$

where B_d is the dth Bernoulli number (with $B_2 = \frac{1}{2}$).

The following corollary is proved in [BS22]. It also follows from the tail decay bound in [HZ15, Lemma 2.8]. We need this for our current investigations for hook length formulas.

Lemma 2.7 (Converse of Frechét–Shohat for CGF's)**.** *Suppose $\mathcal{X}_1, \mathcal{X}_2, \ldots$ is a sequence of random variables corresponding to polynomials of the same form as (2.7). If $\mathcal{X}_n^* \Rightarrow \mathcal{X}$ for some random variable $\mathcal{X}$, then $\mathcal{X}$ is determined by its cumulants and, for all $d \in \mathbb{Z}_{\geqslant 1}$,*

$$\lim_{n\to\infty} \kappa_d^{\mathcal{X}_n^*} = \kappa_d^{\mathcal{X}}.$$

2.2. Semistandard Young tableaux and plane partitions

We briefly recall the definition and notation for Schur functions, semistandard tableaux and plane partitions. For more information on symmetric functions and their connection with the enumeration of plane partitions and tableaux, see [Sta99, Ch. 7].

A partition $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_k)$ is a finite decreasing sequence of positive integers. Let $\ell(\lambda) = k$ denote the *length* of λ. We think of λ in terms of its Young diagram, which is a left justified array of $\ell(\lambda)$ rows with λ_i cells on row i and index the cells in matrix notation.

A *semistandard Young tableau*, or just semistandard tableau for short, of shape λ is a filling of the cells of λ with positive integer labels, possibly repeated, such that the labels weakly increase to the right in rows and strictly increase down columns. The set of semistandard Young tableaux of shape λ is denoted $\mathrm{SSYT}(\lambda)$. The subset of $\mathrm{SSYT}(\lambda)$ filled with integers no greater than m is denoted $\mathrm{SSYT}_{\leqslant m}(\lambda)$, which is a finite set. The *type* of a semistandard tableau T is the composition $\alpha(T) = (\alpha_1, \alpha_2, \ldots)$ where α_i is the number of times i appears in T. The *Schur function*

$$s_\lambda(x_1, x_2, \ldots) \coloneqq \sum_{T \in \mathrm{SSYT}(\lambda)} \mathbf{x}^{\alpha(T)}$$

is the *type generating function* for all semistandard tableaux of shape λ, where $\mathbf{x}^\alpha \coloneqq x_1^{\alpha_1} x_2^{\alpha_2} \cdots$.

The *rank* of a semistandard tableau T is a nonnegative integer statistic depending only on the type. It is defined by

$$\mathrm{rank}(T) \coloneqq \mathrm{rank}(\alpha) \coloneqq \sum_{i \geqslant 1} (i-1)\alpha_i.$$

For example, for a fixed partition λ, the smallest possible rank of any $T \in \mathrm{SSYT}(\lambda)$ occurs for the tableau with all 1's in the first row, all 2's in the second row, etc. in the diagram of λ. Therefore, the minimal rank is $\sum (i-1)\lambda_i$, which we denote as $\mathrm{rank}(\lambda)$. The *rank generating function* for $\mathrm{SSYT}(\lambda)$ is given by the *principal specialization of the Schur function*,

$$s_\lambda(1, q, q^2, \ldots) = \mathrm{SSYT}(\lambda)^{\mathrm{rank}}(q) = \sum_{T \in \mathrm{SSYT}(\lambda)} q^{\mathrm{rank}(T)}$$
$$s_\lambda(1, q, q^2, \ldots, q^{m-1}) = \mathrm{SSYT}_{\leqslant m}(\lambda)^{\mathrm{rank}}(q) \sum_{T \in \mathrm{SSYT}(\lambda)_{\leqslant m}} q^{\mathrm{rank}(T)}.$$

The motivation for considering this particular specialization comes from the q-analog of the Weyl dimension formula in representation theory. Stembridge [Ste94, §2.2-2.3, Prop. 2.4] put a ranked poset structure on the weights of a semisimple Lie algebra, which in type A reduces to $\mathrm{rank}(\alpha)$. The following rational product formula for $s_\lambda(1, q, q^2, \ldots, q^{m-1})$ follows easily from the classical ratio of determinants definition of Schur polynomials.

Theorem 2.8 ([Lit40, §7.1], [Sta99, (7.105)]). *For any partition λ and positive integer $m \geqslant \ell(\lambda)$,*

$$s_\lambda(1, q, q^2, \ldots, q^{m-1}) = q^{\mathrm{rank}(\lambda)} \prod_{1 \leqslant i < j \leqslant m} \frac{[\lambda_i - \lambda_j + j - i]_q}{[j-i]_q}. \tag{2.9}$$

Stanley gave an alternate rational product formula for $s_\lambda(1,q,\dots,q^{m-1})$, which is called the *$q$-hook-content formula*. Here the *content* of a cell u in row i, column j in λ is defined as $c_u := j-i$. Also, the *hook length* of cell u, denoted h_u, is the number of cells directly east of u, plus the number of cells directly south of u in the diagram of λ.

Theorem 2.9 ([Sta99, Thm. 7.21.2])**.** *For any partition λ and positive integer $m \geqslant \ell(\lambda)$,*

$$s_\lambda(1,q,\dots,q^{m-1}) = q^{\operatorname{rank}(\lambda)} \prod_{u\in\lambda} \frac{[m+c_u]_q}{[h_u]_q}. \tag{2.10}$$

The two product formulas for $s_\lambda(1,q,\dots,q^{m-1})$ are each useful in different circumstances. The product in (2.9) involves $\binom{m}{2}$ terms, whereas the product in (2.10) involves $|\lambda|$ terms. One can observe from these formulas that $s_\lambda(1,q,\dots,q^{m-1})$ is symmetric about the mean nonzero coefficient. From the representation theory of $\mathrm{GL}_2(\mathbb{C})$, it is known that $s_\lambda(1,q,\dots,q^{m-1})$ is also unimodal. See [GOS92] for a combinatorial proof relying on the unimodality of the Gaussian polynomials.

Recently, Huh–Matherne–Mészáros–St.Dizier [HMMSD22] showed that Schur polynomials are strongly log-concave. However, we note that $s_\lambda(1,q,\dots,q^{m-1})$ is not always log-concave. For example,

$$s_{(3,1)}(1,q,q^2,q^3) = q^{10}+2q^9+4q^8+5q^7+7q^6+7q^5+7q^4+5q^3+4q^2+2q^1+1,$$

which is not log-concave since $5^2 < 4\cdot 7$.

Combining Theorem 2.6 and Theorem 2.9, we get an exact formula for the cumulants of the random variable associated to the rank function on semi-standard Young tableaux on the alphabet $[m]$ chosen uniformly. This cumulant formula is the key to analyzing the asymptotic distributions.

Corollary 2.10. *Fix a partition λ. If $\kappa_d^{\lambda;m}$ is the dth cumulant of the random variable associated to* rank *on* $\mathrm{SSYT}_{\leqslant m}(\lambda)$*, then, for $d>1$,*

$$\kappa_d^{\lambda;m} = \frac{B_d}{d}\left(\sum_{1\leqslant i<j\leqslant m} (\lambda_i-\lambda_j+j-i)^d - (j-i)^d\right) \tag{2.11}$$

$$= \frac{B_d}{d}\left(\sum_{u\in\lambda} (m+c_u)^d - h_u^d\right). \tag{2.12}$$

Observe, the summands in (2.12) can be negative, but the summands in (2.11) are each clearly positive. Thus, $\kappa_d^{\lambda;m}$ has the same sign as the Bernoulli number B_d, namely it is negative if and only if d is divisible by 4, and $\kappa_d^{\lambda;m} = B_d = 0$ if and only if $d>1$ and odd.

Definition 2.11. A *plane partition* is a finite collection of unit cubes in the positive orthant of $\mathbb{R}^3$ stacked towards the origin. More formally, it is a finite lower order ideal in $\mathbb{Z}^3_{\geqslant 1}$ under the component-wise partial order. We may imagine a plane partition ρ as a matrix with entry ρ_{ij} recording the number of cells with x-coordinate i and y-coordinate j. The *size* of a plane partition ρ is the number of cubes, denoted $|\rho| = \sum \rho_{ij}$. We write $\mathrm{PP}(a\times b\times c)$ for the set of all plane partitions fitting inside an a by b by c rectangular prism.

There is a straightforward bijection between plane partitions and rectangular shape semistandard Young tableaux,

$$\begin{aligned} \mathrm{PP}(a \times b \times c) &\xrightarrow{\sim} \mathrm{SSYT}_{\leqslant a+c}((b^a)) \\ \rho &\mapsto T \text{ where } T_{ij} = c - \rho_{ij} + i. \end{aligned} \tag{2.13}$$

All T and ρ in the bijection are rectangular arrays with a rows and b columns with entries labeled using matrix indexing conventions. Letting $|T| := \sum_{i,j} T_{ij}$, note that $|T| = \mathrm{rank}(T) + ab$ and $|T|+|\rho| = abc+b\binom{a+1}{2}$ is constant. Hence, the unique element of minimal size in $\mathrm{PP}(a\times b\times c)$, namely $\varnothing$, maps to the unique maximal rank tableau in $\mathrm{SSYT}_{\leqslant a+c}((b^a))$ with values $c - i$ in row i for each $1 \leqslant i \leqslant a$.

By Theorem 2.8 and Theorem 2.9, we know $\mathrm{SSYT}(\lambda)^{\mathrm{rank}}(q)$ is symmetric up to an overall q-shift. Similarly, $\mathrm{PP}(a \times b \times c)$ is closed under box complementation, so it follows from the bijection and (2.9) that

$$\mathrm{PP}(a \times b \times c)^{\mathrm{size}}(q) = q^{-\mathrm{rank}(\lambda)}\, \mathrm{SSYT}_{\leqslant a+c}((b^a))^{\mathrm{rank}}(q) \tag{2.14}$$

$$= \prod_{i=1}^{a} \prod_{j=1}^{b} \frac{[a+c+j-i]_q}{[a+b-i-j+1]_q} \tag{2.15}$$

$$= \prod_{i=1}^{a} \prod_{j=1}^{b} \frac{[i+j+c-1]_q}{[i+j-1]_q} \tag{2.16}$$

$$= \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{[i+j+k-1]_q}{[i+j+k-2]_q}. \tag{2.17}$$

The later two product formulas are originally due to MacMahon. See the proof of [Sta99, Thm. 7.21.7] for more details and [Sta99, pp. 402-403] for historical references. In particular, the cumulants of size on $\mathrm{PP}(a\times b\times c)$ are given by (2.11) or (2.12) where $\lambda = (b^a)$ and $m = a + c$.

2.3. Linear extensions of forests

Next, we summarize the relevant terminology and results from [BW89]. Briefly recall, a *tree* is a finite, connected simple graph with no cycles. A *forest* is a finite disjoint union of trees. A tree is *rooted* if it has a distinguished vertex, called the root. A forest is *rooted* if each of its trees is rooted. The *Hasse diagram* of a partially ordered set (*poset*) P is the graph with vertex set P where there is an edge between x and y if y *covers* x, i.e. $x <_P y$ and there does not exist $u \in P$ such that $x <_P u <_P y$. We refer to a poset as a *forest* if its Hasse diagram is a forest with roots as maximal elements, or equivalently if every element of P is covered by at most one element.

Definition 2.12. Let P be a finite partially ordered set. The *rank* of P is the maximum number of elements in any chain $u_1 < u_2 < \cdots < u_k$ in P. For instance, if P is a singleton, its rank is 1. Note that this definition is one larger than the standard definition in [Sta12, Ch.3], but it is more convenient for our purposes.

Definition 2.13. Let P be a poset. A *labeling* of P is a bijection $w\colon P \to [n]$, and a *labeled poset* is a pair (P, w) where w is a labeling of P. A labeling w of P for which $w(p) \leqslant w(q)$ whenever $p \leqslant_P q$ is called a *natural labeling*. A labeling w of P is *regular* if for all $x <_P z$ and $y \in P$, if $w(x) < w(y) < w(z)$ or $w(x) > w(y) > w(z)$ then $x <_P y$ or $y <_P z$. Regular labelings of forests include the postorder, preorder, and inorder labelings, which are commonly used in computer science.

Definition 2.14. A *linear extension* of P is an ordered list $p_1, \ldots, p_n$ of the elements of P such that $i \leqslant j$ whenever $p_i \leqslant_P p_j$. If (P, w) is a labeled poset, a linear extension can be thought of as the permutation $i \mapsto w(p_i)$ of $[n]$. The set $\mathcal{L}(P, w)$ is the set of all permutations obtained in this fashion from linear extensions of the labeled poset (P, w).

It is often convenient to use a natural labeling w of P so that $\mathrm{id} \in \mathcal{L}(P, w)$. Choosing labelings which are not natural forces inversions to appear in any $\sigma \in \mathcal{L}(P, w)$. Finding the minimum number of inversions in any linear extension of an arbitrarily labeled poset motivates the following analogues related to inversions and descents in permutations.

Definition 2.15. Let (P, w) be a labeled poset. Set

$$\begin{aligned}
\mathrm{Inv}(P, w) &:= \{(w(x), w(y)) : x <_P y \text{ and } w(x) > w(y)\} && \textit{(inversion set)}\\
\mathrm{inv}(P, w) &:= |\,\mathrm{Inv}(P, w)| && \textit{(inversion number)}\\
\mathrm{Des}(P, w) &:= \{w(x) : w(x) > w(y), y \text{ covers } x \in P\} && \textit{(descent set)}\\
\mathrm{maj}(P, w) &:= \sum_{x \in \mathrm{Des}(P,w)} h_x && \textit{(major index)}
\end{aligned}$$

where the *hook length* of an element $x \in P$ is

$$h_x := \#\{t \in P : t \leqslant_P x\}. \tag{2.18}$$

Example 2.16. For the first labeled poset (P, v) in Figure 2.1, we have $\mathcal{L}(P, v) = \{1234, 1324\}$, $\mathrm{Inv}(P, v) = \mathrm{Des}(P, v) = \varnothing$, and $\mathrm{inv}(P, v) = \mathrm{maj}(P, v) = 0$. For the second labeled poset (P, w) in Figure 2.1, we have $\mathcal{L}(P, w) = \{3142, 3412\}$, $\mathrm{Inv}(P, w) = \{(3, 1), (3, 2), (4, 2)\}$, $\mathrm{Des}(P, w) = \{3, 4\}$, $\mathrm{inv}(P, w) = \mathrm{maj}(P, w) = 3$. The hook lengths of the diamond poset are $1, 2, 2, 4$.

Remark 2.17. One can consider a partition λ as a poset on its cells where $(u, v) \leqslant (x, y)$ if and only if $u \leqslant x$ and $v \leqslant y$. However, the hook lengths of λ do *not* agree with (2.18) except when λ is a single row or column. For example, the hook lengths for the partition $(2, 2)$ are $1, 2, 2, 3$, while the hook lengths for the diamond poset are $1, 2, 2, 4$.

Mallows and Riordan first studied the inversion enumeration on labeled rooted trees [MR68], and connected it to cumulants of the lognormal distribution. Knuth gave a hook length formula for $|\mathcal{L}(P, w)|$ [Knu73, p. 70] for posets which are forests. Björner–Wachs [BW89] and Stanley [Sta72] generalized Knuth's result to q-hook length formulas using the inv and maj statistics on $\mathcal{L}(P, w)$. Stanley considered only the case when w is natural, i.e. when $\mathrm{inv}(P, w) = \mathrm{maj}(P, w) = 0$, for the maj generating function. Recently Zaguia has studied linear extensions of forests and proved the "1/3-2/3 Conjecture" holds on such posets [Zag19].

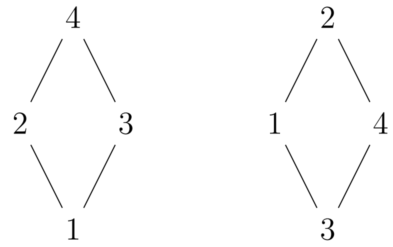

Figure 2.1: A naturally labeled poset (P, v) on the left and another labeling of the same diamond poset (P, w) on the right which is not natural or regular.

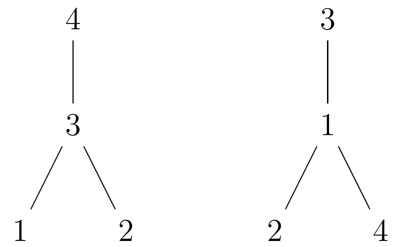

Figure 2.2: A naturally labeled poset (P, v) on the left and another labeling of the same forest poset (P, w) on the right which is not natural.

Theorem 2.18 ([BW89, Thm. 1.1-1.2, Cor. 3.1, Thm. 6.1-6.2])**.** *Let (P, w) be a labeled poset with n elements. Then*

$$\mathcal{L}(P, w)^{\mathrm{maj}}(q) := \sum_{\pi \in \mathcal{L}(P,w)} q^{\mathrm{maj}(\pi)} = q^{\mathrm{maj}(P,w)} \frac{[n]_q!}{\prod_{u \in P} [h_u]_q}$$

if and only if P is a forest. Similarly,

$$\mathcal{L}(P, w)^{\mathrm{inv}}(q) := \sum_{\pi \in \mathcal{L}(P,w)} q^{\mathrm{inv}(\pi)} = q^{\mathrm{inv}(P,w)} \frac{[n]_q!}{\prod_{u \in P} [h_u]_q}$$

if and only if (P, w) is a regularly labeled forest. Moreover, if P is a forest, $\frac{[n]_q!}{\prod_{u \in P}[h_u]_q}$ has symmetric and unimodal coefficients.

Example 2.19. For the first labeled poset (P, v) in Figure 2.2, we have $\mathcal{L}(P, v) = \{1234, 2134\}$, $\mathrm{Inv}(P, v) = \mathrm{Des}(P, v) = \varnothing$, and $\mathrm{inv}(P, v) = \mathrm{maj}(P, v) = 0$. For the second labeled poset (P, w) in Figure 2.2, we have $\mathcal{L}(P, w) = \{2413, 4213\}$, $\mathrm{Inv}(P, w) = \{(2, 1), (4, 1), (4, 3)\}$, $\mathrm{Des}(P, w) = \{2, 4\}$, $\mathrm{inv}(P, w) = 3$, $\mathrm{maj}(P, w) = 2$. Note $\mathcal{L}(P, w)^{\mathrm{maj}} = q^2 + q^3$, and $\mathcal{L}(P, w)^{\mathrm{inv}} = q^3 + q^4$. The hook lengths of the underlying poset are $1, 1, 3, 4$. One can verify the formulas in Theorem 2.18 hold in each of these cases, but they don't hold for the diamond poset.

Given a forest P, define the polynomial

$$\mathcal{L}_P(q) := [n]_q! / \prod_{u \in P} [h_u]_q, \tag{2.19}$$

and let $\mathcal{X}_P$ the associated random variable. Note, the distribution of $\mathcal{X}_P$ does not depend on the choice of labeling of the vertices of P since $\mathcal{L}_P(q)$ depends only on the unlabeled poset structure. We also get simple formulas for the associated cumulants in the next two statements.

Remark 2.20. By the unimodality result in Theorem 2.18, we know $\mathcal{L}_P(q) := [n]_q!/\prod_{u\in P}[h_u]_q$ has nonzero coefficients in an interval, so it has no internal zeros. The degree of $\mathcal{L}_P(q)$ is

$$\sum_{k=1}^{n} k - \sum_{u\in P} h_u,$$

and the mean of $\mathcal{X}_P$ is half the degree.

