Robust Minimum Gain Lemma

Tamás Baár and Tamás Luspay

Abstract— **The computation of the minimum sensitivity of uncertain Linear Time Invariant (LTI) systems is presented in the paper. The system interconnection is given by a generic Linear Fractional Transformation (LFT) of a nominal model and an uncertain block, where the input-output behavior of the latter is described by Integral Quadratic Constraints (IQC). The extension of the Minimum Gain Lemma is presented for such interconnections, resulting in a convex optimization problem subject to Linear Matrix Inequality (LMI) constraints. With the aim of the Generalized-KYP (GKYP) lemma the minimum gain/sensitivity is computed over a certain finite frequency range. Connection with the already existing literature is highlighted, providing an insight on the obtained results. A numerical example is given to illustrate and validate the proposed methodology.**

I. Introduction

The $\mathcal{H}_\infty$ norm is a well-known measure for the maximum sensitivity of dynamical systems [1]; it is defined as the peak value of the largest singular value over the whole frequency range. It can be efficiently computed by using convex optimization subject to Linear Matrix Inequality (LMI) constraints, this is usually referred as the Bounded Real Lemma in the literature [2]. The $\mathcal{H}_\infty$ norm plays a key-role in the theory of robust analysis and synthesis. One particular aspect, that we are interested in, is its extension for uncertain systems. Here, one of the most generic description is the Linear Fractional (LFT) interconnection of a nominal plant G and an uncertain block Δ. A solid theoretical foundation exists for the analysis (and synthesis) of LFT interconnected uncertain systems, however the developed methodologies differ in the underlying assumptions imposed on the Δ block. It has been shown that a wide range of dynamical components can be described by using Integral Quadratic Constraints (IQCs), where the possible combinations of input and output signals are fulfilling an integral formula. Starting from their early frequency domain interpretation [3] several features have been revealed in the past years, including the time-domain interpretation and dissipativity theory of IQCs [4], [5].

On the other hand, the $\mathcal{H}_-$ index characterizes the system's minimum sensitivity, i.e. it is defined as the infinum of the lowest singular value of the system over the whole frequency range. Despite the fact that the $\mathcal{H}_-$ index is not a norm (since it fails to satisfy certain norm properties), it has gained attention in the control community. It has been first introduced in [6] with a convex formulation similar to the Bounded Real Lemma. Alternatively, the Minimum Gain Lemma was introduced in [7], extending the notion of minimal sensitivity for unstable systems with non-zero initial conditions. Furthermore, the Large-gain theorem has been also proposed in [7], providing a stability criteria based on the minimum sensitivities of the components in the feedback loop. Computation of the minimum gain over a finite frequency range was proposed in [8], [9] by using the the Generalized Kalman-Yakubovic-Popov (GKYP) lemma.

Despite its theoretical foundations, the minimum sensitivity has not received nearly as much attention as the $\mathcal{H}_\infty$ norm. It has been applied as a performance measure in Fault Detection algorithms [10], [11], [12], and more recently in the decoupling problems of dynamical systems [13], [14]. The remarkable (and counter-intuitive) findings of [7] are contributing in the field of controller synthesis for unstable plants. In addition, some remarks have also been made on the robustness of the underlying problem in [7], however, no systematic analysis tool has been provided.

Our aim is to cover this gap and offer a convex minimum sensitivity analysis tool for LTI systems containing uncertain elements. Our derivation is based on the formulation of [7], along with the time-domain interpretation of the IQC theory, as presented in [4] and [5]. The obtained results are directly related to previously established theorems from robust control and analysis.

Section II collects the necessary mathematical background. The main contributions of the paper are the convex, IQC based robust minimum sensitivity analysis methods presented in Section III. A demonstrative example is presented in Section IV, and the paper is concluded in Section V.