Corollary 2.21. *Let P be a forest with n elements. Suppose $d \in \mathbb{Z}_{\geqslant 2}$. Let κ_d^P denote the dth cumulant of the random variable $\mathcal{X}_P$. Then,*

$$\kappa_d^P = \frac{B_d}{d}\left(\sum_{k=1}^{n} k^d - \sum_{u\in P} h_u^d\right).$$

Remark 2.22. In order to characterize all possible limit laws for the standardized random variables associated with maj and inv on labeled forests, we only need to consider the set of all distributions associated with standardized trees as follows. Given any forest P, we may turn P into a tree by adding a new vertex covering the roots of all the trees of P. It is easy to see that the quotient in (2.19) is unchanged, so the cumulants and the corresponding distributions are the same. Similarly, if P is a tree and the root has exactly one child, we may delete the root while preserving the fact that P is a tree, and the quotient in (2.19) is again unchanged. Consequently, we say a forest is *standardized* if it is a tree and the root has at least two children. Therefore,

$$\mathbf{M}_{\text{Forest}} := \{\mathcal{X}_P^* : P \text{ is a forest}\} = \{\mathcal{X}_P^* : P \text{ is a standardized tree}\}.$$

2.4. Riemann integral estimates

Many of our theorems depend on approximations using a mixture of combinatorics and analysis. In particular, we return to certain basic sums over and over again. Let $\mathbf{h}_d(a,b) = \sum_{j=0}^{d} a^j b^{d-j}$ denote the complete homogeneous symmetric function on two inputs.

Lemma 2.23. *For positive integers a, b, and $d > 1$, we have*

$$\frac{1}{d}\left[(a+b)^d - a^d\right] < \sum_{j=a+1}^{a+b} j^{d-1} < \frac{1}{d}\left[(a+b)^d - a^d\right] + (a+b)^{d-1} - a^{d-1}.$$

Equivalently,

$$\frac{b}{d}\,\mathbf{h}_{d-1}(a+b,a) < \sum_{j=a+1}^{a+b} j^{d-1} < \frac{b}{d}\,\mathbf{h}_{d-1}(a+b,a) + b\,\mathbf{h}_{d-2}(a+b,a).$$

Proof. Use a Riemann integral estimate. □

2.5. Standard notation for approximations

We use the following standard Bachmann–Landau asymptotic notation without further comment. We write $f(n) = \Theta(g(n))$ to mean there exist constants $a, b > 0$ such that for n large enough, we have $ag(n) \leqslant f(n) \leqslant bg(n)$. If $f(n) = O(g(n))$, then there exists a constant $c > 0$ such that for all n large enough, we have $f(n) \leqslant cg(n)$. On the other hand, if $f(n) = o(g(n))$, then as $n \to \infty$, we have $\frac{f(n)}{g(n)} \to 0$. Similarly, $f(n) = \omega(g(n))$ implies $\frac{f(n)}{g(n)} \to \infty$ as $n \to \infty$, and $f(n) \sim g(n)$ implies $\frac{f(n)}{g(n)} \to 1$ as $n \to \infty$.

3. Metric spaces related to uniform sum distributions

Motivated by applications to $\mathrm{M}_{\mathrm{SSYT}}$ and $\mathrm{M}_{\mathrm{Forest}}$ in the next two sections, we first analyze the distributions of finite and infinite sums of uniform continuous random variables. We parameterize these distributions using certain sequence spaces and precisely relate weak convergence of the underlying distributions to pointwise convergence of the parametrizing sequences. The closure of the space of all possible distributions associated to standardized sums of independent uniform random variables leads us to define the metric space of DUSTPAN distributions. We also describe a closed subset of the DUSTPAN distributions related to distance multisets, which appear in the study of $\mathrm{M}_{\mathrm{SSYT}}$.

3.1. Generalized uniform sum distributions and decreasing sequence space

The Irwin–Hall distributions, also known as *uniform sum distributions*, are the distributions associated to finite sums of independent, identically distributed, uniform random variables supported on $[0, 1]$. First, we relax the requirement that they be identically distributed, and then we relax the requirement that they are finite sums.

Consider a random variable defined as the sum of m independent uniform continuous random variables of the form $\mathcal{S} := \sum_{k=1}^{m} \mathcal{U}[\alpha_k, \beta_k]$ with $\alpha_k \leqslant \beta_k$ for each k. We call the distribution of $\mathcal{S}$ a *generalized uniform sum distribution*. See Figure 3.1 for example density functions. We note that each of the generalized uniform sum distributions is non-normal, though the histograms may look quite similar. By Example 2.3, the dth cumulant of $\mathcal{S}$ for $d \geqslant 2$ is

$$\kappa_d^{\mathcal{S}} = \frac{B_d}{d} \sum_{k=1}^{m} (\beta_k - \alpha_k)^d, \tag{3.1}$$

which only depends on the differences $t_k := \beta_k - \alpha_k$. It is useful to compare (3.1) to the cumulants in (2.8).

The random variable $\mathcal{S}$ can be expressed as a constant overall shift $c = \frac{1}{2} \sum_{k=1}^{m} (\alpha_k + \beta_k)$ plus a *uniform sum random variable associated to* $\mathbf{t}$

$$\mathcal{S}_{\mathbf{t}} := \sum_{k=1}^{m} \mathcal{U}\left[-\frac{t_k}{2}, \frac{t_k}{2}\right], \tag{3.2}$$

where $\mathbf{t} = \{t_1 \geqslant t_2 \geqslant \ldots \geqslant t_m\}$ is a multiset of non-negative real numbers written in decreasing order. Thus, up to an overall constant shift, in order to classify all possible finite generalized uniform sum distributions, it suffices to classify finite sums of independent central continuous uniform random variables of the form (3.2).

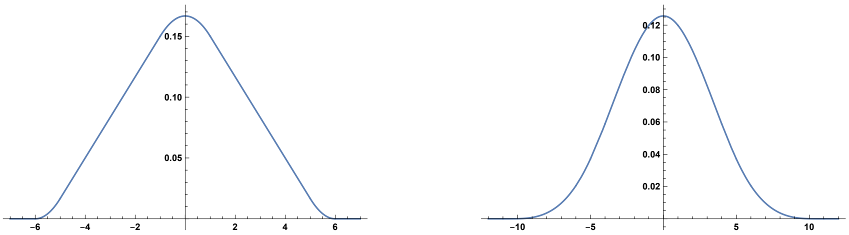

Figure 3.1: Plots of density functions for the distributions $\mathcal{S}_{\mathbf{t}}$ with $\mathbf{t} = (6, 5, 1)$ and $\mathbf{t} = (6, 5, 5, 5, 1)$.

Example 3.1. Consider the $1/2$-power sequence $\mathbf{t} = (1, 1/2, 1/4, 1/8, \ldots)$. The density function for the distribution $\mathcal{S}_{\mathbf{t}}$ in Figure 3.2 has a rather flat top like the sum of two uniform distributions, in contrast to the harmonic sequence $\mathbf{t} = (1, 1/2, 1/3, 1/4, 1/5, \ldots)$.

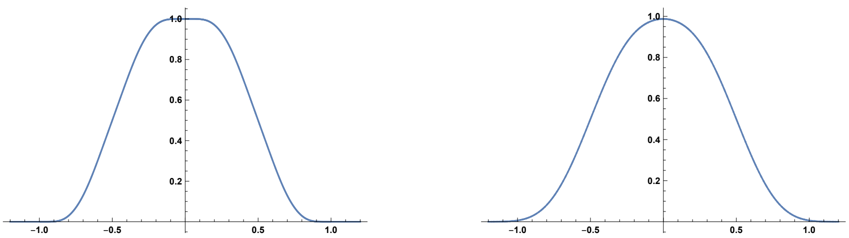

Figure 3.2: Plots of density functions for the distributions $\mathcal{S}_{\mathbf{t}}$ with $\mathbf{t} = (1, 1/2, 1/4, 1/8, 1/16, 1/32, 1/64, 1/128, 1/256)$ and $\mathbf{t} = (1, 1/2, 1/3, 1/4, 1/5, 1/6, 1/7, 1/8, 1/9)$.

We will show below that a similar classification holds for the distributions associated to countable sums of independent continuous uniform random variables, which are defined provided the expectation and variance are finite. Again we have a nice formula for the cumulants of infinite sums of uniform random variables simply by letting $m \to \infty$. Observe that (3.1) is very similar to the definition of the p-norm for a real vector space.

Definition 3.2. Let $\mathbf{t} = (t_1, t_2, \ldots)$ be a sequence of non-negative real numbers. For $p \in \mathbb{R}_{\geqslant 1}$, the *$p$-norm* of $\mathbf{t}$ is $|\mathbf{t}|_p := (\sum_{k=1}^{\infty} t_k^p)^{1/p}$. We also set $|\mathbf{t}|_\infty := \sup_k t_k$.

The p-norm has many nice properties. In particular for $d \geqslant 2$ and $\mathbf{t} = (t_1, \ldots, t_m)$, we have

$$\kappa_d^{\mathcal{S}_{\mathbf{t}}} = \frac{B_d}{d} \sum_{k=1}^{m} (t_k)^d = \frac{B_d}{d} |\mathbf{t}|_d^d. \tag{3.3}$$

It is well-known (e.g. [MV97, Ex. 7.3, p.58]) that if $1 \leqslant p \leqslant q \leqslant \infty$, then $|\mathbf{t}|_p \geqslant |\mathbf{t}|_q$, and that if $|\mathbf{t}|_p < \infty$, then $\lim_{p\to\infty} |\mathbf{t}|_p = |\mathbf{t}|_\infty$. Thus, if $\mathbf{t}$ is weakly decreasing, $|\mathbf{t}|_\infty = \sup_k t_k = t_1$.

The *sequence space with finite p-norm* $\ell_p := \{\mathbf{t} = (t_1, t_2, \ldots) \in \mathbb{R}_{\geqslant 0}^{\mathbb{N}} : |\mathbf{t}|_p < \infty\}$ is commonly used in functional analysis and statistics. Here we define a related concept for analyzing sums of central continuous uniform random variables.

Definition 3.3. The *decreasing sequence space with finite p-norm* is

$$\widetilde{\ell}_p := \{\mathbf{t} = (t_1, t_2, \ldots) : t_1 \geqslant t_2 \geqslant \cdots \geqslant 0, |\mathbf{t}|_p < \infty\}.$$

The elements of $\widetilde{\ell}_p$ may equivalently be thought of as the set of *countable multisets of non-negative real numbers with finite p-norm.* Any finite multiset of non-negative real numbers can be considered as an element of $\widetilde{\ell}_p$ with finite support by sorting the multiset and appending 0's. The multisets in $\widetilde{\ell}_p$ are uniquely determined by their p-norms. In fact, any subsequence of p-norm values injectively determines the multiset provided the sequence goes to infinity.

Lemma 3.4. *Let $\mathbf{t}, \mathbf{u} \in \widetilde{\ell}_p$ for some $1 \leqslant p \leqslant \infty$. Suppose $|\mathbf{t}|_{p_j} = |\mathbf{u}|_{p_j}$ for some sequence $p_j \to \infty$. Then $\mathbf{t} = \mathbf{u}$.*

Proof. We have

$$t_1 = \sup_k t_k = |\mathbf{t}|_\infty = \lim_{j\to\infty} |\mathbf{t}|_{p_j} = \lim_{j\to\infty} |\mathbf{u}|_{p_j} = |\mathbf{u}|_\infty = \sup_k u_k = u_1.$$

We may remove the first elements from both $\mathbf{t}$ and $\mathbf{u}$ to obtain the multisets $(t_2, t_3, \ldots)$ and $(u_2, u_3, \ldots)$ which are both in $\widetilde{\ell}_p$ and have equal p_j-norms again. While removing these largest elements alters the p_j-norms, it does so by the same amount for both $\mathbf{t}$ and $\mathbf{u}$. Repeating the argument, $t_i = u_i$ for all i, so $\mathbf{t} = \mathbf{u}$. □

Theorem 3.5. *Finite generalized uniform sum distributions are bijectively parameterized by*

$$\mathbb{R} \times \{\mathbf{t} \in \widetilde{\ell}_2 : \mathbf{t} \textit{ has finite support}\}.$$

Proof. As noted above, every such distribution is defined by a random variable of the form $c + \mathcal{S}_\mathbf{t}$ for some $c \in \mathbb{R}$ and $\mathbf{t} = (t_1, \ldots, t_m, 0, 0, \ldots) \in \widetilde{\ell}_2$. To show uniqueness, suppose $\mathcal{S}_\mathbf{t} = \mathcal{S}_\mathbf{u}$. By (3.3), we know

$$\frac{B_d}{d}|\mathbf{t}|_d^d = \frac{B_d}{d}\sum_{k=1}^m t_k^d = \kappa_d^{\mathcal{S}_\mathbf{t}} = \kappa_d^{\mathcal{S}_\mathbf{u}} = \frac{B_d}{d}\sum_{k=1}^m u_k^d = \frac{B_d}{d}|\mathbf{u}|_d^d.$$

Therefore, since the even Bernoulli numbers are non-zero, we have $|\mathbf{t}|_d = |\mathbf{u}|_d$ for each d even, which is a sequence approaching infinity. Hence, by Lemma 3.4, $\mathbf{t} = \mathbf{u}$. □

The probability density functions (PDF) for any finite generalized uniform sum distributions can be determined as a convolution. We will not need this formula in the rest of this paper, but we note it here for completeness. It was used to generate Figure 3.2.

Lemma 3.6. *Let $\mathbf{t} = \{t_1 \geqslant \ldots \geqslant t_m > 0\}$. Then $\mathrm{PDF}(\mathcal{S}_\mathbf{t}; x)$ is given by*

$$\frac{1}{2(m-1)!t_1\cdots t_m}\sum_{\epsilon_1,\ldots,\epsilon_m\in\{\pm1\}} \epsilon_1\cdots\epsilon_m \left(x + \frac{\epsilon_1 t_1 + \cdots + \epsilon_m t_m}{2}\right)^{m-1} \cdot \operatorname{sgn}\left(x + \frac{\epsilon_1 t_1 + \cdots + \epsilon_m t_m}{2}\right).$$

Proof. For the case $m = 1$,

$$\mathrm{PDF}(\mathcal{U}[-t_1/2, t_1/2]; x) = \frac{1}{2t_1}\left(\mathrm{sgn}\left(x + \frac{t_1}{2}\right) - \mathrm{sgn}\left(x - \frac{t_1}{2}\right)\right).$$

Let $*$ denote convolution. One can check that for all $u > 0$, we have the convolution identity

$$\begin{aligned} x^k \mathrm{sgn}(x) * \frac{1}{2}\left(\mathrm{sgn}(x+u) - \mathrm{sgn}(x-u)\right) \\ = \frac{1}{k+1}\left((x+u)^{k+1}\mathrm{sgn}(x+u) - (x-u)^{k+1}\mathrm{sgn}(x-u)\right). \end{aligned}$$

The probability density function of the sum of independent random variables is the convolution of their density functions. Therefore, the general case of the lemma now follows by applying the $m = 1$ case and the convolution identity inductively. □

Remark 3.7. When $t_1 = \cdots = t_m = 1$, the formula in Lemma 3.6 collapses to

$$\begin{aligned} &\mathrm{PDF}(\mathcal{IH}_m - m/2; x) \\ &= \frac{1}{2(m-1)!}\sum_{k=0}^{m}(-1)^k\binom{m}{k}\left(x + \frac{m-k}{2} - \frac{k}{2}\right)^{m-1}\mathrm{sgn}\left(x + \frac{m-k}{2} - \frac{k}{2}\right). \end{aligned}$$

Hence we recover the density formula for the Irwin–Hall distributions [JKB94, p. 296]

$$\mathrm{PDF}(\mathcal{IH}_m; x) = \frac{1}{2(m-1)!}\sum_{k=0}^{m}(-1)^k\binom{m}{k}(x-k)^{m-1}\mathrm{sgn}(x-k).$$

Remark 3.8. A similar formula for the cumulative distribution function of $\mathcal{S}_{\mathbf{t}}$ as a sum over the vertices of the hypercube is given in [BS79]. See also [JKB94, p. 298-300] for relevant discussion.

We now turn to infinite sums of independent uniform continuous random variables. Our next goal is to generalize Theorem 3.5 to this setting. To do so, we must first extend the uniform-sum distributions $\mathcal{S}_{\mathbf{t}}$ to countably infinite multisets $\mathbf{t}$, and discuss the basic properties of these random variables including existence, characteristic functions, and cumulants. Existence depends on the following well-known result, which often appears in treatments of the law of large numbers. See, for example, [Dur10, Thm. 2.5.3].

Theorem 3.9 (Kolmogorov's Two-Series Theorem)**.** *Let $\mathcal{X}_1, \mathcal{X}_2, \ldots$ be a sequence of independent real-valued random variables. Suppose $\mathbb{E}[\mathcal{X}_k] = 0$ and $\sum_{k=1}^{\infty}\mathrm{Var}[\mathcal{X}_k] < \infty$. Then $\sum_{k=1}^{\infty}\mathcal{X}_k$ converges almost surely.*

Almost sure convergence implies convergence in distribution. Therefore, by Kolmogorov's Two-Series Theorem, we are lead to the following definition.

Definition 3.10. A *generalized uniform sum distribution* is any distribution associated to a random variable with finite mean and variance given as a countable sum of independent continuous uniform random variables. As in the finite case, such random variables are given by a constant overall shift plus a *uniform sum random variable*

$$\mathcal{S}_{\mathbf{t}} := \mathcal{U}\left[-\frac{t_1}{2}, \frac{t_1}{2}\right] + \mathcal{U}\left[-\frac{t_2}{2}, \frac{t_2}{2}\right] + \cdots$$

for some $\mathbf{t} = (t_1, t_2, \ldots) \in \widetilde{\ell}_2$. Kolmogorov's Theorem applies since $\mathrm{Var}[\mathcal{U}[-t/2, t/2]] = \frac{B_2}{2}t^2$ and $\sum_{k=1}^{\infty} \mathrm{Var}[\mathcal{U}[-t_k/2, t_k/2]] = \frac{B_2}{2}|\mathbf{t}|_2^2 < \infty$.

Conversely, Kolmogorov's stronger Three-Series Theorem [Dur10, Thm. 2.5.4] shows that if $\sum_{i=1}^{\infty} t_i^2 = \infty$, then $\sum_{i=1}^{\infty} \mathcal{U}[-t_i/2, t_i/2]$ diverges with positive probability, so the assumption $|\mathbf{t}|_2^2 < \infty$ is essential. In this way we also see that uncountably many non-zero summands of independent continuous uniform random variables must diverge. Thus, we cannot extend Definition 3.10 beyond countable sums.

We claim that each uniform sum random variable $\mathcal{S}_{\mathbf{t}}$ for $\mathbf{t} \in \widetilde{\ell}_2$ gives rise to a distinct distribution. In order to prove the claim, we need to verify the relationship between the p-norms and the cumulants of the infinite sums is as expected. To do so, we describe the characteristic and moment-generating functions of $\mathcal{S}_{\mathbf{t}}$.