II. Mathematical preliminaries

The mathematical notations of the paper is fairly standard. $\mathbb{R}$ and $\mathbb{C}$ denote the set of real and complex numbers, respectively. $\mathbb{RL}_\infty$ is the set of rational numbers with real coefficients that are proper and have no poles on the imaginary axis. $\mathbb{RH}_\infty$ is the subset of functions in $\mathbb{RL}_\infty$ that are analytic in the closed right half complex plane. $\mathbb{R}^{m\times n}$, $\mathbb{C}^{m\times n}$, $\mathbb{RL}_\infty^{m\times n}$, $\mathbb{RH}_\infty^{m\times n}$ denote the sets of $m\times n$ matrices that are in $\mathbb{R}$, $\mathbb{C}$, $\mathbb{RL}_\infty$ and $\mathbb{RH}_\infty$, respectively.

Furthermore $y \in \mathcal{L}_2$ if $||y||_2^2 = \int_0^\infty |y(t)|^2 dt < \infty$, and $y \in \mathcal{L}_{2e}$ if $||y||_2^2 = \int_0^\infty |y_T(t)|^2 dt < \infty$, $T \in \mathbb{R}^+$ and $y_T(t) = y(t)$ for $0 \le t \le T$ and $y_T(t) = 0$ for $t \ge T$.

$M \prec 0$ and $M \succ 0$ denotes the negative (positive) definiteness of the matrix M, respectively. S^m denotes the

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The authors are with the ELKH Institute for Computer Science and Control, 1111 Budapest, Hungary baartamas@sztaki.hu, tluspay@sztaki.hu

set of a symmetric $m \times m$ matrices. Symmetric matrix terms in inequalities are denoted by $\star$.

A. Minimum and Maximum Sensitivities

Consider a continuous time Linear Time Invariant (LTI) system G, with state space representation:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{1}$$

where: $x \in \mathbb{R}^{n_x}$ is the state vector, $u \in \mathbb{R}^{n_u}$ is the input vector and $y \in \mathbb{R}^{n_y}$ is the output vector of the system, while the constant system matrices (A, B, C, D) are of appropriate dimensions.

The maximum sensitivity of a stable system is characterized by a positive scalar γ such that

$$\frac{||y(t)||_2^2}{||u(t)||_2^2} \leq \gamma. \tag{2}$$

For LTI systems the peak sensitivity is the $\mathcal{H}_\infty$ norm and defined as

$$||G||_\infty^{[0,\infty)} := \sup_{\omega \in [0,\infty)} \bar{\sigma}\left[G\right], \tag{3}$$

where $\bar{\sigma}$ denotes the maximum singular value. The $\mathcal{H}_\infty$ norm can be computed through various numerical techniques, from which we only refer to the Bounded Real Lemma (BRL) which is a convex optimization subject to Linear Matrix Inequality (LMI) constraints [2].

In a similar way, the minimum sensitivity of a system can be characterized by a positive scalar β such that

$$\frac{||y(t)||_2^2}{||u(t)||_2^2} \geq \beta. \tag{4}$$

Again, in the LTI case, this minimum sensitivity is called the $\mathcal{H}_-$ index and defined as:

$$||G||_-^{[0,\infty)} := \inf_{\omega \in [0,\infty)} \underline{\sigma}\left[G\right], \tag{5}$$

with $\underline{\sigma}$ denoting the minimal singular value. Note that, at the presence of transmission zeros the $G(s)$ system has zero output despite that the transfer function matrix itself is not zero. This shows that the $\mathcal{H}_-$ index is not a norm, as it fails to satisfy certain norm properties [10]. There are also different algorithms for the computation of the $\mathcal{H}_-$ index, which are presented next.

B. $\mathcal{H}_-$ index over infinite frequency range

For stable LTI systems, the following optimization problem was presented in [6] for the computation of the minimum sensitivity:

Lemma 2.1: Let $\beta > 0$ be a constant scalar, and denote the system given in (1) by G. Then $||G||_-^{[0,\infty)} > \beta$, if and only if there exists a symmetric matrix P such that

$$\begin{bmatrix} PA + A^T P + C^T C & PB + C^T D \\ (PB + C^T D)^T & D^T D - \beta^2 I \end{bmatrix} \succ 0. \tag{6}$$

Proof: The proof can be found in [6]. ∎

Note that (6) has a similar structure as the BRL, without the additional restriction on the definiteness of the matrix variable P.