Lemma 3.11. *Let $\mathbf{t} = (t_1, t_2, \ldots) \in \widetilde{\ell}_2$. Then $\mathcal{S}_{\mathbf{t}}$ exists, has moments of all orders, and is determined by its moments. The characteristic function is the entire function*

$$\phi_{\mathcal{S}_{\mathbf{t}}}(s) = \prod_{k=1}^{\infty} \mathrm{sinc}(st_k/2), \qquad s \in \mathbb{C}. \tag{3.4}$$

Moreover, $\mathbb{E}[\mathcal{S}_{\mathbf{t}}] = 0$, $\mathrm{Var}[\mathcal{S}_{\mathbf{t}}] < \infty$, *and for each* $d \in \mathbb{Z}_{\geqslant 2}$,

$$\kappa_d^{\mathcal{S}_{\mathbf{t}}} = \frac{B_d}{d} \sum_{k=1}^{\infty} t_k^d = \frac{B_d}{d} |\mathbf{t}|_d^d. \tag{3.5}$$

Proof. As mentioned above, the assumption $\mathbf{t} \in \widetilde{\ell}_2$ and Theorem 3.9 together imply $\mathcal{S}_{\mathbf{t}}$ exists. The characteristic function of $\mathcal{U}[-x, x]$ is

$$\phi_{\mathcal{U}[-x,x]}(s) = \frac{1}{2x} \int_{-x}^{x} e^{ist}\, dt = \frac{e^{isx} - e^{-isx}}{2isx} = \frac{\sin(sx)}{sx} := \mathrm{sinc}(sx), \tag{3.6}$$

where $\mathrm{sinc}(0) := 1$. Consequently, the nth partial sum $\mathcal{S}_n = \sum_{k=1}^{n} \mathcal{U}\left[-\frac{t_k}{2}, \frac{t_k}{2}\right]$ has characteristic function $\phi_{\mathcal{S}_n}(s) = \prod_{k=1}^{n} \mathrm{sinc}(st_k/2)$. Almost sure convergence implies convergence in distribution, so $\mathcal{S}_n \Rightarrow \mathcal{S}_{\mathbf{t}}$. Thus, by Lévy's Continuity Theorem, we have for each $s \in \mathbb{R}$ that

$$\phi_{\mathcal{S}_{\mathbf{t}}}(s) = \prod_{k=1}^{\infty} \mathrm{sinc}(st_k/2).$$

By Lemma 3.12 below, the product form for $\phi_{\mathcal{S}_{\mathbf{t}}}$ is entire and hence complex analytic on an open ball, so (3.5) follows from Remark 2.1. Likewise, $\mathcal{S}_{\mathbf{t}}$ has moments of all orders and $\mathcal{S}_{\mathbf{t}}$ is determined by its moments.

Since the entire functions $\phi_{\mathcal{S}_n}(s)$ converge uniformly on compact subsets of $\mathbb{C}$ to $\phi_{\mathcal{S}_{\mathbf{t}}}(s)$, it follows that the d^{th} moment can be determined by the constant term of the d^{th} derivative of the characteristic function

$$\lim_{n\to\infty} \mathbb{E}[\mathcal{S}_n^d] = \lim_{n\to\infty} i^{-d}\phi_{\mathcal{S}_n}^{(d)}(0) = i^{-d}\phi_{\mathcal{S}_{\mathbf{t}}}^{(d)}(0) = \mathbb{E}[\mathcal{S}_{\mathbf{t}}^d]$$

for all $d \geqslant 1$. The moments of any random variable determine its cumulants and vice versa. Therefore, the cumulant formula now follows from (3.3), including the first two moments. □

Lemma 3.12. *Let* $\mathbf{t} = (t_1, t_2, \ldots) \in \widetilde{\ell}_2$. *As a function of* s, *the infinite product*

$$\prod_{i=1}^{\infty} \operatorname{sinc}(st_i/2)$$

converges to an entire function in the complex plane. Moreover, for $|s| < 1/|\mathbf{t}|_2$,

$$\left|\prod_{i=1}^{\infty} \operatorname{sinc}(st_i/2)\right| \leqslant e.$$

Proof. For each $D > 0$, the entire function $\frac{1-\operatorname{sinc}(z)}{z^2}$ is bounded on $|z| < D$ by some constant $C > 0$. Thus

$$|1 - \operatorname{sinc}(z)| \leqslant C|z|^2 \qquad \text{for } |z| < D.$$

Consequently, for $|s| < 2D/\sup\{t_i\}$, we have

$$|1 - \operatorname{sinc}(st_i/2)| < \frac{C}{4}|s|^2 t_i^2.$$

Hence

$$\sum_{i=1}^{\infty} |1 - \operatorname{sinc}(st_i/2)| \leqslant \frac{C}{4}|s|^2|\mathbf{t}|_2^2 < \infty.$$

Thus, the sum converges uniformly on compact subsets of $\{|s| < 2D/\sup\{t_i\}\}$. Taking $D \to \infty$, the sum converges uniformly on compact subsets of all of $\mathbb{C}$. The result now follows by standard criteria for infinite product convergence such as [Rud87, Thm. 15.6].

For the growth rate bound, it is straightforward to check that when $D = 1/2$, we may use $C = 4$. Since $|\mathbf{t}|_2 \geqslant |\mathbf{t}|_\infty = \sup\{t_i\}$, for $|s| < 1/|\mathbf{t}|_2$, we have

$$\begin{aligned}\left|\prod_{i=1}^{\infty} \operatorname{sinc}(st_i/2)\right| &= \prod_{i=1}^{\infty} |1 - (1 - \operatorname{sinc}(st_i/2))| \leqslant \prod_{i=1}^{\infty} (1 + |1 - \operatorname{sinc}(st_i/2)|) \\ &\leqslant \prod_{i=1}^{\infty} \left(1 + |s|^2 t_i^2\right) \leqslant \prod_{i=1}^{\infty} \exp\left(|s|^2 t_i^2\right) \\ &= \exp\left(|s|^2 |\mathbf{t}|_2^2\right) \leqslant \exp(1) = e.\end{aligned}$$

□

Theorem 3.13. *Generalized uniform sum distributions are bijectively parameterized by* $\mathbb{R} \times \widetilde{\ell}_2$. *In particular, if* $\mathbf{t}, \mathbf{u} \in \widetilde{\ell}_2$ *with* $\mathbf{t} \neq \mathbf{u}$, *then* $\mathcal{S}_{\mathbf{t}} \neq \mathcal{S}_{\mathbf{u}}$. *Furthermore,* $\mathcal{S}_{\mathbf{t}}^* = \mathcal{S}_{\mathbf{u}}^*$ *if and only if* $\mathbf{t}, \mathbf{u}$ *differ by a scalar multiple.*

Proof. The first and second claims follow exactly as in Theorem 3.5 using the cumulant formula in Lemma 3.11. For the third claim, we can assume $|\mathbf{t}|_\infty = |\mathbf{u}|_\infty$ by rescaling if necessary and $\mathcal{S}_{\mathbf{t}}^* = \mathcal{S}_{\mathbf{u}}^*$. From Lemma 3.11 and the general properties of cumulants, it follows that for all d even,

$$|\mathbf{t}|_d^d / |\mathbf{t}|_2^{d/2} = |\mathbf{u}|_d^d / |\mathbf{u}|_2^{d/2}.$$

Taking dth roots and the limiting sequence of positive even integers d, this implies

$$\frac{|\mathbf{t}|_\infty}{|\mathbf{t}|_2^{1/2}} = \lim_{d\to\infty} \frac{|\mathbf{t}|_d}{|\mathbf{t}|_2^{1/2}} = \lim_{d\to\infty} \frac{|\mathbf{u}|_d}{|\mathbf{u}|_2^{1/2}} = \frac{|\mathbf{u}|_\infty}{|\mathbf{u}|_2^{1/2}}.$$

Since $|\mathbf{t}|_\infty = |\mathbf{u}|_\infty$, we have $|\mathbf{t}|_2 = |\mathbf{u}|_2$, which hence gives $|\mathbf{t}|_d = |\mathbf{u}|_d$ for all d even. Again by Lemma 3.4, we have $\mathbf{t} = \mathbf{u}$. □

Example 3.14. Infinite sums of independent continuous uniform random variables have appeared elsewhere in the literature, though rarely. For instance, when $\mathbf{t} = (1, 1/2, 1/4, 1/8, \ldots)$, the cumulative distribution function of $\mathcal{S}_{\mathbf{t}} = \sum_{k=1}^\infty \mathcal{U}[-1/2^k, 1/2^k]$ is the so-called *Fabius function*, [Fab66], which is a known example of a C^∞-function on an interval which is nowhere analytic. The characteristic function is nonetheless entire by Lemma 3.12.

Example 3.15. Another interesting case arises from $\mathbf{t} = (1, 1/2, 1/3, 1/4, \ldots)$. Since $|\mathbf{t}|_2 = \sum_{k=1}^\infty 1/k^2 < \infty$, $\mathcal{S}_{\mathbf{t}} = \sum_{k=1}^\infty \mathcal{U}[-1/(2k), 1/(2k)]$ converges almost surely. For $d \geqslant 1$, we have

$$\kappa_{2d} = \frac{B_{2d}}{2d} \sum_{k=1}^\infty \frac{1}{k^{2d}} = \frac{B_{2d}}{2d} \zeta(2d).$$

Using the known identity

$$\zeta(2d) = (-1)^{d+1} \frac{B_{2d}(2\pi)^{2d}}{2(2d)!},$$

it follows that

$$\begin{aligned} \log \phi_{\mathcal{S}_{\mathbf{t}}}(s) &= \sum_{k=1}^\infty \kappa_k \frac{s^k}{k!} \\ &= -\sum_{d=1}^\infty \frac{\zeta(2d)^2}{d} \left(\frac{s}{2\pi}\right)^{2d}, \end{aligned}$$

which is valid in a complex neighborhood of $s = 0$. This last expression is similar to the left-hand side of the known identity

$$\sum_{d=0}^\infty \zeta(2d) s^{2d} = -\frac{\pi s}{2} \cot(\pi s).$$

Example 3.16. Let $\alpha \in \mathbb{R}_{>0}$ and set $\mathbf{t}^{(N)} = (1/N^\alpha, 1/N^\alpha, \ldots, 1/N^\alpha, 0, \ldots)$ where there are N non-zero terms. Then $|\mathbf{t}^{(N)}|_p = N^{\frac{1}{p}-\alpha}$. So, for $1 \leqslant p < \infty$,

$$\lim_{N\to\infty} |\mathbf{t}^{(N)}|_p = \begin{cases} 0 & \text{if } p > 1/\alpha \\ 1 & \text{if } p = 1/\alpha \\ \infty & \text{if } p < 1/\alpha. \end{cases}$$

On the other hand, for each k we have $\lim_{N\to\infty} t_k^{(N)} = 0$, independent of α. Hence we have a large family of sequences which each converges pointwise to $(0, 0, \ldots)$, but which have different limiting p-norms. In particular, when $\alpha = 1/2$ we have $\lim_{N\to\infty} |\mathbf{t}^{(N)}|_2 = 1 \neq 0 = |(0, 0, \ldots)|_2$, so the limit of the 2-norms is not the 2-norm of the limit. The interplay between convergence in $\widetilde{\ell}_2$ and convergence of generalized uniform sum distributions is consequently somewhat subtle, which we treat in the next subsection.

3.2. Pointwise convergence and convergence in even norms

The decreasing sequence space $\widetilde{\ell}_2$ has a natural notion of pointwise convergence. In this subsection, we relate pointwise convergence to convergence of p-norms for all positive even $p \geqslant 4$, assuming the 2-norms are bounded.

Lemma 3.17. *Fix $M \in \mathbb{R}$. Let $\mathbf{t}^{(N)} \in \widetilde{\ell}_2$ be a countable sequence of sequences such that $|\mathbf{t}^{(N)}|_2^2 \leqslant M$ for each N and*

$$\lim_{N\to\infty} |\mathbf{t}^{(N)}|_{2d} = \tau_{2d}$$

exists for all $d \in \mathbb{Z}_{\geqslant 2}$. Then

(i) $\lim_{d\to\infty} \tau_{2d}$ *exists,*

(ii) $\lim_{N\to\infty} t_1^{(N)}$ *exists,*

(iii) $\lim_{d\to\infty} \tau_{2d} = \lim_{N\to\infty} t_1^{(N)} = \lim_{N\to\infty} |\mathbf{t}^{(N)}|_\infty$*, and*

(iv) $\mathbf{t}^{(N)}$ *converges pointwise to* $\mathbf{t} = (t_1, t_2, \ldots) \in \widetilde{\ell}_2$ *where* $t_i = \lim_{N\to\infty} t_i^{(N)}$.

Proof. For (i), if $d \leqslant e \leqslant \infty$ then $|\mathbf{t}^{(N)}|_{2d} \geqslant |\mathbf{t}^{(N)}|_{2e}$ by properties of the p-norm. Therefore, $\tau_{2d} \geqslant \tau_{2e} \geqslant 0$ and $\lim_{d\to\infty} \tau_{2d}$ exists.

For (ii), observe that since $\mathbf{t}^{(N)}$ is a decreasing sequence in $\widetilde{\ell}_2$, we know $|\mathbf{t}^{(N)}|_\infty = t_1^{(N)} \geqslant t_i^{(N)}$ for all i. Therefore, for all $d \in \mathbb{Z}_{\geqslant 1}$, we have

$$\begin{aligned} |\mathbf{t}^{(N)}|_{2d}^{2d} &= \sum_i (t_i^{(N)})^{2d} \\ &\leqslant \sum_i (t_1^{(N)})^{2(d-1)} (t_i^{(N)})^2 \\ &\leqslant (t_1^{(N)})^{2(d-1)} \cdot M. \end{aligned}$$

Combining this with the fact that $t_1^{(N)} \leqslant |\mathbf{t}^{(N)}|_{2d}$ by definition of the p-norm, one has

$$t_1^{(N)} \leqslant |\mathbf{t}^{(N)}|_{2d} \leqslant (t_1^{(N)})^{1-\frac{1}{d}} \cdot M^{\frac{1}{2d}}. \tag{3.7}$$

Taking $N \to \infty$ in (3.7) gives

$$\limsup_{N\to\infty}(t_1^{(N)}) \leqslant \tau_{2d} \leqslant \liminf_{N\to\infty}(t_1^{(N)})^{1-\frac{1}{d}} \cdot M^{\frac{1}{2d}}. \tag{3.8}$$

Taking $d \to \infty$ in (3.8) gives

$$\limsup_{N\to\infty}(t_1^{(N)}) \leqslant \lim_{d\to\infty} \tau_{2d} \leqslant \liminf_{N\to\infty}(t_1^{(N)}), \tag{3.9}$$

so $\lim_{N\to\infty} t_1^{(N)} = \lim_{d\to\infty} \tau_{2d}$ which implies the limit exists by (i). Part (iii) also follows from (3.9) and the fact that $|\mathbf{t}^{(N)}|_\infty = t_1^{(N)}$.

Part (iv) follows by an inductive argument. By (ii), $t_1 = \lim_{N\to\infty} t_1^{(N)}$ exists. Define another sequence of sequences $\mathbf{u}^{(N)} := \{t_2^{(N)} \geqslant t_3^{(N)} \geqslant \cdots\}$, so that $|\mathbf{u}^{(N)}|_2^2 = |\mathbf{t}^{(N)}|_2^2 - (t_1^{(N)})^2 \leqslant M$ and

$$|\mathbf{u}^{(N)}|_{2d}^{2d} = |\mathbf{t}^{(N)}|_{2d}^{2d} - (t_1^{(N)})^{2d} \qquad \Rightarrow \qquad \lim_{N\to\infty} |\mathbf{u}^{(N)}|_{2d} = \left(\tau_{2d}^{2d} - t_1^{2d}\right)^{\frac{1}{2d}} \text{ exists}$$

by the hypotheses on $\mathbf{t}^{(\mathbf{N})}$. By (iii) applied to $\mathbf{u}^{(N)}$, $t_2 := \lim_{N\to\infty} u_1^{(N)} = \lim_{N\to\infty} t_2^{(N)}$ exists. Repeating the argument, $\mathbf{t}^{(N)}$ converges pointwise to $(t_1, t_2, \ldots)$. □

Lemma 3.18. *Suppose* $\mathbf{t}^{(N)} \in \widetilde{\ell}_2$ *with* $|\mathbf{t}^{(N)}|_2^2 \leqslant M$ *converges pointwise to* $\mathbf{t} \in \widetilde{\ell}_2$*. Then* $|\mathbf{t}|_2^2 \leqslant M$ *and for all* $d \geqslant 2$*,*

$$|\mathbf{t}|_{2d} = \lim_{N\to\infty} |\mathbf{t}^{(N)}|_{2d}.$$

Proof. By Fatou's Lemma applied to the counting measure on $\mathbb{Z}_{\geqslant 1}$,

$$|\mathbf{t}|_2^2 \leqslant \liminf_{N\to\infty} |\mathbf{t}^{(N)}|_2^2 \leqslant M.$$

Fix $d \geqslant 2$. For each N, we have $t_1^{(N)} \geqslant t_2^{(N)} \geqslant \cdots \geqslant t_i^{(N)} \geqslant \cdots$. Thus

$$M \geqslant (t_1^{(N)})^2 + \cdots + (t_i^{(N)})^2 \geqslant i(t_i^{(N)})^2,$$

which implies

$$(t_i^{(N)})^2 \leqslant \frac{M}{i} \;\Rightarrow\; (t_i^{(N)})^{2d} \leqslant \left(\frac{M}{i}\right)^d.$$

Since $\sum_{i=1}^\infty \frac{1}{i^d}$ converges for $d \geqslant 2$, the sequence $(t_i^{(N)})^{2d}$ is dominated by the integrable function $\left(\frac{M}{i}\right)^d$ over the positive integers. By Lebesgue's Dominated Convergence Theorem, since $\lim_{N\to\infty}(t_i^{(N)})^{2d} = t_i^{2d}$, we have

$$\lim_{N\to\infty} |\mathbf{t} - \mathbf{t}^{(N)}|_{2d} = 0 \qquad \Rightarrow \qquad \lim_{N\to\infty} |\mathbf{t}^{(N)}|_{2d} = |\mathbf{t}|_{2d}.$$ □

Corollary 3.19. *Suppose* $\mathbf{t}^{(N)} \in \widetilde{\ell}_2$ *with* $|\mathbf{t}^{(N)}|_2^2 \leqslant M$. *Then* $\mathbf{t}^{(N)}$ *converges pointwise to* $\mathbf{t}$ *if and only if* $|\mathbf{t}|_{2d} = \lim_{N\to\infty} |\mathbf{t}^{(N)}|_{2d}$ *for all* $d \geqslant 2$.

Proof. The proof follows directly from Lemma 3.17 and Lemma 3.18. □

Observe that Corollary 3.19 says nothing about the 2-norm of the sequences. It is possible for $\mathbf{t}^{(N)} \to \mathbf{t}$ pointwise, even if $|\mathbf{t}|_2^2 \neq \lim_{N\to\infty} |\mathbf{t}^{(N)}|_2^2$, as the next example and lemma illustrate.

Example 3.20. In the Irwin–Hall case, we have $\mathcal{IH}_N = \mathcal{S}_{\mathbf{t}^{(N)}} + N/2$ where

$$\mathbf{t}^{(N)} = (\underbrace{1, \ldots, 1}_{N \text{ copies}}, 0, \ldots).$$

Since $|\mathbf{t}^{(N)}|_2^2 = N$, after standardizing, $\mathcal{IH}_N^* = \mathcal{S}_{\widehat{\mathbf{t}^{(N)}}}$ where

$$\widehat{\mathbf{t}^{(N)}} = (\underbrace{\sqrt{12/N}, \ldots, \sqrt{12/N}}_{N \text{ copies}}, 0, \ldots),$$

which converges pointwise to $\mathbf{t} = (0, 0, \ldots)$. Nonetheless, $|\mathbf{t}|_2^2 = 0 < 12 = |\widehat{\mathbf{t}^{(N)}}|_2^2$ and $\mathcal{IH}_N^* \Rightarrow \mathcal{N}(0, 1)$.