The authors in [7] proposed an alternative, yet similar definition for the minimum gain of a system, defined as follows:

Definition 2.2: A causal system $\mathcal{G} : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$, has minimum gain $0 \leq \beta \leq \infty$ if there exists ν, depending only on the initial conditions, such that

$$||Gu||_{2T} - \beta||u||_{2T} \geq \nu, \ \forall u \in \mathcal{L}_{2e}, \ \forall T \in \mathbb{R}^+. \tag{7}$$

For LTI systems, an LMI-based computation has also been derived in [7], which is referred as the 'Minimum Gain Lemma':

Lemma 2.3: The LTI system given in (1) has minimum gain $0 \leq \beta \leq \infty$ if there exists $P = P^T \succeq 0$ such that

$$\begin{bmatrix} PA + A^T P - C^T C & PB - C^T D \\ (PB - C^T D)^T & \beta^2 I - D^T D \end{bmatrix} \preceq 0. \tag{8}$$

Proof: The detailed proof can be found in [7] and hence omitted here. ∎

Nevertheless, a few remarks have to be given regarding the Minimum Gain Lemma. First, (6) is restricted to stable plants, while the definition and the computation of the minimum gain extends to unstable systems as well. Second, the resulting LMI constraints are structurally similar and connected. In order to show this, we borrow the argument presented in [15]. In particular, [15] is using an auxiliary description for unstable (sub)systems, which is defined by $\tilde{G} = (-A, -B, C, D)$. The time-domain interpretation of the auxiliary system is given by reversing the time variable t. For this $t = \tau$ is introduced and the signals are rewritten: $\tilde{x}(\tau) = x(-t)$. For the computation of the unstable Gramians in [15], it is then showed that they are the solution of a minimal energy problem for the corresponding auxiliary system. What is interesting for our case is that the $\mathcal{H}_-$ index for an unstable system can be computed by using the same arguments and the auxiliary description. Namely: following the same train of thoughts (6) yields to (8) for unstable systems.

C. $\mathcal{H}_-$ index over a finite frequency range

The computation of the minimum sensitivity can be carried out also over a finite frequency range $[\underline{\omega}, \bar{\omega}]$ by the aid of the Generalized Kalman-Yakubovic-Popov lemma, as discussed in [8]. This is summarized in the following lemma:

Lemma 2.4: Consider the LTI system in (1). Let $\Theta = \begin{bmatrix} -I & 0 \\ 0 & \beta^2 I \end{bmatrix} \in \mathbb{R}^{(n_x+n_y)\times(n_x+n_y)}$ and $\underline{\omega}$, $\bar{\omega}$ denote the minimum and maximum frequencies respectively in the interested frequency range, with $\tilde{\omega} = \frac{\underline{\omega}+\bar{\omega}}{2}$. Then $||G||_-^{[\underline{\omega},\bar{\omega}]} > \beta$ if and only if there exists Hermitian P and Q, with $Q \succ 0$ satisfying

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \Xi \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^T \Theta \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \prec 0, \tag{9}$$

where $\Xi = \begin{bmatrix} -Q & P + j\frac{\tilde{\omega}}{2}Q \\ P - j\frac{\tilde{\omega}}{2}Q & -\underline{\omega}\bar{\omega}Q \end{bmatrix}$.

Proof: The proof is available in [8], [9]. ∎

Remark 2.5: [16] shows that in Theorem 3., that Lemma 2.4 holds for all solutions of (1) with $u \in \mathcal{L}_2$ such that

$$\int_0^\infty \left(-\dot{x}\dot{x}^T + i\tilde{\omega}x\dot{x}^T - i\tilde{\omega}\dot{x}x^T - \underline{\omega}\bar{\omega}xx^T\right) dt \geq 0. \tag{10}$$

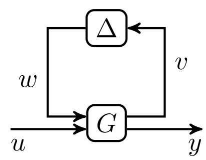

Fig. 1: LFT description of an uncertain system

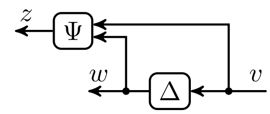

Fig. 2: Graphical interpretation of an IQC

This means that the system (1) possesses a β minimum gain for input signals with spectral contents in the targeted $\begin{bmatrix}\underline{\omega} & \bar{\omega}\end{bmatrix}$ frequency range.