Lemma 3.21. *For every* $\mathbf{t} = (t_1, t_2, \ldots) \in \widetilde{\ell}_2$ *and every* $M \geqslant |\mathbf{t}|_2^2$, *there exists a sequence* $\mathbf{t}^{(N)}$ *of finitely supported decreasing sequences such that* $|\mathbf{t}^{(N)}|_2^2 = M$ *and* $\mathbf{t}^{(N)} \to \mathbf{t}$ *pointwise.*

Proof. Define a sequence of sequences $\mathbf{t}^{(N)} \in \widetilde{\ell}_2$ with $|\mathbf{t}^{(N)}|_2^2 = M$ as follows. Let

$$\epsilon_N := \sqrt{M - \sum_{i=1}^{N} t_i^2}.$$

For each $N \geqslant 1$, choose $m_N \in \mathbb{Z}_{\geqslant 1}$ large enough so that $\epsilon_N / m_N \leqslant \frac{1}{N}$. Set

$$\mathbf{t}^{(N)} = (t_1, t_2, \ldots, t_N, \underbrace{\epsilon_N/m_N, \ldots, \epsilon_N/m_N}_{m_N^2 \text{ copies}}, 0, 0, \ldots).$$

As claimed, $\mathbf{t}^{(N)} \to \mathbf{t}$ pointwise and

$$|\mathbf{t}^{(N)}|_2^2 = \sum_{i=1}^{N} t_i^2 + m_N^2 \cdot \left(\frac{\epsilon_N}{m_N}\right)^2 = M. \qquad \square$$

Example 3.22. Consider again $\mathbf{t} = (1, 1/2, 1/3, \ldots)$ so $|\mathbf{t}|_2^2 = \sum_{i=1}^{\infty} (\frac{1}{i})^2 = \pi^2/6 \approx 1.6449$. Let $\epsilon_N := \sqrt{2 - \sum_{i=1}^{N} (\frac{1}{i})^2}$. For each $N \geqslant 1$, set

$$\mathbf{t}^{(N)} = (1, 1/2, \ldots, 1/N, \underbrace{\epsilon_N/N, \ldots, \epsilon_N/N}_{N^2 \text{ copies}}, 0, 0, \ldots).$$

Clearly $\mathbf{t}^{(N)} \to \mathbf{t}$ pointwise and

$$|\mathbf{t}^{(N)}|_2^2 = \sum_{i=1}^{N} t_i^2 + N^2 \cdot \left(\frac{\epsilon_N}{N}\right)^2 = 2.$$

However, $|\mathbf{t}|_2^2 = \pi^2/6 \neq 2 = \lim_{N\to\infty} |\mathbf{t}^{(N)}|_2^2$.

Lemma 3.23. *Suppose $\mathbf{t}^{(N)} \in \widetilde{\ell}_2$ converges pointwise to $\mathbf{t} \in \widetilde{\ell}_2$ with $|\mathbf{t}^{(N)}|_2^2 \to \tau_2 < \infty$. Then*

$$\mathcal{S}_{\mathbf{t}^{(N)}} \Rightarrow \mathcal{S}_{\mathbf{t}} + \mathcal{N}(0, \sigma^2)$$

where $\sigma = \sqrt{(\tau_2 - |\mathbf{t}|_2^2)/12}$ and the sum is independent.

Proof. By Lemma 3.18, $\lim_{N\to\infty} |\mathbf{t}^{(N)}|_{2d} = |\mathbf{t}|_{2d}$ for all $d \in \mathbb{Z}_{\geqslant 2}$, so for all $d \geqslant 3$,

$$\kappa_d^{\mathcal{S}_{\mathbf{t}^{(N)}}} \to \kappa_d^{\mathcal{S}_{\mathbf{t}}} = \kappa_d^{\mathcal{S}_{\mathbf{t}}+\mathcal{N}(0,\sigma^2)}$$

since $\kappa_d^{\mathcal{N}(0,\sigma^2)} = 0$. As for $d = 2$,

$$\kappa_2^{\mathcal{S}_{\mathbf{t}^{(N)}}} \to \frac{\tau_2}{12} = \frac{|\mathbf{t}|_2^2}{12} + \sigma^2 = \kappa_2^{\mathcal{S}_{\mathbf{t}}+\mathcal{N}(0,\sigma^2)}.$$

The result follows by the Method of Moments/Cumulants. □

In light of Lemma 3.23, pointwise convergence in $\widetilde{\ell}_2$ leads to us to study an additional family of sums of random variables. Note, the sum of two generalized uniform sum random variables is another generalized uniform sum of random variables. Also, the sum of two normal distributions is normal, so we have reached a natural limit to the generalizations.

Definition 3.24. A *DUSTPAN distribution* is a *distribution associated to a uniform sum for* **t** *plus an independent normal distribution* $\mathcal{S}_{\mathbf{t}} + \mathcal{N}(0, \sigma^2)$, assuming the two random variables are independent, $\mathbf{t} \in \widetilde{\ell}_2$, and $\sigma \in \mathbb{R}_{\geqslant 0}$.

Example 3.25. Consider the $1/n$-sequence $\mathbf{t} = (1, 1/2, 1/3, \dots)$ again. Let $\sigma = \sqrt{12 - \pi^2/6}$. The distribution $\mathcal{S}_{\mathbf{t}}$ has a small variance compared to $\mathcal{N}(0, \sigma^2)$, so $\mathcal{S}_{\mathbf{t}} + \mathcal{N}(0, \sigma^2)$ looks like a fat normal distribution. See the approximation in Figure 3.3.

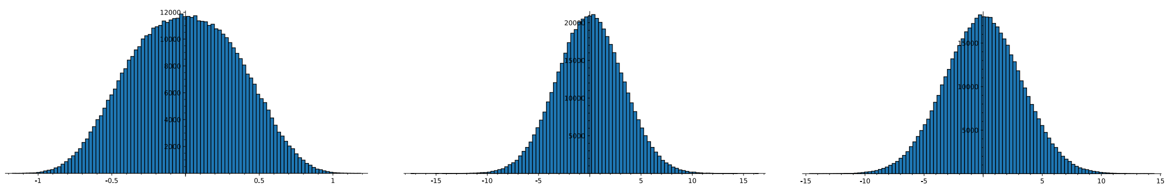

Figure 3.3: Histograms obtained by sampling from the distributions $\mathcal{S}_{\mathbf{t}}$, $\mathcal{N}(0, \sigma)$, and $\mathcal{S}_{\mathbf{t}} + \mathcal{N}(0, \sigma)$ with $\mathbf{t} = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8})$ and $\sigma \approx 3.22$.

3.3. The metric space of DUSTPAN distributions

Recall the *metric space of DUSTPAN distributions*,

$$\mathbf{M}_{\mathrm{DUST}} \coloneqq \{\mathcal{S}_{\mathbf{t}} + \mathcal{N}(0,\sigma) : |\mathbf{t}|_2^2/12 + \sigma^2 = 1\},$$

along with the *DUSTPAN parameter space*

$$\mathbf{P}_{\mathrm{DUST}} \coloneqq \left\{\mathbf{t} \in \widetilde{\ell}_2 : |\mathbf{t}|_2^2 \leqslant 12\right\}.$$

We will show below that $\mathbf{P}_{\mathrm{DUST}}$ and $\mathbf{M}_{\mathrm{DUST}}$ are homeomorphic closed sets in their respective topologies of pointwise convergence and convergence in distribution, thus completing the task of completely characterizing all possible limit laws of standardized general uniform sum distributions.

From Definition 3.24, it follows that the characteristic functions of DUSTPAN distributions have nice properties. Recall that a *normal family* of holomorphic functions in some open set $U \subset \mathbb{C}$ is one where every infinite sequence has a subsequence which converges uniformly on compact subsets of U.

Lemma 3.26. *The set of characteristic functions $\{\phi_{\mathcal{S}}(s) : \mathcal{S} \in \mathbf{M}_{\mathrm{DUST}}\}$ is a normal family of entire functions.*

Proof. Let $\mathcal{S} = \mathcal{S}_{\mathbf{t}} + \mathcal{N}(0,\sigma) \in \mathbf{M}_{\mathrm{DUST}}$. By definition, the characteristic function of a DUSTPAN distribution is the product of the corresponding characteristic functions for the normal and generalized uniform sum distributions,

$$\phi_{\mathcal{S}}(s) = \exp(-\sigma^2/2)\prod_{i=1}^{\infty}\mathrm{sinc}(st_k)/2.$$

By the growth bound in Lemma 3.12, for $|s| < \frac{1}{12}$, we have

$$|\exp(-\sigma^2/2)\prod_{i=1}^{\infty}\mathrm{sinc}(st_k)/2| \leqslant \exp(1).$$

Thus $\{\phi_{\mathcal{S}}(s) : \mathcal{S} \in \mathbf{M}_{\mathrm{DUST}}\}$ is a family of bounded analytic functions on $|s| < \frac{1}{12}$. By Montel's Theorem, it is a normal family in that domain. The bound in Lemma 3.12 may be extended to any bounded domain using the same argument, so it is in fact a normal family of entire functions. □

Lemma 3.27 (Converse of Frechét–Shohat for DUSTPAN's)**.** *Suppose a sequence of DUSTPAN distributions $\mathcal{X}_N \coloneqq \mathcal{S}_{\mathbf{t}^{(N)}} + \mathcal{N}(0,\sigma^{(N)}) \in \mathbf{M}_{\mathrm{DUST}}$ converges in distribution to some $\mathcal{X}$. Then $\mathbb{E}[\mathcal{X}^d] < \infty$ exists for all $d \in \mathbb{Z}_{\geqslant 1}$, $\mathcal{X}$ is determined by its moments, and* $\lim_{N\to\infty}\mathbb{E}[\mathcal{X}_N^d] = \mathbb{E}[\mathcal{X}^d]$.

Proof. By Lévy's Continuity Theorem, $\phi_{\mathcal{X}_N}(s) \to \phi_{\mathcal{X}}(s)$ for all $s \in \mathbb{R}$. By Lemma 3.26, we may replace $\mathcal{X}_N$ if necessary with a subsequence for which $\phi_{\mathcal{X}_N}(s)$ converges uniformly on compact subsets so that we can assume $\phi_{\mathcal{X}}(s)$ is entire. Therefore, the moment generating function of $\mathcal{X}$ has positive radius of convergence, moments of all order exist, $\mathcal{X}$ is determined by its moments, and the limit of the moments is the moment of the limit. □

We may now restate and prove Theorem 1.15 from the introduction.

Theorem 1.15. The map $\Phi\colon \mathbf{P}_{\mathrm{DUST}} \to \mathbf{M}_{\mathrm{DUST}}$ given by $\mathbf{t} \mapsto \mathcal{S}_{\mathbf{t}} + \mathcal{N}(0,\sigma)$ where $\sigma := \sqrt{1 - |\mathbf{t}|_2^2/12}$ is a homeomorphism between sequentially compact spaces.

Proof. The parameter space $\mathbf{P}_{\mathrm{DUST}}$ is closed under pointwise convergence by Lemma 3.18. Moreover, it is sequentially compact under pointwise convergence, either by Tychonoff's Theorem applied to $[0,\sqrt{12}]^{\mathbb{N}}$ or by a simple diagonalization argument. Since $\mathbf{P}_{\mathrm{DUST}}$ and $\mathbf{M}_{\mathrm{DUST}}$ are metrizable and Φ is a bijection by Theorem 3.13, we need only show that

$$\mathbf{t}^{(N)} \to \mathbf{t} \text{ in } \mathcal{P} \text{ pointwise} \qquad \Leftrightarrow \qquad \mathcal{S}_{\mathbf{t}^{(N)}} + \mathcal{N}(0,\sigma^{(N)}) \Rightarrow \mathcal{S}_{\mathbf{t}} + \mathcal{N}(0,\sigma).$$

The forwards direction follows from Lemma 3.18 and the Method of Moments/Cumulants exactly as in the proof of Lemma 3.23. The backwards direction follows from Lemma 3.27 and Lemma 3.17. ☐

Corollary 3.28. *The metric space of DUSTPAN distributions* $\mathbf{M}_{\mathrm{DUST}}$ *is compact, hence it is closed and bounded in the space of distributions under the Lévy metric.*

Proof. $\mathbf{P}_{\mathrm{DUST}}$ is a compact subset of $\widetilde{\ell}_2$ under pointwise convergence, so $\mathbf{M}_{\mathrm{DUST}}$ is compact under the Lévy metric as well by Theorem 1.15. ☐

Corollary 3.29. *The closure of the metric space* $\{\mathcal{S}_{\mathbf{t}} : \mathbf{t} \in \widetilde{\ell}_2, |\mathbf{t}|_2^2 = 12, \mathbf{t} \text{ is finite}\}$ *in the Lévy metric is* $\mathbf{M}_{\mathrm{DUST}}$.

Proof. Since $\{\mathcal{S}_{\mathbf{t}} : \mathbf{t} \in \widetilde{\ell}_2, |\mathbf{t}|_2^2 = 12, \mathbf{t} \text{ is finite}\} \subset \mathbf{M}_{\mathrm{DUST}}$ by definition and $\mathbf{M}_{\mathrm{DUST}}$ is closed by Corollary 3.28, we know

$$\overline{\{\mathcal{S}_{\mathbf{t}} : \mathbf{t} \in \widetilde{\ell}_2, |\mathbf{t}|_2^2 = 12, \mathbf{t} \text{ is finite}\}} \subset \overline{\mathbf{M}_{\mathrm{DUST}}} = \mathbf{M}_{\mathrm{DUST}}.$$

For the other inclusion, we just need to show each $\mathbf{t} \in \widetilde{\ell}_2$ with $|\mathbf{t}|_2^2 \leqslant 12$ is the pointwise limit of a sequence $\mathbf{t}^{(N)} \in \widetilde{\ell}_2$ with $|\mathbf{t}^{(N)}|_2^2 = 12$ and $\mathbf{t}$ finite by Theorem 1.15. As noted above, this follows from Lemma 3.21. ☐

3.4. The metric space of distance distributions

For convenience, we recall some of the definitions and notation from the introduction. For each $\mathbf{t} \in \widetilde{\ell}_2$ with $|\mathbf{t}|_2 > 0$, let

$$\widehat{\mathbf{t}} := \frac{\sqrt{12}\cdot\mathbf{t}}{|\mathbf{t}|_2}$$

be the rescaled sequence such that $|\widehat{\mathbf{t}}|_2^2 = 12$ and $\mathcal{S}_{\widehat{\mathbf{t}}} = \mathcal{S}_{\widehat{\mathbf{t}}}^*$. By definition of the hat-operation, $\widehat{\mathbf{t}} \in \mathbf{P}_{\mathrm{DUST}}$ and $\Phi(\widehat{\mathbf{t}}) = \mathcal{S}_{\widehat{\mathbf{t}}} + \mathcal{N}(0,0) = \mathcal{S}_{\widehat{\mathbf{t}}} = \mathcal{S}_{\widehat{\mathbf{t}}}^*$. The *distance multiset* of $\mathbf{t} = \{t_1 \geqslant t_2 \geqslant \cdots \geqslant t_m\}$ is the multiset

$$\Delta\mathbf{t} := \{t_i - t_j : 1 \leqslant i < j \leqslant m\},$$

and the *metric space of distance distributions* is

$$\mathbf{M}_{\mathrm{DIST}} = \left\{\mathcal{S}_{\widehat{\Delta\mathbf{t}}} : \mathbf{t} = \{1 = t_1 \geqslant \cdots \geqslant t_m = 0\}\right\}. \tag{3.10}$$

Thus, the *parameter space of distance multisets*, mentioned in Section 1, is defined as

$$\mathbf{P}_{\mathrm{DIST}} := \left\{ \widehat{\Delta \mathbf{t}} : \mathbf{t} = \{1 = t_1 \geqslant \cdots \geqslant t_m = 0\} \right\} \tag{3.11}$$

By padding with 0's, consider $\mathbf{P}_{\mathrm{DIST}} \subset \mathbf{P}_{\mathrm{DUST}} \subset \widetilde{\ell}_2$ as a sequence space with the topology of pointwise convergence.

Lemma 3.30. *The closure of* $\mathbf{P}_{\mathrm{DIST}}$ *is* $\mathbf{P}_{\mathrm{DIST}} \sqcup \{\mathbf{0}\}$.

Proof. Let $\mathbf{d}^{(N)} \in \mathbf{P}_{\mathrm{DIST}}$ be a sequence converging pointwise to $\mathbf{d}$. By Theorem 1.15, we can assume $\mathbf{d} \in \mathbf{P}_{\mathrm{DUST}}$. By definition, each $\mathbf{d}^{(N)} = \widehat{\Delta \mathbf{t}^{(N)}}$ for some finite sequence of real numbers $\mathbf{t}^{(N)} = \{1 = t_1^{(N)} \geqslant \cdots \geqslant t_{m^{(N)}}^{(N)} = 0\}$.

Suppose $\limsup_{N\to\infty} m^{(N)} < \infty$. We may pass to a subsequence for which $m^{(N)} = m$ is constant. We may pass to a further subsequence for which $\mathbf{t}^{(N)} \in [0,1]^m$ converges pointwise to some $\mathbf{t} = \{1 = t_1 \geqslant \cdots \geqslant t_m = 0\} \in [0,1]^m$ and where $|\mathbf{t}^{(N)}|_2$ converges. Clearly the distance multiset operator $\Delta \colon [0,1]^m \to [0,1]^{\binom{m}{2}}$ is continuous, so $\Delta \mathbf{t}^{(N)} \to \Delta \mathbf{t}$, and moreover $\mathbf{d}^{(N)} = \widehat{\Delta \mathbf{t}^{(N)}} \to \widehat{\Delta \mathbf{t}}$, so $\widehat{\Delta \mathbf{t}} = \mathbf{d}$ which implies $\mathbf{d} \in \mathbf{P}_{\mathrm{DIST}}$.

Now suppose $\limsup_{N\to\infty} m^{(N)} = \infty$. Again, we may pass to a subsequence if necessary so we may assume $m^{(N)} \to \infty$. Since $t_1^{(N)} = 1$ and $t_{m^{(N)}}^{(N)} = 0$ for each N, we have

$$\begin{aligned} |\Delta \mathbf{t}^{(N)}|_2^2 &= \sum_{1 \leqslant i < j \leqslant m^{(N)}} (t_i^{(N)} - t_j^{(N)})^2 \\ &\geqslant \sum_{1 < \ell < m^{(N)}} \left[(1 - t_\ell^{(N)})^2 + (t_\ell^{(N)} - 0)^2 \right] \\ &\geqslant \sum_{1 < \ell < m^{(N)}} \frac{1}{2} = \frac{m^{(N)}}{2} - 1. \end{aligned}$$

Therefore, $\lim_{N\to\infty} \frac{\sqrt{12}}{|\Delta \mathbf{t}^{(N)}|_2} \to 0$, so pointwise $\widehat{\Delta \mathbf{t}^{(N)}} \to \mathbf{0}$. ☐

Corollary 3.31. *Any pointwise convergent sequence* $\widehat{\Delta \mathbf{t}^{(N)}}$ *with* $\mathbf{t}^{(N)} = \{1 = t_1^{(N)} \geqslant \cdots \geqslant t_{m^{(N)}}^{(N)} = 0\}$ *converges to* $\mathbf{0}$ *if and only if* $m^{(N)} \to \infty$.

Theorem 3.32. *The map* $\Phi_{\mathrm{DIST}} \colon \overline{\mathbf{P}_{\mathrm{DIST}}} \to \overline{\mathbf{M}_{\mathrm{DIST}}} = \mathbf{M}_{\mathrm{DIST}} \sqcup \{\mathcal{N}(0,1)\}$ *given by* $\mathbf{d} \mapsto \mathcal{S}_{\mathbf{d}}$ *and* $\mathbf{0} \mapsto \mathcal{N}(0,1)$ *is a homeomorphism between (sequentially) compact spaces.*

Proof. First note that $\mathbf{P}_{\mathrm{DIST}} \subset \mathbf{P}_{\mathrm{DUST}}$ and $\mathbf{M}_{\mathrm{DIST}} \subset \mathbf{M}_{\mathrm{DUST}}$ by construction, so Φ_{DIST} is the restriction of $\Phi : \mathbf{P}_{\mathrm{DUST}} \longrightarrow \mathbf{M}_{\mathrm{DUST}}$. Therefore, by Theorem 1.15, Φ_{DIST} is a homeomorphism. Since closed subsets of compact spaces are compact, we know $\overline{\mathbf{P}_{\mathrm{DIST}}}$ is compact by Lemma 3.30. Furthermore, $\Phi(\overline{\mathbf{P}_{\mathrm{DIST}}}) = \mathbf{M}_{\mathrm{DIST}} \sqcup \{\mathcal{N}(0,1)\}$ is closed and compact. ☐

4. Metric spaces related to $\mathrm{SSYT}_{\leqslant m}(\lambda)$ distributions

We next consider the family of generating functions for semistandard tableaux given by the principal specialization of Schur polynomials, or equivalently the rank statistic on $\mathrm{SSYT}_{\leqslant m}(\lambda)$, as described in Section 1.2 and Section 2.2. An interesting special case is given by MacMahon's formula for the size statistic on the set $\mathrm{PP}(a \times b \times c)$ of plane partitions inside an $(a \times b \times c)$ box, given in (2.14). In particular, we will prove Theorem 1.8 and Theorem 1.11. We provide a wide variety of limit law classification results for these statistics in various regimes. The subsections are divided into four natural special cases: $n/m \to 0$, $n/m \to \infty$, cases based on the number of distinct parts of λ, and plane partitions. See Summary 4.20 for a summary.

4.1. Limit laws with $|\lambda|/m \to 0$ and uniform sums

We begin classifying the limit laws for semistandard Young tableaux. Throughout this section, we tacitly assume $\ell(\lambda) \leqslant m$, so $\mathrm{SSYT}_{\leqslant m}(\lambda) \neq \varnothing$. Furthermore, if $\lambda_m > 0$, the first λ_m columns of $T \in \mathrm{SSYT}_{\leqslant m}(\lambda)$ are forced to each be $1, 2, \ldots, m$. Hence, up to a q shift, $\mathrm{SSYT}(\lambda)^{\mathrm{rank}}(q)$ equals $\mathrm{SSYT}(\mu)^{\mathrm{rank}}(q)$ where $\mu_i = \lambda_i - \lambda_m$. In order to classify limit laws for $\mathrm{SSYT}_{\leqslant m}(\lambda)^{\mathrm{rank}}(q)$, it thus suffices to assume throughout that $\ell(\lambda) < m$ and $\lambda_m = 0$.

We begin with a simple analogue of Theorem 1.2. This will be our only use of the hook-content-based cumulant formula; all of our other results rely on the q-Weyl dimension-based cumulant formula.

Theorem 4.1. *Let λ denote an infinite sequence of partitions with $|\lambda| = n$. If $\frac{n}{m} \to 0$, then for each fixed $d \in \mathbb{Z}_{\geqslant 2}$, the corresponding sequence of cumulants is*

$$\kappa_d^{\lambda;m} \sim \frac{B_d}{d} n m^d. \tag{4.1}$$

Furthermore, we can characterize convergence in distribution in the case $\frac{n}{m} \to 0$ depending on the limiting value of n.

(i) If n converges to a finite value N, then $\mathcal{X}_\lambda[\mathrm{rank}]^*$ converges in distribution to $\mathcal{IH}_N^*$.

(ii) If $n \to \infty$, then $\mathcal{X}_\lambda[\mathrm{rank}]$ is asymptotically normal.

Proof. For each cell $u \in \lambda$, the trivial bounds $0 \leqslant c_u \leqslant n$ and $1 \leqslant h_u \leqslant n$ give

$$m^d - n^d \leqslant (m + c_u)^d - h_u^d \leqslant (m+n)^d.$$

Summing over all $u \in \lambda$ and dividing through by nm^d gives

$$1 - \left(\frac{n}{m}\right)^d \leqslant \frac{\sum_{u \in \lambda} (m + c_u)^d - h_u^d}{nm^d} \leqslant \left(1 + \frac{n}{m}\right)^d.$$

When $n/m \to 0$, the lower and upper bounds each tend to 1. The cumulant formula (4.1) now follows from (2.12).

By (4.1), $(\kappa_d^{\lambda;m})^* \sim (B_d/d)/(B_2/2)^{d/2} \cdot n^{1-d/2}$, which eliminates m from the limits. If $n \to N$, then $(B_d/d)/(B_2/2)^{d/2} \cdot n^{1-d/2}$ approaches the d^{th} cumulant of $\mathcal{IH}_N^*$ by (2.2) and Example 2.3, proving (i). If $n \to \infty$, then $(B_d/d)/(B_2/2)^{d/2} \cdot n^{1-d/2}$ tends to 0 for $d \geqslant 3$, which are the cumulants of $\mathcal{N}(0,1)$, hence (ii) follows from Corollary 2.5. □

Example 4.2. Consider a constant sequence of partitions $\lambda^{(N)} = \lambda$ and let $m \to \infty$. By Theorem 4.1(i), $\mathcal{X}_{\lambda;m}[\text{rank}]^* \Rightarrow \mathcal{IH}^*_{|\lambda|}$, which depends only on $|\lambda|$. On the other hand, if the sequence $\lambda^{(N)}$ is chosen such that $|\lambda^{(N)}| \to \infty$ and $m^{(N)} \sim |\lambda^{(N)}|^2$, the limit is $\mathcal{N}(0,1)$ by Theorem 4.1(ii).

Corollary 4.3. *For any fixed $\epsilon > 0$, let*

$$\mathbf{M}_\epsilon \coloneqq \{\mathcal{X}_{\lambda;m}[\text{rank}]^* : |\lambda| < m^{1-\epsilon}\} \subset \mathbf{M}_{\text{SSYT}}.$$

In the Lévy metric,

$$\overline{\mathbf{M}_\epsilon} = \mathbf{M}_\epsilon \sqcup \overline{\mathbf{M}_{\mathcal{IH}}}, \tag{4.2}$$

which is (sequentially) compact. Moreover, the set of limit points of $\mathbf{M}_\epsilon$ is $\overline{\mathbf{M}_{\mathcal{IH}}}$.

Proof. Given a sequence in $\mathbf{M}_\epsilon$, if m is bounded, then so is $n = |\lambda|$, so there are only finitely many distinct (λ, m) in the sequence and convergence occurs if and only if the sequence is eventually constant. On the other hand, if $m \to \infty$, then $n < m^{1-\epsilon}$ yields $\frac{n}{m} < m^{-\epsilon} \to 0$, so (4.2) follows immediately from Theorem 4.1.

By these observations, every infinite sequence of distinct points in $\mathbf{M}_\epsilon$ has a limit point in $\overline{\mathbf{M}_{\mathcal{IH}}}$, so $\mathbf{M}_\epsilon$ consists entirely of isolated points and $\overline{\mathbf{M}_{\mathcal{IH}}}$ consists entirely of limit points. Sequential compactness is similarly clear. □

4.2. Limit laws with $|\lambda|/m \to \infty$ and distance distributions

At the other extreme, we may consider the case when $|\lambda|/m \to \infty$. As we will see, the possible behavior is vastly more varied in this limit. Among the sequences of partitions λ with $|\lambda|/m \to \infty$, the easiest case to consider is when $\lambda_1/m^3 \to \infty$. This includes the case where m converges to a fixed finite value and $|\lambda| \to \infty$.

For a partition $\lambda = (\lambda_1, \ldots, \lambda_m)$, recall from Theorem 1.8 that

$$\mathbf{t}(\lambda) = (t_1, \ldots, t_m) \in [0,1]^m$$

is the finite multiset with $t_j \coloneqq \frac{\lambda_j}{\lambda_1}$ for $1 \leqslant j \leqslant m$. By Definition 1.7, the corresponding distance multiset is

$$\Delta\mathbf{t}(\lambda) \coloneqq \{t_i - t_j : 1 \leqslant i < j \leqslant m\}.$$

Lemma 4.4. *Let λ denote an infinite sequence of partitions with $\ell(\lambda) < m$ and $|\lambda| = n$. If $\frac{n}{m} \to \infty$ in such a way that $\lambda_1/m^3 \to \infty$, then for each fixed $d \in \mathbb{Z}_{\geqslant 2}$,*

$$\frac{\kappa_d^{\lambda;m}}{\lambda_1^d} \sim (B_d/d)|\Delta\mathbf{t}(\lambda)|_d^d,$$

which is the d^{th} cumulant of the rescaled uniform sum $\mathcal{S}_{\Delta\mathbf{t}(\lambda)}/\lambda_1$.

Proof. Note that

$$(\lambda_i - \lambda_j)^d - m^d \leqslant (\lambda_i - \lambda_j + j - i)^d - (j-i)^d \leqslant (\lambda_i - \lambda_j + m)^d.$$

Divide through by λ_1^d and consider the upper bound. Setting $t_j \coloneqq \frac{\lambda_j}{\lambda_1}$ for $1 \leqslant j \leqslant m$, we find

$$\begin{aligned}\frac{(\lambda_i - \lambda_j + m)^d}{\lambda_1^d} &= (t_i - t_j + m/\lambda_1)^d \\ &= \sum_{k=0}^{d} \binom{d}{k} (t_i - t_j)^{d-k} \left(\frac{m}{\lambda_1}\right)^k \\ &\leqslant (t_i - t_j)^d + \sum_{k=1}^{d} \binom{d}{k} \left(\frac{m}{\lambda_1}\right)^k.\end{aligned}$$

Summing over all $1 \leqslant i < j \leqslant m$ and considering both bounds gives

$$\begin{aligned}\sum_{1 \leqslant i<j \leqslant m} (t_i - t_j)^d - \binom{m}{2} \cdot \left(\frac{m}{\lambda_1}\right)^d &\leqslant \frac{\sum_{1\leqslant i<j\leqslant m} (\lambda_i - \lambda_j + j - i)^d - (j-i)^d}{\lambda_1^d} \\ &\leqslant \sum_{1 \leqslant i<j \leqslant m} (t_i - t_j)^d + \binom{m}{2} \sum_{k=1}^{d} \binom{d}{k} \left(\frac{m}{\lambda_1}\right)^k.\end{aligned}$$

Since $\lambda_1/m^3 \to \infty$, we have $\binom{m}{2}(m/\lambda_1) \to 0$. Furthermore, for each $k \geqslant 1$,

$$\binom{m}{2} (m/\lambda_1)^k \leqslant m^{2+k}/\lambda_1^k \leqslant m^{3k}/\lambda_1^k \to 0.$$

It follows that

$$\frac{\sum_{1\leqslant i<j\leqslant m} (\lambda_i - \lambda_j + j - i)^d - (j-i)}{\lambda_1^d} \sim \sum_{1 \leqslant i<j \leqslant m} (t_i - t_j)^d.$$

The result now follows from (2.11) and (3.3). □

We use the results on generalized uniform sum distributions from Section 3 to characterize convergence in distribution in the next theorem. It is a more explicit statement of Theorem 1.8.

Theorem 4.5. *Let λ denote an infinite sequence of partitions, with $\ell(\lambda) < m$ and $|\lambda| = n$. If $\frac{n}{m} \to \infty$ in such a way that $\lambda_1/m^3 \to \infty$, then for each fixed $d \in \mathbb{Z}_{\geqslant 2}$, the standardized cumulants are approximately*

$$(\kappa_d^{\lambda;m})^* \sim \frac{B_d}{d} |\widehat{\Delta \mathbf{t}(\lambda)}|_d^d = \kappa_d^{\mathcal{S}_{\widehat{\Delta \mathbf{t}(\lambda)}}}. \tag{4.3}$$

Furthermore, we can characterize convergence in distribution when it occurs.

(i) If m is bounded, then $\mathcal{X}_{\lambda;m}[\mathrm{rank}]^*$ converges in distribution if and only if the multisets $\widehat{\Delta \mathbf{t}(\lambda)}$ converge pointwise to some multiset $\mathbf{d}$, in which case, the limiting distribution is $\mathcal{S}_{\mathbf{d}}$ and $\mathbf{d} \in \mathbf{P}_{\mathrm{DIST}}$,

(ii) The sequence $m \to \infty$ if and only if $\mathcal{X}_{\lambda;m}[\mathrm{rank}]^*$ is asymptotically normal.

Proof. By hypothesis, $\lambda_1/m^3 \to \infty$, so Lemma 4.4 implies $\kappa_d^{\lambda;m} \sim (B_d/d)|\Delta\mathbf{t}(\lambda)|_d^d$ for all $d \geqslant 2$. Thus, the standardized cumulants are given by

$$(\kappa_d^{\lambda;m})^* \sim \frac{(B_d/d)|\Delta\mathbf{t}(\lambda)|_d^d}{((B_2/2)|\Delta\mathbf{t}(\lambda)|_2^2)^{d/2}} = \frac{B_d/d}{(B_2/2)^{d/2}}\left(\frac{|\Delta\mathbf{t}(\lambda)|_d}{|\Delta\mathbf{t}(\lambda)|_2}\right)^d = \frac{B_d}{d}|\widehat{\Delta\mathbf{t}(\lambda)}|_d^d$$

by the definition of the hat-operation (1.8). By (3.3), $\frac{B_d}{d}|\widehat{\Delta\mathbf{t}(\lambda)}|_d^d$ is the d^{th} cumulant for the uniform sum random variable $\mathcal{S}_{\widehat{\Delta\mathbf{t}(\lambda)}}$.

By the Method of Moments/Cumulants (Theorem 2.4) together with its converse in this context (Lemma 2.7), the sequence $\mathcal{X}_{\lambda;m}[\mathrm{rank}]^*$ converges in distribution to some $\mathcal{X}$ if and only if the limit of the standardized cumulants $(\kappa_d^{\lambda;m})^* \to \kappa_d^{\mathcal{X}} < \infty$ for each $d \geqslant 1$, which happens if and only if $\kappa_d^{\mathcal{S}_{\widehat{\Delta\mathbf{t}(\lambda)}}} \to \kappa_d^{\mathcal{X}}$ for each $d \geqslant 1$. By the Method of Moments/Cumulants and its converse for DUSTPAN distributions (Lemma 3.27), this occurs if and only if $\mathcal{S}_{\widehat{\Delta\mathbf{t}(\lambda)}} \Rightarrow \mathcal{X}$. Finally, by Theorem 3.32, this occurs if and only if $\widehat{\Delta\mathbf{t}(\lambda)}$ converges pointwise to some $\mathbf{d} \in \overline{\mathbf{P}_{\mathrm{DIST}}}$. The result follows from Corollary 3.31. In particular, if m is bounded (i) holds, and if $m \to \infty$ (ii) holds. □

Example 4.6. Fix a partition λ and a positive integer $m > \ell(\lambda)$. Pick a sequence $r^{(N)} \to \infty$ of row scale factors, so that $\lambda_i^{(N)} = r^{(N)}\lambda_i$ and $m^{(N)} = m$. Clearly $\lambda_1^{(N)}/(m^{(N)})^3 \to \infty$, so by Theorem 4.5(i), we have $\mathcal{X}_{r^{(N)}\lambda;m}[\mathrm{rank}]^* \Rightarrow \mathcal{S}_{\Delta\lambda}^*$.

Example 4.7. Consider the sequence of partitions with $\lambda^{(N)} = (2^{N-1}, 2^{N-2}, \ldots, 1)$ and $m^{(N)} = N$. Strictly speaking, $\ell(\lambda^{(N)}) = N = m^{(N)}$ here, so recall we can delete the first column and consider the auxiliary sequence $\mu^{(N)} = (2^{N-1}-1, 2^{N-2}-1, \ldots, 0)$. Now

$$\frac{\mu_1^{(N)}}{(m^{(N)})^3} = \frac{2^{N-1}-1}{N^3} \to \infty$$

and $m^{(N)} = N \to \infty$. Thus the sequence $\mathcal{X}_{(2^{N-1},2^{N-2},\ldots,1);N}[\mathrm{rank}]$ is asymptotically normal by Theorem 4.5(ii).

4.3. Limit laws based on distinct values in λ and the weft statistic

We now describe a very general test for asymptotic normality of $\mathcal{X}_{\lambda;m}[\mathrm{rank}]$ based on a new statistic we call weft in analogy with aft for standard Young tableaux. This test depends on the number of distinct values in a partition, so we switch to exponential notation. Note, throughout the rest of this section k will denote the number of distinct values in λ.

Definition 4.8. We may write a nonempty partition in *exponential notation* $\lambda = \ell_1^{e_1} \cdots \ell_k^{e_k}$ where $\ell_1 > \cdots > \ell_k \geqslant 0$ and $e_i > 0$, meaning λ has e_i rows of length ℓ_i. In our earlier notation, $m = e_1 + \cdots + e_k$ and $n = e_1\ell_1 + \cdots + e_k\ell_k$.

Lemma 4.9. *Take a partition $\lambda = (\lambda_1, \ldots, \lambda_m) = \ell_1^{e_1} \cdots \ell_k^{e_k}$. Then, uniformly for all $d \geqslant 2$,*

$$\sum_{1 \leqslant i < j \leqslant m} (\lambda_i - \lambda_j)(\lambda_i - \lambda_j + j - i)^{d-1} = \Theta\left(\sum_{1 \leqslant a < b \leqslant k} (\ell_a - \ell_b) e_a e_b (\ell_a - \ell_b - 1 + e_a + \cdots + e_b)^{d-1}\right). \tag{4.4}$$

Proof. Observe that we may restrict the sum in (4.4) to just the indices with $\lambda_i \neq \lambda_j$. Hence, we group the terms according to the distinct values $\lambda_i = \ell_a$ and $\lambda_j = \ell_b$ for $1 \leqslant a < b \leqslant k$. The contribution to the sum in (4.4) for all $\lambda_i = \ell_a$ and $\lambda_j = \ell_b$ for a fixed $a < b$ is

$$\sum (\ell_a - \ell_b)(\ell_a - \ell_b + j - i)^{d-1} \tag{4.5}$$

where the sum is over i, j such that $e_1 + \cdots + e_{a-1} + 1 \leqslant i \leqslant e_1 + \cdots + e_a$ and $e_1 + \cdots + e_{b-1} + 1 \leqslant j \leqslant e_1 + \cdots + e_b$. Reindexing with $p = e_a - (i - e_1 - \cdots - e_{a-1}) + 1$ and $q = j - e_1 - \cdots - e_{b-1}$, the sum in (4.5) becomes

$$(\ell_a - \ell_b) \sum_{\substack{1 \leqslant p \leqslant e_a \\ 1 \leqslant q \leqslant e_b}} (\ell_a - \ell_b + p + q - 1 + e_{a+1} + \cdots + e_{b-1})^{d-1}. \tag{4.6}$$

Next, note that for fixed $d \geqslant 2$, $(u+v+w)^d = \Theta(u^d + v^d + w^d)$ uniformly for all $u, v, w \geqslant 0$, since then

$$\begin{aligned} u^d + v^d + w^d &\leqslant (u + v + w)^d \\ &\leqslant (3 \max\{u, v, w\})^d = 3^d \max\{u^d, v^d, w^d\} \\ &\leqslant 3^d (u^d + v^d + w^d). \end{aligned}$$

Letting $u = p$, $v = \ell_a - \ell_b + e_{a+1} + \cdots + e_{b-1} - 1$, and $w = q$, we see the sum in (4.5) and (4.6) is Θ of

$$\begin{aligned} &(\ell_a - \ell_b) \sum_{\substack{1 \leqslant p \leqslant e_a \\ 1 \leqslant q \leqslant e_b}} \left[(\ell_a - \ell_b + e_{a+1} + \cdots + e_{b-1} - 1)^{d-1} + p^{d-1} + q^{d-1}\right] \\ &= (\ell_a - \ell_b) \left[e_a e_b (\ell_a - \ell_b + e_{a+1} + \cdots + e_{b-1} - 1)^{d-1} + e_b \sum_{1 \leqslant p \leqslant e_a} p^{d-1} + e_a \sum_{1 \leqslant q \leqslant e_b} q^{d-1}\right]. \end{aligned}$$