Remark 2.6: In [9] the authors also present the LMI formulation of the maximum sensitivity (the $\mathcal{H}_\infty$ norm), by setting $\Theta = \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix}$.

Remark 2.7: Note also that, by $Q = 0$, (9) yields the LMI condition of (6) for the whole frequency range (which was shown to be equivalent to (8)).

III. Robust Minimum Gain

Having introduced the definition and various computational aspects of the minimum gain, now we extend these results for uncertain dynamical systems. For this purpose, our starting point will be a generic LFT interconnection of a nominal LTI plant G and the perturbation block Δ, as illustrated in Figure 1. The interconnection is denoted by $\mathcal{F}_u(G, \Delta)$ and can be computed by using the upper LFT of the two blocks.

Using this setting, our aim is to compute the minimum gain of $\mathcal{F}_u(G, \Delta)$, from the input u to the output y, i.e.:,

$$||G||_{\Delta-} = \inf||\mathcal{F}_u(G, \Delta)||. \tag{11}$$

A crucial point in the analysis and synthesis of uncertain dynamics is the available knowledge regarding the perturbation block Δ. Generally, the exact description of Δ is unknown, but some assumptions can be given. Then (11) has to be evaluated over all the possible uncertainties satisfying the assumptions.

Among the different uncertainty handling methodologies, the Integral Quadratic Constraint (IQC) based framework received the most attention due to the fact that numerous dynamical components (e.g. norm-bounded or polytopic uncertainty, time delay, saturation, various types of nonlinearities, etc.) can be covered by this formalism. The basic idea in the IQC framework is that the input and output signals of the uncertainty satisfy an integral formula. We follow the terminology of [5], but the interested reader is referred to [3], [4] for a more detailed presentation and discussion about IQCs.

The signals $v \in \mathcal{L}_2^{n_v}$, $w \in \mathcal{L}_2^{n_w}$ in the interconnection depicted in Figure 2 are satisfying the IQC defined by Π if

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix} d\omega \leq 0 \tag{12}$$

in the frequency domain, where $\hat{v}$ and $\hat{w}$ are the Fourier transforms of v, and w respectively. A time-domain alternative is constructed by calculating a $(\Psi,\ M)$ factorization of Π, where $M \in \mathbb{S}^{n_z}$ and $\Psi \in \mathbb{RH}_\infty^{n_z \times (n_v + n_w)}$ is a stable invertible linear system with the following frequency domain realization:

$$\Psi(j\omega) := C_\Psi(j\omega I - A_\Psi)^{-1} \begin{bmatrix} B_{\Psi v} & B_{\Psi w} \end{bmatrix} + \begin{bmatrix} D_{\Psi v} & D_{\Psi w} \end{bmatrix}. \tag{13}$$

The state-space representation of Ψ is:

$$\begin{aligned} \dot{x}_\Psi &= A_\Psi x_\Psi(t) + B_{\Psi v} v(t) + B_{\Psi w} w(t), \\ z(t) &= C_\Psi x_\Psi(t) + D_{\Psi v} v(t) + D_{\Psi w} w(t). \end{aligned} \tag{14}$$

This $(\Psi,\ M)$ factorization allows to express (12) in the time domain as

$$\int_0^\infty z(t)^T M z(t) dt \geq 0. \tag{15}$$

This factozization is called a soft IQC factorization.

If $\Pi \in \mathbb{RL}_\infty$ can be factorized as $\Pi = \tilde{\Psi} M \Psi$ where $\{\tilde{\cdot}\}$ denotes the para-Hermitian conjugate, then (Ψ, M) is a hard factorization of Π and

$$\int_0^T z(t)^T M z(t) dt \geq 0. \tag{16}$$

Throughout the paper we use hard factorization. Hard and soft IQC factorizations are discussed in [3], and [4] in more details. Furthermore if Δ satisfies an IQC constraint given by its hard factorization $(\Psi,\ M)$, then it will be denoted by $\Delta \in \text{IQC}(\Psi,\ M)$.

A. Robust Minimum Gain over the entire frequency domain

We are now in the position to derive analysis conditions for the robust minimum gain over the entire frequency domain. The discussion closely follows the results presented in [5] corresponding to the worst-case gain calculation. The system interconnection used for the analysis is shown in Figure 3, with the extended dynamics written in state-space:

$$\begin{aligned} \dot{x} &= Ax + B_w w + B_u u, \\ z &= C_z x + D_{zw} w + D_{zu} u, \\ y &= C_y x + D_{yw} w + D_{yu} u, \end{aligned} \tag{17}$$

where the $x = \begin{bmatrix} x_G^T & x_\Psi^T \end{bmatrix}^T$ state vector is composed of the states of the G system and the Ψ filter. The signal w is treated as an external signal and (16) is used for replacing the $w = \Delta(z)$ relationship.

Then, the following lemma provides the computation of the robust minimum gain over the entire frequency domain.

Theorem 3.1: Assume that $\mathcal{F}_u(G, \Delta)$ is well posed for all $\Delta \in \text{IQC}(\Psi, M)$, and the interconnection is stable. Then the minimum gain is finite and larger than β, if there exists a

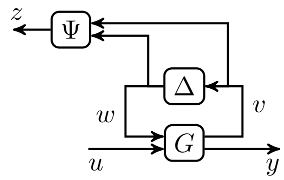

Fig. 3: Analysis Interconnection Structure

$P = P^T$ and $\lambda > 0$ such that

$$\begin{bmatrix} PA + A^T P & PB_w & PB_u \\ B_w^T P & 0 & 0 \\ B_u^T P & 0 & \beta^2 I \end{bmatrix} + \lambda \begin{bmatrix} C_z^T \\ D_{zw}^T \\ D_{zu}^T \end{bmatrix} M \left[\star\right] - \begin{bmatrix} C_y^T \\ D_{yw}^T \\ D_{yu}^T \end{bmatrix} \left[\star\right] \prec 0. \tag{18}$$

is satisfied.

Proof: We start by repeating the definition of the minimum gain from Definition 2.2, where it is given as

$$||Gu||_{2T} - \beta ||u||_{2T} \geq \nu, \; \forall u \in \mathcal{L}_{2e}, \; \forall T \in \mathbb{R}^+. \tag{19}$$

Rewriting the lefthand side gives

$$||y||_{2T}^2 - \beta^2 ||u||_{2T}^2 = \int_0^T \left(|y|^2 - \beta^2 |u|^2 \right) dt. \tag{20}$$

The integral term in (20) can be trivially extended with the storage function and the IQC condition as:

$$\int_0^T \Big(|y|^2 - \beta^2 |u|^2 + \frac{d}{dt}(x^T P x) - \frac{d}{dt}(x^T P x) + + \lambda z^T M z - \lambda z^T M z \Big) \, dt. \tag{21}$$

After introducing the following notations by using the state space representation in (17):

$$\Gamma_1 = \begin{bmatrix} PA + A^T P & PB_w & PB_u \\ B_w^T P & 0 & 0 \\ B_u^T P & 0 & 0 \end{bmatrix}, \; \Gamma_2 = \begin{bmatrix} C_z^T \\ D_{zw}^T \\ D_{zu}^T \end{bmatrix} M \left[\star\right], \; \Gamma_3 = \begin{bmatrix} C_y^T \\ D_{yw}^T \\ D_{yu}^T \end{bmatrix} \left[\star\right], \; \Gamma_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \beta^2 I \end{bmatrix}, \tag{22}$$

and re-arranging the terms we get:

$$\int_0^T \begin{bmatrix} x \\ w \\ u \end{bmatrix}^T (-\Gamma_1 - \lambda \Gamma_2 + \Gamma_3 - \Gamma_4) \begin{bmatrix} x \\ w \\ u \end{bmatrix} dt + + \int_0^T \begin{bmatrix} x \\ w \\ u \end{bmatrix}^T (\Gamma_1 + \lambda \Gamma_2) \begin{bmatrix} x \\ w \\ u \end{bmatrix} dt. \tag{23}$$