Since $d \geqslant 2$, $\sum_{1 \leqslant p \leqslant e_a} p^{d-1} = \Theta(e_a^d)$ uniformly for all $u \in \mathbb{Z}_{\geqslant 1}$ by the bounds in Lemma 2.23, and similarly $\sum_{1 \leqslant p \leqslant e_b} q^{d-1} = \Theta(e_b^d)$. Consequently, the preceding sum and also the sum in (4.5) are Θ of

$$\begin{aligned} &(\ell_a - \ell_b) \left[e_a e_b (\ell_a - \ell_b + e_{a+1} + \cdots + e_{b-1} - 1)^{d-1} + e_b e_a^d + e_a e_b^d\right] \\ &\quad = (\ell_a - \ell_b) e_a e_b \left[(\ell_a - \ell_b + e_{a+1} + \cdots + e_{b-1} - 1)^{d-1} + e_a^{d-1} + e_b^{d-1}\right] \\ &\quad = \Theta\left((\ell_a - \ell_b) e_a e_b (\ell_a - \ell_b - 1 + e_a + \cdots + e_b)^{d-1}\right). \end{aligned}$$

The result follows by summing over all $1 \leqslant a < b \leqslant k$, since the preceding bounds were all uniform. □

Theorem 4.10. *Let $\lambda = \ell_1^{e_1} \cdots \ell_k^{e_k}$ denote an infinite sequence of partitions with $\ell(\lambda) \leqslant m$, $\ell_1 > \ell_2 > \cdots > \ell_k \geqslant 0$ and each $e_i > 0$. Then, for $d \geqslant 2$ even,*

$$\kappa_d^{\lambda;m} = \Theta\left(\sum_{1\leqslant a<b\leqslant k} (\ell_a - \ell_b)e_a e_b(\ell_a - \ell_b - 1 + e_a + \cdots + e_b)^{d-1}\right). \tag{4.7}$$

Furthermore, $\mathcal{X}_{\lambda;m}[\mathrm{rank}]$ is asymptotically normal if

$$\mathrm{weft}(\lambda) := \frac{\sum_{1\leqslant a<b\leqslant k}(\ell_a - \ell_b)e_a e_b(\ell_a - \ell_b - 1 + e_a + \cdots + e_b)}{(\ell_1 - \ell_k - 1 + m)^2} \to \infty. \tag{4.8}$$

Proof. In general, $u^d - v^d = (u-v)\sum_{i=0}^{d-1} u^i v^{d-i-1} = (u-v)\mathbf{h}_{d-1}(u,v)$, so (2.11) gives

$$\kappa_d^{\lambda;m} = \sum_{1\leqslant i<j\leqslant m} (\lambda_i - \lambda_j)\mathbf{h}_{d-1}(\lambda_i - \lambda_j + j - i, j - i).$$

For fixed $d \geqslant 2$ even and $u \geqslant v \geqslant 0$, we have $\mathbf{h}_{d-1}(u,v) = \Theta(u^{d-1})$ since

$$u^{d-1} \leqslant u^{d-1} + u^{d-2}v + \cdots + v^{d-1} \leqslant u^{d-1} + u^{d-1} + \cdots + u^{d-1} = du^{d-1}.$$

Consequently,

$$\kappa_d^{\lambda;m} = \Theta\left(\sum_{1\leqslant i<j\leqslant m} (\lambda_i - \lambda_j)(\lambda_i - \lambda_j + j - i)^{d-1}\right).$$

Hence, (4.7) holds by Lemma 4.9.

We use the cumulant formula in (4.7) to prove the asymptotic normality result. Write $x_{ab} := (\ell_a - \ell_b)e_a e_b$ and $y_{ab} := \ell_a - \ell_b - 1 + e_a + \cdots + e_b$ for $1 \leqslant a < b \leqslant k$. By (4.7), we have for all $d \geqslant 2$ even

$$(\kappa_d^{\lambda;m})^* = \Theta\left(\frac{\sum_{1\leqslant a<b\leqslant k} x_{ab} y_{ab}^{d-1}}{\left(\sum_{1\leqslant a<b\leqslant k} x_{ab} y_{ab}\right)^{d/2}}\right).$$

Note that $y_{1k} \geqslant y_{ab}$, so $\hat{y}_{ab} := y_{ab}/y_{1k} \leqslant 1$. Hence $\hat{y}_{ab}^{d-1} \leqslant \hat{y}_{ab}$ and

$$\sum_{1\leqslant a<b\leqslant k} x_{ab}\hat{y}_{ab}^{d-1} \leqslant \sum_{1\leqslant a<b\leqslant k} x_{ab}\hat{y}_{ab}.$$

Consequently,

$$\begin{aligned}
\frac{\sum_{1\leqslant a<b\leqslant k} x_{ab} y_{ab}^{d-1}}{\left(\sum_{1\leqslant a<b\leqslant k} x_{ab} y_{ab}\right)^{d/2}} &= \frac{y_{1k}^{d-1}}{y_{1k}^{d/2}} \frac{\sum_{1\leqslant a<b\leqslant k} x_{ab}\hat{y}_{ab}^{d-1}}{\left(\sum_{1\leqslant a<b\leqslant k} x_{ab}\hat{y}_{ab}\right)^{d/2}} \\
&\leqslant \frac{1}{y_{1k}^{1-d/2}}\left(\sum_{1\leqslant a<b\leqslant k} x_{ab}\hat{y}_{ab}\right)^{1-d/2} \\
&= \left(\sum_{1\leqslant a<b\leqslant k} x_{ab} y_{ab}/y_{1k}^2\right)^{1-d/2}.
\end{aligned}$$

The latter parenthesized quantity equals the $\mathrm{weft}(\lambda)$ statistic in (4.8) by construction. Thus,

$$(\kappa_d^{\lambda;m})^* = O\left(\frac{1}{\mathrm{weft}(\lambda)^{d/2-1}}\right). \tag{4.9}$$

When $d \geqslant 4$, $\mathrm{weft}(\lambda) \to \infty$ implies $(\kappa_d^{\lambda;m})^* \to 0$. Thus, asymptotic normality follows from Corollary 2.5 since all of the odd Bernoulli numbers for $d \geqslant 3$ are zero. □

Example 4.11. Let $\lambda^{(N)} = \delta_N := (N-1, N-2, \ldots, 2, 1, 0)$ be the staircase partition for $N > 1$. We have $e_1 = \cdots = e_N = 1$ and $\ell_1 = N-1, \ldots, \ell_N = 0$. In this case, (4.8) simplifies to

$$\mathrm{weft}(\lambda^{(N)}) = \frac{\sum_{1 \leqslant a < b \leqslant N} 2(b-a)^2}{(2N-2)^2} = N^2 \frac{N+1}{24(N-1)}.$$

Thus, as $N \to \infty$ this statistic goes to infinity, so $\mathcal{X}_{\delta_N;N}[\mathrm{rank}]$ is asymptotically normal by Theorem 4.10.

The characterization in Theorem 4.10 is powerful enough to prove asymptotic normality in many cases of interest. We will use the criteria in the next corollary to further simplify the arguments in the examples below and the applications to plane partitions. As mentioned in the introduction to this section, we can assume $\ell(\lambda) < m$ without loss of generality. We may include the case $\ell(\lambda) = m$ if desired by replacing ℓ_1 with $\ell_1 - \ell_k$ in the following result.

Corollary 4.12. *Let $\lambda = \ell_1^{e_1} \cdots \ell_k^{e_k}$ denote an infinite sequence of partitions with $\ell(\lambda) < m$, so $\lambda_1 = \ell_1 > \ell_2 > \cdots > \ell_k = 0$, and each $e_i > 0$. Then $\mathcal{X}_{\lambda;m}[\mathrm{rank}]$ is asymptotically normal in the following situations.*

(i) $\frac{m^2}{k\ell_1(k+\ell_1)} \to 0$ *and* $k \to \infty$.

(ii) $\frac{e^{[2]}}{k(\ell_1/m+1)^2} \to \infty$, *where $e^{[2]}$ denotes the second largest element among $e_1, \ldots, e_k$.*

(iii) $\frac{\ell_1 e_1 e_k}{\ell_1 + m} \to \infty$.

Proof. For (i), suppose $\frac{m^2}{k\ell_1(k+\ell_1)} \to 0$ and $k \to \infty$. We have

$$\begin{aligned}
&\sum_{1 \leqslant a < b \leqslant k} (\ell_a - \ell_b) e_a e_b (\ell_a - \ell_b - 1 + e_a + \cdots + e_b) \\
&\qquad \geqslant \sum_{1 < p < k} [(\ell_1 - \ell_p) e_1 e_p (\ell_1 - \ell_p - 1 + e_1 + \cdots + e_p) + \ell_p e_p e_k (\ell_p - 1 + e_p + \cdots + e_k)] \\
&\qquad \geqslant \sum_{1 < p < k} [(\ell_1 - \ell_p)(\ell_1 - \ell_p + p - 1) + \ell_p(\ell_p + k - p)] \\
&\qquad \geqslant \sum_{\frac{k}{4}+1 < p < \frac{3k}{4}} [(\ell_1 - \ell_p)(\ell_1 - \ell_p + k/4) + \ell_p(\ell_p + k/4)] \\
&\qquad = \sum_{\frac{k}{4}+1 < p < \frac{3k}{4}} [\ell_1^2 - 2\ell_1\ell_p + 2\ell_p^2 + k\ell_1/4].
\end{aligned}$$

Set $x_p := \ell_p/\ell_1$ and divide the preceding inequality by ℓ_1^2. Suppose $k \geqslant 4$. The final expression becomes

$$\begin{aligned}\sum_{\frac{k}{4}+1<p<\frac{3k}{4}} [1 - 2x_p + 2x_p^2 + k/(4\ell_1)] &\geqslant \sum_{\frac{k}{4}+1<p<\frac{3k}{4}} [1/2 + k/(4\ell_1)] \\ &\geqslant (k/2-2)(1/2 + k/(4\ell_1)) \\ &\geqslant k/16 \cdot (1 + k/\ell_1).\end{aligned}$$

Consequently,

$$\begin{aligned}\mathrm{weft}(\lambda) &\geqslant \frac{k/16 \cdot (1 + k/\ell_1)}{(1 + (m-1)/\ell_1)^2} \\ &\geqslant \frac{1}{16} \frac{k\ell_1(k+\ell_1)}{(\ell_1+m)^2} \\ &\geqslant \frac{1}{16} \min\left\{ \frac{k\ell_1(k+\ell_1)}{(2\ell_1)^2}, \frac{k\ell_1(k+\ell_1)}{(2m)^2} \right\} \\ &\geqslant \frac{1}{64} \min\left\{ k, \frac{k\ell_1(k+\ell_1)}{m^2} \right\} \to \infty,\end{aligned}$$

since $\frac{m^2}{k\ell_1(k+\ell_1)} \to 0$ and $k \to \infty$ by hypothesis. The result now follows from Theorem 4.10.

For (ii), suppose $\frac{e^{[2]}}{k(\ell_1/m+1)^2} \to \infty$. By definition, $e^{[2]} \leqslant m - e_i$ for all i. Thus,

$$\begin{aligned}&\sum_{1\leqslant a<b\leqslant k} (\ell_a - \ell_b)e_a e_b(\ell_a - \ell_b - 1 + e_a + \cdots + e_b) \\ &\qquad \geqslant \sum_{1\leqslant a<b\leqslant k} e_a^2 e_b + e_a e_b^2 \\ &\qquad = \sum_{1\leqslant i\leqslant k} e_i^2(e_{i+1} + \cdots + e_k) + \sum_{1\leqslant i\leqslant k} (e_1 + \cdots + e_{i-1})e_i^2 \\ &\qquad = \sum_{1\leqslant i\leqslant k} e_i^2(m - e_i) \\ &\qquad \geqslant e^{[2]} \sum_{1\leqslant i\leqslant k} e_i^2.\end{aligned}$$

If $f_1 \geqslant \cdots \geqslant f_k \geqslant 0$, then

$$\begin{aligned}(f_1 + \cdots + f_k)^2 &= f_1^2 + \cdots + f_k^2 + 2 \sum_{1 \leqslant i < j \leqslant k} f_i f_j \\ &\leqslant f_1^2 + \cdots + f_k^2 + 2 \sum_{1 \leqslant i < j \leqslant k} f_i^2 \\ &= \sum_{i=1}^{k} (1 + 2(k-i)) f_i^2 \\ &\leqslant 2k \sum_{i=1}^{k} f_i^2.\end{aligned}$$

This latter bound is independent of the actual order of the f_i. Consequently,

$$e^{[2]} \sum_{1 \leqslant i \leqslant k} e_i^2 \geqslant e^{[2]} \frac{m^2}{2k}.$$

Clearly $(\ell_1 - \ell_k - 1 + m)^2 \leqslant (\ell_1 + m)^2$. Hence

$$\begin{aligned}\operatorname{weft}(\lambda) &= \frac{\sum_{1 \leqslant a < b \leqslant k} (\ell_a - \ell_b) e_a e_b (\ell_a - \ell_b - 1 + e_a + \cdots + e_b)}{(\ell_1 - \ell_k - 1 + m)^2} \\ &\geqslant \frac{e^{[2]} m^2}{2k(\ell_1 + m)^2} = \frac{e^{[2]}}{2k(\ell_1/m + 1)^2} \to \infty,\end{aligned}$$

since $\frac{e^{[2]}}{k(\ell_1/m+1)^2} \to \infty$. The result again follows from Theorem 4.10.

For (iii), suppose $\frac{\ell_1 e_1 e_k}{\ell_1 + m} \to \infty$. We have

$$\begin{aligned}\operatorname{weft}(\lambda) &\geqslant \frac{(\ell_1 - \ell_k) e_1 e_k (\ell_1 - \ell_k - 1 + m)}{(\ell_1 - \ell_k - 1 + m)^2} \\ &\geqslant \frac{\ell_1 e_1 e_k}{\ell_1 + m} \to \infty.\end{aligned}$$

The result again follows from Theorem 4.10. □

Example 4.13. Suppose λ is a sequence of partitions with distinct parts and $\ell(\lambda) \to \infty$. Then $\ell_1 \geqslant k = m$ and $\frac{m^2}{k\ell_1(k+\ell_1)} \leqslant \frac{1}{k} \to 0$. By Corollary 4.12(i), the sequence $\mathcal{X}_{\lambda;m}[\mathrm{rank}]$ is asymptotically normal.

Example 4.14. Suppose λ is a sequence of partitions with $m = \ell_1$ and $k \to \infty$. Then $\frac{m^2}{k\ell_1(k+\ell_1)} \leqslant \frac{1}{k} \to 0$. Again by Corollary 4.12(i), the sequence $\mathcal{X}_{\lambda;m}[\mathrm{rank}]$ is asymptotically normal.

Remark 4.15. The limit shape of a randomly chosen partition of n as $n \to \infty$ is well-known to be the curve

$$e^{-\frac{\pi}{\sqrt{6}}x} + e^{-\frac{\pi}{\sqrt{6}}y} = 1$$

where (x, y) corresponds to $(i/\sqrt{n}, \lambda_i/\sqrt{n})$ [Ver96, Thm. 4.4, p.99]. One consequently expects $\lambda_1 \approx \sqrt{n}$, and certainly $k \to \infty$. It seems natural to use $m = \lambda'_1 \approx \sqrt{n}$, in which case

$$\frac{m^2}{k\lambda_1(k+\lambda_1)} \approx \frac{\sqrt{n}^2}{k\sqrt{n}(k+\sqrt{n})} \leqslant \frac{1}{k} \to 0.$$

Thus, one heuristically expects $\mathcal{X}_{\lambda;m}[\mathrm{rank}]$ to be asymptotically normal for randomly chosen partitions. We do not attempt to make this precise.

Question 4.16. *Suppose λ is a sequence of partitions with $\ell(\lambda) < m$ and k is the number of distinct parts of λ. Does $k \to \infty$ ensure $\mathcal{X}_{\lambda;m}[\mathrm{rank}]$ is asymptotically normal?*

4.4. Limit laws for plane partitions

We may use Corollary 4.12(ii) to deduce the complete characterization of the asymptotic limits for plane partitions in a box from the introduction. The following is a restatement of Theorem 1.11.

Theorem 4.17. *The* size *statistic on $\mathrm{PP}(a \times b \times c)$ is asymptotically normal if and only if*

$$\mathrm{median}\{a, b, c\} \to \infty.$$

If ab converges and $c \to \infty$, the normalized limit law is the Irwin–Hall distribution $\mathcal{IH}^*_{ab}$.

Proof. From the discussion in Section 2.2, we have

$$\mathcal{X}_{\mathrm{PP}(a\times b\times c)}[\mathrm{size}]^* = \mathcal{X}_{\mathrm{SSYT}_{\leqslant a+c}((b^a))}[\mathrm{rank}]^*.$$

Let $\lambda = (b^a) = b^a 0^c$, so $n = ab$, $k = 2$, $\ell_1 = b$, $\ell_2 = 0$, $e_1 = a$, $e_2 = c$, and $m = a + c$. Suppose $\mathrm{median}\{a, b, c\} \to \infty$. Without loss of generality, we may suppose $b \leqslant a \leqslant c$, so $a \to \infty$. In this case, $e^{[2]} = a$ and $b/(a + c) \leqslant 1/2$. Hence $\frac{e^{[2]}}{k(\ell_1/m+1)^2} = \frac{a}{2(b/(a+c)+1)^2} \geqslant \frac{a}{2(3/2)^2} \to \infty$ and asymptotic normality follows from Corollary 4.12(ii).

On the other hand, if $\mathrm{median}\{a, b, c\}$ is bounded, we may suppose $a \leqslant b \leqslant c$, so that $n = ab$ is bounded. If $c \to \infty$, then the standardized limit distribution is $\mathcal{IH}^*_{ab}$ provided ab converges by Theorem 4.1(i). The result follows. □

We conclude this section by giving some sample applications of the preceding results to three natural scaling limits of partitions obtained by stretching rows and/or columns by scale factors tending to ∞.

Example 4.18. Continuing Example 4.6, instead pick a sequence $c^{(N)} \to \infty$ of column scale factors, so that

$$\lambda^{(N)} = (\underbrace{\lambda_1, \ldots, \lambda_1}_{c^{(N)}}, \cdots, \underbrace{\lambda_m, \ldots, \lambda_m}_{c^{(N)}}),$$

$m^{(N)} = c^{(N)}m$, $\ell_1^{(N)} = \lambda_1$, $e_i^{(N)} = c^{(N)}e_i$, and $(e^{(N)})^{[2]} = c^{(N)}e^{[2]}$. Thus

$$\frac{(e^{(N)})^{[2]}}{k^{(N)}(\ell_1^{(N)}/m^{(N)}+1)^2} = \frac{c^{(N)}e^{[2]}}{k(\lambda_1/(c^{(N)}m)+1)^2} \sim \frac{c^{(N)}e^{[2]}}{k} \to \infty,$$

so by Corollary 4.12(ii), $\mathcal{X}_{(c^{(N)}\lambda')';c^{(N)}m}[\mathrm{rank}]$ is asymptotically normal.

Example 4.19. Combining Example 4.6 and Example 4.18, use both row and column scale factors simultaneously. We see

$$\frac{\ell_1^{(N)} e_1^{(N)} e_k^{(N)}}{\ell_1^{(N)} + m^{(N)}} = \frac{r^{(N)} (c^{(N)})^2 \lambda_1 e_1 e_k}{r^{(N)} \lambda_1 + c^{(N)} m} \to \infty,$$

so by Corollary 4.12(iii), $\mathcal{X}_{r^{(N)}(c^{(N)}\lambda')';c^{(N)}m}[\mathrm{rank}]$ is asymptotically normal. In particular, this includes the case when $c^{(N)} = r^{(N)} \to \infty$ and $\lambda^{(N)}$ is obtained from λ by replacing each cell with a $c^{(N)} \times c^{(N)}$ grid of cells.

4.5. Summary

Here we collect the known cases when $\mathcal{X}_{\lambda;m}[\mathrm{rank}]^*$ converges in distribution. Let $n = |\lambda|$, without loss of generality suppose $\ell(\lambda) < m$, let k be the number of distinct row lengths of λ (including 0 since $\ell(\lambda) < m$), let e_i be the multiplicity of the ith largest row length, and let $e^{[2]}$ be the second-largest element amongst $e_1, e_2, \ldots, e_k$.

Summary 4.20.

(i) In the following situations, $\mathcal{X}_{\lambda;m}[\mathrm{rank}]^* \Rightarrow \mathcal{N}(0,1)$.

- *(a)* $\frac{n}{m} \to 0$ *and* $n \to \infty$ *(Theorem 4.1(ii))*
- *(b)* $\lambda_1/m^3 \to \infty$ *and* $m \to \infty$. *Moreover, a converse holds. (Theorem 4.5(ii))*
- *(c)* $\lambda^{(N)} = (2^{N-1}, 2^{N-2}, \ldots, 1)$ *and* $m^{(N)} = N$ *(Example 4.7)*
- *(d)* $\lambda^{(N)} = \delta_N = (N-1, N-2, \ldots, 2, 1, 0)$ *and* $m^{(N)} = N$ *(Example 4.11)*
- *(e)* $\mathrm{weft}(\lambda) \to \infty$ *(Theorem 4.10)*
- *(f)* $\frac{m^2}{k\ell_1(k+\ell_1)} \to 0$ *and* $k \to \infty$ *(Corollary 4.12(i))*
- *(g)* $\frac{e^{[2]}}{k(\ell_1/m+1)^2} \to \infty$ *(Corollary 4.12(ii))*
- *(h)* $\frac{\ell_1 e_1 e_k}{\ell_1 + m} \to \infty$ *(Corollary 4.12(iii))*
- *(i)* $e_1 = \cdots = e_k = 1$ *and* $k \to \infty$ *(Example 4.13)*
- *(j)* $m = \lambda_1$ *and* $k \to \infty$ *(Example 4.14)*
- *(k)* $\lambda = (b^a)$, $m = a + c$, *and* $\mathrm{median}\{a, b, c\} \to \infty$ *(Theorem 1.11)*
- *(l) If the sequence* λ *is obtained by successively scaling the columns by a factor* $c \to \infty$. *(Example 4.18)*
- *(m) If the sequence* λ *is obtained by successively scaling the rows and columns by factors of* $r, c \to \infty$. *(Example 4.19)*

(ii) In the following situations, $\mathcal{X}_{\lambda;m}[\mathrm{rank}]^* \Rightarrow \mathcal{IH}_M^*$.

- *(a)* $n/m \to 0$ *and* $n \to M$. *(Theorem 4.1(i))*
- *(b)* $\lambda = (b^a)$, $m = a + c$, $ab \to M$, *and* $c \to \infty$ *(Theorem 1.11)*

(iii) In the following situations, $\mathcal{X}_{\lambda;m}[\mathrm{rank}]^* \Rightarrow \mathcal{S}^*_{\mathbf{d}}$.

(a) $\lambda_1 \to \infty$, *m is bounded, and* $\Delta\mathbf{t}(\lambda) \to \mathbf{d}$ *where* $x_i := \lambda_i/\lambda_1$. *Moreover, a converse holds. (Theorem 4.5(i))*

(b) If the sequence λ is obtained by successively scaling the rows by a factor $r \to \infty$, and $\mathbf{d} = \Delta\lambda$. *(Example 4.6)*

5. Metric spaces related to forest distributions

In this section, we consider the two q-analogs of the number of linear extensions of posets which come from trees and forests using variations on the inv and maj statistics for permutations as given by Björner–Wachs in [BW89]. Recall the background for these q-analogs from Section 2.3. As summarized in Section 1.3, we will show that the coefficients in the corresponding polynomials "generically" are asymptotically normal, but that the metric space of DUSTPAN distributions $\mathrm{M}_{\mathrm{DUST}}$ characterizes all possible limit laws in a certain degenerate regime. In particular, we prove Theorem 1.13, Theorem 1.14, and Corollary 1.17.

5.1. Generic asymptotic normality for trees and forests

Recall from Section 2.3 that for any forest P, there is an associated q-hook length polynomial

$$\mathcal{L}_P(q) := [n]_q!/\prod_{u\in P}[h_u]_q$$

and random variable $\mathcal{X}_P$. Here we show that the sequences of random variables $\mathcal{X}_P$ for forests P are asymptotically normal if certain numerical conditions hold; see Theorem 1.13. This covers the "generic cases". We begin by describing a family of trees which maximize the sum of the hook lengths over all trees of rank r with n elements. We use this family of trees to identify good approximations for the cumulants corresponding with all trees.

Definition 5.1. Suppose $n \in \mathbb{Z}_{\geqslant 1}$ and $1 < r \leqslant n$. Let $H_{n,r}$ be the tree obtained by starting with a rooted chain C with r elements and adding $n-r$ elements each as children of the second-smallest node in the chain. See Figure 5.1.

Lemma 5.2. *Among all trees P with n elements and rank $1 < r \leqslant n$, $H_{n,r}$ is the unique maximizer of $\sum_{u\in P} h_u$. Consequently, the degree of $\mathcal{L}_P(q)$ is*

$$\sum_{k=1}^{n} k - \sum_{u\in P} h_u \geqslant \sum_{k=1}^{n} k - \sum_{v\in H_{n,r}} h_v = \binom{n-r+1}{2} \tag{5.1}$$

Proof. Let C be a maximal chain of P with $r > 1$ elements and second-smallest element y. If $P \neq H_{n,r}$, let $x \in P - C$ be a leaf of P which is not a child of y. Let P' be the result of moving x to be a descendant x' of y, which preserves the rank and number of vertices. Since C is maximal, we can easily determine the change in the sum of the hook lengths: it increases by

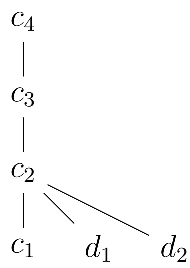

Figure 5.1: The poset $H_{6,4}$. The chain $C = \{c_1, c_2, c_3, c_4\}$ of length 4 has 2 additional descendants added to the second-smallest element c_2.

$\#\{v' \in P' : v' \in C, v' \geqslant y\} = |C| - 1 = r - 1$ and decreases by $\#\{v \in P : v > x\} \leqslant r - 1$. This procedure always weakly increases the sum of the hook lengths and arrives at $H_{n,r}$ after a finite number of iterations, so the maximality claim follows.

Observe the procedure strictly increases the sum of the hook lengths unless $\#\{v \in P : v > x\} = r - 1$. In this case, let z be the unique cover of x in P. By construction, $z \notin C$. After applying the procedure to x to get P', applying the procedure again to all of z's children and then to z will strictly increase the sum of hook lengths. Thus, P has strictly smaller sum of hook lengths than $H_{n,r}$, and the uniqueness claim follows.

For the equality in (5.1), we find

$$\sum_{v \in H_{n,r}} h_u = 1 \cdot (n - r + 1) + \sum_{k=n-r+2}^{n} k = \sum_{k=n-r+1}^{n} k.$$

Therefore,

$$\sum_{k=1}^{n} k - \sum_{v \in H_{n,r}} h_v = \sum_{k=1}^{n} k - \sum_{k=n-r+1}^{n} k = \sum_{k=1}^{n-r} k = \binom{n-r+1}{2}. \qquad \square$$

Lemma 5.3. *Suppose $0 \leqslant \alpha < 1$ and fix $d \in \mathbb{Z}_{\geqslant 2}$ even. Uniformly for all trees P with n elements and rank $1 < r \leqslant \alpha n$, we have*

$$|\kappa_d^P| = \Theta(n^{d+1}).$$

Explicitly, for a fixed $d \in \mathbb{Z}_{\geqslant 1}$,

$$\frac{ba^d}{d} n^{d+1} \leqslant \sum_{k=1}^{n} k^d - \sum_{u \in P} h_u^d \leqslant \left(\frac{1}{d+1} + \frac{1}{n}\right) n^{d+1} \tag{5.2}$$

where $x := \left[\left(\frac{2}{1-\alpha}\right)^2 - 1\right] > 1$, $a := 1/x$, and $b := 1/(x+1)$, so $0 < a, b < 1$.

Proof. Recall from Corollary 2.21 that $\kappa_d^P = \frac{B_d}{d}\left(\sum_{k=1}^{n} k^d - \sum_{u \in P} h_u^d\right)$ so $|\kappa_d^P| = \Theta(n^{d+1})$ provided the lower bound and upper bound in (5.2) hold. The upper bound follows from the upper bound in Lemma 2.23.

For the lower bound, construct a labeling w of P by iteratively building up P as follows. Begin by labeling the root of P with 1 in w. At each step, increment all existing labels in w, pick an element of P which has not been labeled whose parent has been labeled, and label it with 1. Observe that the resulting labeling $w\colon P \to [n]$ is natural. Consider the quantity $w(u) - h_u$ during this procedure. When u has initially been labeled, we have $w(u) - h_u = 1 - 1 = 0$. After u has been labeled, when adding a new vertex v, if $v \leqslant u$ then both $w(u)$ and h_u are incremented, while if $v \nleqslant u$ then only $w(u)$ in incremented. Consequently, the final value of $w(u) - h_u$ counts the number of elements v added after u such that $v \nleqslant u$. In particular, $w(u) - h_u \geqslant 0$.

Using the real numbers a, b, x defined in the statement of the lemma, let $M := \{u \in P : w(u) - h_u \geqslant bn\}$. We claim $\#M \geqslant an$. To prove the claim, suppose to the contrary that $\#M < an$. By definition, $0 < a, b < 1$. Consequently,

$$\begin{aligned}\sum_{u\in P} w(u) - h_u &= \sum_{u\in M}(w(u) - h_u) + \sum_{u\notin M}(w(u) - h_u)\\ &\leqslant \#M \cdot n + (n - \#M)\cdot bn\\ &= bn^2 + \#M \cdot (1-b)n\\ &< bn^2 + a(1-b)n^2 = (a + b - ab)n^2.\end{aligned}$$

One may easily check that $a + b - ab = 2/(x + 1) = (1 - \alpha)^2/2$. Since $r \leqslant \alpha n$, we have $n - r \geqslant (1 - \alpha)n$, so that

$$\sum_{u\in P} w(u) - h_u < \frac{(1-\alpha)^2}{2}n^2 \leqslant \frac{(n-r)^2}{2} \leqslant \binom{n-r+1}{2},$$

contradicting Lemma 5.2 and verifying the claim. Using the claim and the lower bound on the sum in Lemma 2.23, we now find

$$\begin{aligned}\sum_{j=1}^{n} j^d - \sum_{u\in P} h_u^d &= \sum_{u\in P}(w(u)^d - h_u^d)\\ &\geqslant \sum_{u\in M}(w(u) - h_u)\mathbf{h}_{d-1}(w(u), h_u) \geqslant \sum_{u\in M}(bn)w(u)^{d-1}\\ &\geqslant bn \cdot \sum_{j=1}^{\#M} j^{d-1}\\ &\geqslant bn \cdot (\#M)^d/d \geqslant \left(ba^d/d\right)\, n^{d+1}.\end{aligned}$$

□

Now, we are prepared to address the question of asymptotic normality for sequences of random variables associated to trees and forests. Recall the following theorem from the introduction.

Theorem 1.13. Given a sequence of forests P, the corresponding sequence of random variables $\mathcal{X}_P^*$ is asymptotically normal if

$$|P| \to \infty \qquad \text{and} \qquad \limsup \frac{\operatorname{rank}(P)}{|P|} < 1.$$

Proof. By Remark 2.22, it suffices to assume P is a tree. For $d \geqslant 2$ even, we know $|\kappa_d^P| = \Theta(n^{d+1})$ by Lemma 5.3, so $|(\kappa_d^P)^*| = |\kappa_d^P|/|\kappa_2^P|^{d/2} = \Theta(n^{1-d/2}) \to 0$. By Corollary 2.21, the odd cumulants vanish. Therefore, the result again follows from Corollary 2.5. □

Remark 5.4. One expects most random forest generation techniques to yield a rank which is logarithmic in the number of nodes with high probability, in which case Theorem 1.13 applies. This is the sense in which we consider Theorem 1.13 to cover "generic" trees and forests.

Remark 5.5. More precisely, we may use the explicit bounds in Lemma 5.3. Setting $\alpha := r/n$, the lower bound becomes $\frac{(1-\alpha)^{2(d+1)}}{4(1+\alpha)^d(3-\alpha)^d} n^{d+1}$. Since $0 \leqslant \alpha \leqslant 1$, the denominator can be ignored. Considering the $d = 4$ case for simplicity, we find

$$\kappa_4^* = O\left(\frac{n^5}{((1-\alpha)^6 n^3)^2}\right) = O\left(\frac{n^{-1}}{(1-r/n)^{12}}\right) = O\left(\frac{n^{11}}{(n-r)^{12}}\right).$$

Thus asymptotic normality follows when $\frac{n-r}{n^{11/12}} \to \infty$, or equivalently when $n - r = \omega(n^{11/12})$. By contrast, Theorem 1.14 classifies limit laws when $n - r = o(n^{1/2})$. Analyzing the possible asymptotic behavior between these extremes is still an open problem.

5.2. Degenerate forests and DUSTPAN distributions

We now consider sequences of random variables associated to the "degenerate" trees with $n - r = o(n^{1/2})$. Note, $n - r = o(n^{1/2})$ implies $r/n \to 1$, so these sequences are not covered by Theorem 1.13. For such trees, we give a simple numerical estimate for the cumulants in terms of *multisets of elevations*, and use them to characterize asymptotic normality as well as the other limiting distributions in terms of the metric space of DUSTPAN distributions $\mathrm{M}_{\mathrm{DUST}}$.

Remark 5.6. To avoid certain redundancies, we restrict to standardized trees in the sense of Remark 2.22. As an example of behavior which is prohibited by this assumption, consider the trees $H_{n,n-k}$ for fixed k, which are not standardized. This sequence of trees has rank $r = n - k$, so $\lim r/n = 1$ as $n \to \infty$, and Theorem 1.13 does not apply. Indeed, it is easy to see that

$$\mathcal{L}_{H_{n,n-k}}(q) = \frac{[n]_q!}{\prod_{u \in H_{n,n-k}} [h_u]_q} = [k+1]_q!.$$

Therefore, $\mathcal{X}^*_{H_{n,n-k}}$ has the same discrete distribution for all $n > k$, so the limit distribution is discrete.

On the other hand, if $n - r \to \infty$, the length of the support of $\mathcal{X}_P$ tends to ∞ by Remark 2.20 and Lemma 5.2. Hence each distribution appears only finitely many times in such a sequence. Moreover, since the coefficients are unimodal, any sequence $\mathcal{X}_P^*$ with $n - r \to \infty$ cannot converge to a discrete distribution.

We begin with a series of estimates relating the cumulants κ_d^P to the following auxiliary combinatorial quantity on P.

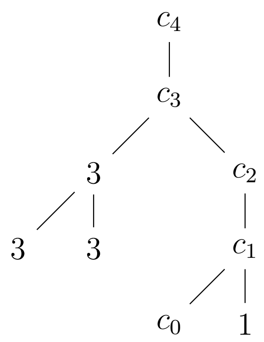

Figure 5.2: A tree P with maximal chain $C = \{c_0 < c_1 < c_2 < c_3 < c_4\}$ and elevations of $P - C$ labeled.

Definition 5.7. Let C be a fixed maximal chain in a forest P with $|C| = r$. For each $u \in P - C$, define the *elevation* of u to be

$$e_u := \#\{v \in C : u \not\leqslant v\}.$$

See Figure 5.2. Let $s_k(P, C)$ be the number of elements in $P - C$ with elevation at least $k - n + r$,

$$s_k(P, C) := \#\{u \in P - C : e_u \geqslant k - n + r\}.$$

For example, if u is attached to the root of the tree which is the maximal element of C, then the elevation is $e_u = r - 1$. If u is attached to the second-smallest element of C, then $e_u = 1$. We see that $e_u = r$ if and only if u is not connected by a path to C. Thus, if P is a tree, then $1 \leqslant e_u < r$, so $s_{n-r}(P, C) = n - r$, and $s_n(P, C) = 0$.

If P is a tree and C is a chain in P, then $P - C$ is a forest so both have associated cumulants. We may relate κ_d^P and κ_d^{P-C} using the numbers $s_k(P, C)$ as follows.

Lemma 5.8. *Let C be a maximal chain in a tree P with n elements and $|C| = r$. Then for each $d \in \mathbb{Z}_{\geqslant 1}$,*

$$\kappa_d^P = \kappa_d^{P-C} + \frac{B_d}{d} \sum_{u \in P-C} \sum_{k=n-r+1}^{n-r+e_u} \mathbf{h}_{d-1}(k, k - s_k) \tag{5.3}$$

Proof. Let $C = v_n > v_{n-1} > \cdots > v_{n-r+1}$. Note that for $u \in P - C$, we have $u < v_k$ if and only if $e_u < k - n + r$. Consequently, for all $n - r < k \leqslant n$ we have

$$\begin{aligned} h_{v_k} &= k - n + r + \#\{u \in P - C : u < v_k\} \\ &= k - n + r + \#\{u \in P - C : e_u < k - n + r\} \\ &= k - n + r + (n - r - \#\{u \in P - C : e_u \geqslant k - n + r\}) \\ &= k - s_k. \end{aligned}$$

Thus,

$$\begin{aligned}\sum_{k=n-r+1}^{n} k^d - h_{v_k}^d &= \sum_{k=n-r+1}^{n} (k - h_{v_k})\mathbf{h}_{d-1}(k, h_{v_k}) \\ &= \sum_{k=n-r+1}^{n} s_k \mathbf{h}_{d-1}(k, k - s_k) \\ &= \sum_{k=n-r+1}^{n} \#\{u \in P - C : e_u \geqslant k - n + r\} \cdot \mathbf{h}_{d-1}(k, k - s_k) \\ &= \sum_{u \in P-C} \sum_{k=n-r+1}^{n-r+e_u} \mathbf{h}_{d-1}(k, k - s_k).\end{aligned}$$

Therefore, (5.3) follows from the cumulant formula in Corollary 2.21. □

If P is a standardized tree with maximal chain C of size $|C| = r > 1$, it has an element $u \in P - C$ with $e_u = r - 1$, so $e_u/r \sim 1$ for r large. As we saw in Section 3.1, renormalizing a multiset by the maximum value is a useful technique while not changing the corresponding standardized general uniform sum distribution. Consequently, we consider the re-scaled multiset of elevations $\mathbf{e}/r = \{e_u/r : u \in P-C\}$, which are then related to the rescaled cumulants κ_d^P/r^d.

Lemma 5.9. *Suppose we have a sequence of standardized trees P such that the number of elements $n \to \infty$ and the rank r satisfies $n - r = o(n^{1/2})$, i.e. $(n - r)/n^{1/2} \to 0$. Let C be a maximal length chain in P. Then, for each $d \in \mathbb{Z}_{\geqslant 1}$,*

$$\frac{\kappa_d^{P-C}}{r^d} = O\left(\frac{(n-r)^{d+1}}{r^d}\right) \to 0 \quad \textit{and} \quad \frac{\kappa_d^P}{r^d} \sim \sum_{u \in P-C} \left(\frac{e_u}{r}\right)^d = |\mathbf{e}/r|_d^d.$$

Proof. Since $n-r = o(n^{1/2})$ and $n \to \infty$, we find $n \sim r$, and so $n-r = o(r^{1/2})$. Consequently, $(n - r)^2/r \to 0$, and more generally $(n - r)^{d+1}/r^d \to 0$ for all $d \geqslant 1$. Therefore,

$$\frac{\kappa_d^{P-C}}{r^d} = \frac{1}{r^d}\sum_{k=1}^{n-r} k^d - \frac{1}{r^d}\sum_{u \in P-C} h_u^d = O\left(\frac{(n-r)^{d+1}}{r^d}\right) \to 0.$$

Consider the formula for κ_d^P/r^d obtained from (5.3) by dividing both sides by r^d. The first term goes to 0 by the argument above. The second term is bounded above and below by

$$d \sum_{u \in P-C} \sum_{k=n-r+1}^{n-r+e_u} (k - s_k)^{d-1} \leqslant \sum_{u \in P-C} \sum_{k=n-r+1}^{n-r+e_u} \mathbf{h}_{d-1}(k, k - s_k) \leqslant d \sum_{u \in P-C} \sum_{k=n-r+1}^{n-r+e_u} k^{d-1}. \quad (5.4)$$

In Lemma 5.10 and Lemma 5.11 below, we will show that, after dividing by r^d, both bounds in (5.4) are asymptotic to $\sum_{u \in P-C} \left(\frac{e_u}{r}\right)^d$. Thus, $\kappa_d^P/r^d \sim \sum_{u \in P-C} \left(\frac{e_u}{r}\right)^d$. □

Lemma 5.10. *With the same hypotheses as Lemma 5.9,*

$$\frac{d}{r^d}\sum_{u\in P-C}\sum_{k=n-r+1}^{n-r+e_u}k^{d-1}\sim\sum_{u\in P-C}\left(\frac{e_u}{r}\right)^d.$$

Proof. From Lemma 2.23, we have

$$\sum_{u\in P-C}\left[\left(\frac{n-r}{r}+\frac{e_u}{r}\right)^d-\left(\frac{n-r}{r}\right)^d\right]\leqslant\frac{d}{r^d}\sum_{u\in P-C}\sum_{k=n-r+1}^{n-r+e_u}k^{d-1}$$
$$\leqslant\sum_{u\in P-C}\left[\left(\frac{n-r}{r}+\frac{e_u}{r}\right)^d-\left(\frac{n-r}{r}\right)^d\right]+\frac{d}{r}\sum_{u\in P-C}\left[\left(\frac{n-r}{r}+\frac{e_u}{r}\right)^{d-1}-\left(\frac{n-r}{r}\right)^{d-1}\right]. \tag{5.5}$$

Consider the lower bound in (5.5). By Lemma 5.9, $\sum_{u\in P-C}\left(\frac{n-r}{r}\right)^d=\frac{(n-r)^{d+1}}{r^d}\to 0$ for all $d\geqslant 1$. Furthermore,

$$\begin{aligned}\sum_{u\in P-C}\left(\frac{n-r}{r}+\frac{e_u}{r}\right)^d&=\sum_{u\in P-C}\sum_{i=0}^{d}\binom{d}{i}\left(\frac{n-r}{r}\right)^i\left(\frac{e_u}{r}\right)^{d-i}\\&=\sum_{i=0}^{d}\binom{d}{i}\left(\frac{n-r}{r}\right)^i\sum_{u\in P-C}\left(\frac{e_u}{r}\right)^{d-i}\\&\leqslant\sum_{u\in P-C}\left[\left(\frac{e_u}{r}\right)^d+\sum_{i=1}^{d}\binom{d}{i}\left(\frac{n-r}{r}\right)^i\cdot 1^d\right]\\&\sim\sum_{u\in P-C}\left(\frac{e_u}{r}\right)^d.\end{aligned}$$

The first term in the upper bound in (5.5) is dominant by a similar argument. Therefore, since the upper and lower bound in (5.5) asymptotically converge to the same sum, it follows that

$$\frac{d}{r^d}\sum_{u\in P-C}\sum_{k=n-r+1}^{n-r+e_u}k^{d-1}\sim\sum_{u\in P-C}\left(\frac{e_u}{r}\right)^d. \qquad\square$$

Lemma 5.11. *With the same hypotheses as Lemma 5.9,*

$$\frac{d}{r^d}\sum_{u\in P-C}\sum_{k=n-r+1}^{n-r+e_u}(k-s_k)^{d-1}\sim\sum_{u\in P-C}\left(\frac{e_u}{r}\right)^d.$$

Proof. Consider the expansion

$$\begin{aligned}\frac{d}{r^d}\sum_{u\in P-C}\sum_{k=n-r+1}^{n-r+e_u}(k-s_k)^{d-1}&=\sum_{u\in P-C}\frac{d}{r^d}\sum_{k=n-r+1}^{n-r+e_u}k^{d-1}\\&\quad+\sum_{i=1}^{d-1}(-1)^i\binom{d-1}{i}\sum_{u\in P-C}\frac{d}{r^d}\sum_{k=n-r+1}^{n-r+e_u}k^{d-1-i}s_k^i.\end{aligned}$$

Since s_k by definition counts a subset of $P-C$, we have $s_k \leqslant n-r$. Thus, for each $1 \leqslant i \leqslant d-1$, we have

$$\begin{aligned}\sum_{u\in P-C} \frac{d}{r^d} \sum_{k=n-r+1}^{n-r+e_u} k^{d-1-i} s_k^i &\leqslant \sum_{u\in P-C} \frac{d(n-r)^i}{r^d} \cdot \sum_{k=n-r+1}^{n} k^{d-i-1} \\ &= \frac{d(n-r)^{i+1}}{r^d} \cdot \sum_{k=n-r+1}^{n} k^{d-i-1} \\ &= O\left(\frac{(n-r)^{i+1}}{r^d} \cdot r^{d-i}\right) = O\left(\frac{(n-r)^{i+1}}{r^i}\right).\end{aligned}$$

By Lemma 5.9, $\frac{(n-r)^{i+1}}{r^i} \to 0$, and so by Lemma 5.10, it follows that

$$\frac{d}{r^d} \sum_{u\in P-C} \sum_{k=n-r+1}^{n-r+e_u} (k-s_k)^{d-1} \sim \frac{d}{r^d} \sum_{u\in P-C} \sum_{k=n-r+1}^{n-r+e_u} k^{d-1} \sim \sum_{u\in P-C} \left(\frac{e_u}{r}\right)^d. \qquad \square$$

We may combine the preceding results to prove the following more explicit form of Theorem 1.14 from the introduction.

Theorem 5.12. *Let P denote an infinite sequence of standardized trees with n elements and maximal chains C of rank r such that $n \to \infty$ and $n - r = o(n^{1/2})$. Let $\mathbf{e} = \{e_u : u \in P - C\}$ be the multiset of elevations for P and C. Then for each fixed $d \in \mathbb{Z}_{\geqslant 2}$ even, the cumulants of $\mathcal{X}_P^*$ are approximately*

$$(\kappa_d^P)^* \sim \frac{B_d/d}{(B_2/2)^{d/2}} \left(\frac{|\mathbf{e}/r|_d}{|\mathbf{e}/r|_2}\right)^d = \frac{B_d}{d} |\widehat{\mathbf{e}}|_d^d. \tag{5.6}$$

The sequence of random variables $\mathcal{X}_P^*$ converges in distribution if and only if the multisets $\widehat{\mathbf{e}}$ converge pointwise to some multiset $\mathbf{t} \in \mathbf{P}_{\text{DUST}}$, in which case the limiting distribution is $\mathcal{S}_{\mathbf{t}} + \mathcal{N}(0,\sigma) \in \mathbf{M}_{\text{DUST}}$ where $\sigma := \sqrt{1 - |\mathbf{t}|_2^2/12}$. In particular, the sequence of random variables $\mathcal{X}_P$ are asymptotically normal if and only if

$$|\mathbf{e}/r|_2^2 := \sum_{u\in P-C} \left(\frac{e_u}{r}\right)^2 \to \infty. \tag{5.7}$$

Proof. Fix $d \geqslant 2$ even. By hypothesis, $n - r = o(n^{1/2})$, so Lemma 5.9 shows that

$$\frac{\kappa_d^P}{r^d} \sim \frac{B_d}{d} |\mathbf{e}/r|_d^d. \tag{5.8}$$

Therefore, by (2.2)

$$(\kappa_d^P)^* \sim \frac{B_d/d}{(B_2/2)^{d/2}} \left(\frac{|\mathbf{e}/r|_d}{|\mathbf{e}/r|_2}\right)^d.$$

Since $\mathbf{e}/r$ is finite, $|\mathbf{e}/r|_2$ exists, so the hat-operation is defined on $\mathbf{e}/r$ and $\widehat{\mathbf{e}/r} = \widehat{\mathbf{e}}$ after cancellation. Hence, (5.6) follows from the definition of the hat-operation in (1.8).

By the Method of Moments/Cumulants (Theorem 2.4) together with Lemma 2.7, the sequence $\mathcal{X}_P^*$ converges in distribution to some $\mathcal{X}$ if and only if $\frac{B_d}{d}|\widehat{\mathbf{e}}|_d^d$ converges to $\kappa_d^{\mathcal{X}}$ for each $d \in \mathbb{Z}_{\geqslant 1}$. By Corollary 3.19 and the fact that $|\widehat{\mathbf{e}}|_2^2 = 12$ by definition, this occurs if and only if $\widehat{\mathbf{e}}$ converges pointwise to some $\mathbf{t}$. Therefore, by Theorem 3.32, we have $\mathbf{t} \in \mathbf{P}_{\text{DUST}}$ and $\mathcal{X}$ has the associated DUSTPAN distribution $\Phi(\mathbf{t}) = \mathcal{S}_{\mathbf{t}} + \mathcal{N}(0, \sigma)$.

In particular, the limiting distribution of $\mathcal{X}_P^*$ is $\mathcal{N}(0,1)$ if and only if $\widehat{\mathbf{e}} \to \mathbf{0}$. Now $\widehat{\mathbf{e}} \to \mathbf{0}$ if and only if $|\mathbf{e}/r|_\infty / |\mathbf{e}/r|_2 = 1/|\mathbf{e}/r|_2 \to 0$ since standardized trees have an element of elevation $r-1$. In particular, the limit is $\mathcal{N}(0,1)$ if and only if $|\mathbf{e}/r|_2 \to \infty$. □

Remark 5.13. We note that considering only standardized trees in Theorem 5.12 is essential for the "if and only if" conditions to hold. For example, consider a sequence of trees $H_{n,r}$ with maximal chain C of size r such that $n \to \infty$ and $n - r = o(n^{1/2})$. Since $\mathcal{L}_{H_{n,r}}(q) = [n-r+1]_q!$ and $n - r \to \infty$, $\mathcal{X}_{H_{n,r}}$ is asymptotically normal by [Fel45]. However, we have elevation $e_u = 1$ for all $u \in H_{n,r} - C$. Therefore, $\sum_{u \in H_{n,r} - C} (e_u/r)^2 = (n-r)/r^2 \to 0$ rather than ∞.

Remark 5.14. One can construct sequences of standardized trees with $n - r = o(n^{1/2})$ where $\mathbf{e}/r$ converges to any prescribed finite multiset $\mathbf{t} = (t_1 \geqslant t_2 \geqslant \cdots \geqslant t_m) \in \widetilde{\ell}_2$ with $|\mathbf{t}|_\infty = 1$. For each $N = m+3, m+4, \ldots$, let $r_N = N - m$. To construct the tree P_N, start with a chain $C_N = (v_0 < v_1 < \cdots < v_{r_N - 1})$, and for each nonzero value $1 = t_1 \geqslant t_2 \geqslant \cdots \geqslant t_m$, add a child to $v_{\lceil (r_n - 1) t_i \rceil}$. Finally, for each $t_i = 0$, add one additional child to v_1. As constructed $n = |P_N| = N$, $r = r_N - 1$ and $n - r = m$ is constant. Since $t_1 = 1$ by assumption, the root of P_N has at least one child so it is a standard tree. Furthermore, $m = |P_N - C_N|$ so the elevation multiset of P_N has exactly m elements. By construction, the multisets $\mathbf{e}/r = \{e_N / r_N : u \in P_N - C_N\}$ approaches $\mathbf{t}$ as $N \to \infty$. Therefore, $\mathcal{S}_{\widehat{\mathbf{t}}}$ is the limiting distribution of $\mathcal{X}_{P_N}^*$. By Corollary 3.29, we know that the closure of $\{\mathcal{S}_{\widehat{\mathbf{t}}} : \mathbf{t} \in \widetilde{\ell}_2, \mathbf{t} \text{ is finite}\}$ is $\mathbf{M}_{\text{DUST}}$. Thus, $\mathbf{M}_{\text{Forest}} \cup \mathbf{M}_{\text{DUST}} \subset \overline{\mathbf{M}_{\text{Forest}}}$ as claimed in Section 1.

Corollary 1.17. Let $\epsilon\,$TREE be the set of standardized trees P for which $|P| - \operatorname{rank}(P) \leqslant |P|^{\frac{1}{2}}$. Let $\mathbf{M}_{\epsilon\,\text{TREE}} := \{\mathcal{X}_P^* : P \in \epsilon\,\text{TREE}\} \subset \mathbf{M}_{\text{Forest}}$ be the corresponding metric space of distributions. Then

$$\overline{\mathbf{M}_{\epsilon\,\text{TREE}}} = \mathbf{M}_{\epsilon\,\text{TREE}} \sqcup \mathbf{M}_{\text{DUST}}, \tag{5.9}$$

which is (sequentially) compact. Moreover, the set of limit points of $\mathbf{M}_{\epsilon\,\text{TREE}}$ is $\mathbf{M}_{\text{DUST}}$.

Proof. By the construction in Remark 5.14, we know $\overline{\mathbf{M}_{\epsilon\,\text{TREE}}} \supset \mathbf{M}_{\text{DUST}}$, and $\mathbf{M}_{\text{DUST}}$ is closed by Corollary 3.28. Furthermore, we have $(|P| - \operatorname{rank}(P))/|P|^{1/2} < |P|^{-\epsilon} \to 0$, so Theorem 5.12 applies. Thus, for every sequence of trees $P \in \epsilon\,\text{TREE}$ with $|P| \to \infty$ such that the corresponding random variables $\mathcal{X}_P^* \in \mathbf{M}_{\epsilon\,\text{TREE}}$ converge in distribution, we know the distribution must be a DUSTPAN distribution. On the other hand, for every sequence of trees $P \in \epsilon\,\text{TREE}$ such that the corresponding random variables $\mathcal{X}_P^* \in \mathbf{M}_{\epsilon\,\text{TREE}}$ converge in distribution but $|P|$ is bounded, we must have a subsequence where $n = |P|$ is eventually constant. There are only a finite number of standardized trees of size n in $\epsilon\,$TREE, so we can further restrict to a sequence where each P is a particular tree, in which case the limiting of $\mathcal{X}_P^*$ is itself $\mathcal{X}_P^* \in \mathbf{M}_{\epsilon\,\text{TREE}}$. □

6. Future work

In addition to the open problems mentioned in Section 1 and Question 4.16, we pose the following questions for future study.

Question 6.1. *Suppose we have a sequence of standardized trees such that $n \to \infty$ where $n-r$ grows at least as fast as $n^{1/2}$ but no faster than $n^{11/12}$ in the sense that $n-r \notin o(n^{1/2})$ and $n-r \notin \omega(n^{11/12})$. When is the corresponding sequence of distributions asymptotically normal? What non-normal limit laws are possible?*

Question 6.2. *Does $\mathrm{weft}(\lambda) \to \infty$ if and only if $\mathcal{X}_{\lambda;m}[\mathrm{rank}]$ is asymptotically normal? See (4.8).*

Question 6.3. *Consider the set of rooted, unlabeled forests with n vertices, sampled uniformly at random. What is the expected value of the rank r, i.e. the maximum length of a path starting at a root of a tree in the forest? How does r compare to n asymptotically as $n \to \infty$?*

See [Pit94] for growth rates of the form $r \approx \log n$ for certain random tree generation techniques. For the number of rooted, unlabeled forests with n vertices, t trees, and rank r, see [OEI22, A291336]. Broutin–Flajolet [BF12, Thm. 3] showed that $\mathbb{E}[r] \sim C\sqrt{n}$ for an explicit constant $C > 0$ when considering rooted, unlabeled *binary* trees. The corresponding problem when order is imposed either by labeling the vertices (resulting in labeled trees) or by ordering the children (resulting in planar trees) is older, though the $\mathbb{E}[r] \sim D\sqrt{n}$ behavior is common throughout; see [BF12, p.1] for a summary and further references.

In [Swa22], the following q,t-analogue of the hook length formula (1.1) is given. Let $(r,c) \in \lambda$ denote a cell in row r and column c. Then

$$[n]_q! \prod_{(r,c)\in\lambda} \frac{q^{r-1} + tq^{c-1}}{[h(r,c)]_q} \tag{6.1}$$

is the generating function for a pair of statistics $(\mathrm{maj}, \mathrm{neg})$ on *standard supertableaux* of shape λ. The $t = 0$ case of (6.1) yields (1.1). While (6.1) is not literally a quotient of q-integers, it is evidently "nearly" such a quotient. Computational evidence suggests the distributions are "typically" bivariate normal with non-trivial covariance, which is strikingly similar to the distributions encountered by Kim–Lee [KL21] for $(\mathrm{des}, \mathrm{maj})$ on permutations in fixed conjugacy classes. See Figure 6.1 for sample data.

Question 6.4. *What are the possible limiting distributions of the coefficients of the q,t-hook length formula (6.1)? What is the support of (6.1)?*

One referee asked the following natural question, saying "A result of this form could give a *conceptual* explanation for some of the results." The authors regard this as an important question, but we do not expect a simple answer.

Question 6.5. *Can one give a formula for the statistic rank on $\mathrm{SSYT}_{\lambda,m}$ as a natural sum of m natural statistics on the tableaux, and then to show that they are (asymptotically) independent, converging to the uniform law?*

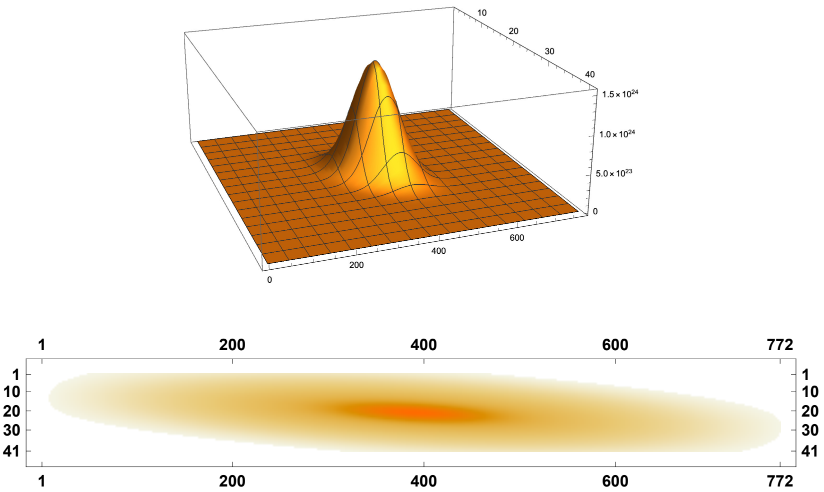

Figure 6.1: Plots of coefficients of the q, t-hook length formula (6.1) with $\lambda = (25, 4, 3, 3, 1, 1, 1, 1, 1)$.

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