If one enforces the first term to be positive (i.e. $[-\Gamma_1 - \lambda\Gamma_2 + \Gamma_3 - \Gamma_4] \succ 0$), then by neglecting the first integral a lower approximation of the $||y||_{2T}^2 - \beta^2 ||u||_{2T}^2$ term in (20) is obtained, i.e.:

$$||Gu||_{2T}^2 - \beta^2 ||u||_{2T}^2 \geq \int_0^T \begin{bmatrix} x \\ w \\ u \end{bmatrix}^T (\Gamma_1 + \lambda \Gamma_2) \begin{bmatrix} x \\ w \\ u \end{bmatrix} dt, \tag{24}$$

where the integral's value is:

$$\int_0^T \begin{bmatrix} x \\ w \\ u \end{bmatrix}^T (\Gamma_1 + \lambda \Gamma_2) \begin{bmatrix} x \\ w \\ u \end{bmatrix} dt = = -x^T(0) P x(0) - \lambda z^T(0) M z(0). \tag{25}$$

Note that the latter is finite, therefore the system possesses a finite minimum gain. At the same time, the technical condition on the positive definiteness of $-\Gamma_1 - \lambda\Gamma_2 + \Gamma_3 - \Gamma_4$ can be easily verified as the LMI condition in (18).

■

Remark 3.2: The worst case induced $\mathcal{L}_2$ gain, $||G||_{\Delta\infty} > \gamma$ can be calculated by replacing $\beta^2 I$ by $-\gamma^2 I$, and changing the sign of the last term (corresponding to $y^T y$) to + in (18). For more details we refer to [5]. Note that [17] showed that this LMI condition can be satisfied by an indefinit $P = P^T$ as well.

Remark 3.3: As [5] shows, the presented method allows the treatment of several uncertainties in the analysis problem. In this case Δ has a block diagonal structure with $\Delta = \text{diag}\{\Delta_1, ..., \Delta_N\}$, where each block satisfies a corresponding IQC constraint (Ψ_k, M_k). These Ψ_k filters are connected to the v_k and w_k signals corresponding to Δ_k and generate the z_k virtual outputs. The second term in (18) then has to be modified to

$$\sum_{k=1}^{N} \lambda_k \begin{bmatrix} C_{zk}^T \\ D_{zwk}^T \\ D_{zuk}^T \end{bmatrix} M_k \left[\star\right], \tag{26}$$

with $\lambda_k \geq 0$. The conservativeness of the analysis test can be reduced by using several IQCs for the same uncertainty block in a similar fashion.

B. Robust minimum gain over finite frequency range

So far we have been assuming that the interconnected system is proper and possesses a direct feed-through term from u to y. However, it is possible to calculate the minimum gain for systems where this condition is not fulfilled by the aid of the Generalized Kalman-Yakubovic-Popov (GKYP) lemma [8], [9]. In this case the minimum sensitivity is computed over a selected frequency range of interest. The following lemma extends the previous results for systems without direct feed-through:

Theorem 3.4: Assume that $\mathcal{F}_u(G, \Delta)$ is well posed for all $\Delta \in \text{IQC}(\Psi, M)$. Let $\underset{\sim}{\omega}$, $\bar{\omega}$ denote the minimum and maximum frequencies respectively in the interested frequency range, with $\tilde{\omega} = \frac{\underset{\sim}{\omega} + \bar{\omega}}{2}$. Then $||\mathcal{F}_u(G, \Delta)||_{\Delta-} > \beta$ if there exists a Hermitian P, Q and real $\lambda > 0$ such that $Q \succ 0$ and

$$\begin{bmatrix} A & B_w & B_u \\ 0 & 0 & 0 \\ I & 0 & 0 \end{bmatrix}^T \Xi \left[\star\right] + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \beta^2 I \end{bmatrix} + \lambda \begin{bmatrix} C_z^T \\ D_{zw}^T \\ D_{zu}^T \end{bmatrix} M \left[\star\right] - \begin{bmatrix} C_y^T \\ D_{yw}^T \\ D_{yu}^T \end{bmatrix} \left[\star\right] \prec 0, \tag{27}$$

where $\Xi = \begin{bmatrix} \Phi_{11}Q & 0 & P+\Phi_{12}Q \\ 0 & 0 & 0 \\ P+\Phi_{21}Q & 0 & \Phi_{22}Q \end{bmatrix}$, with $\Phi = \begin{bmatrix} -1 & j\tilde{\omega} \\ -j\tilde{\omega} & -\underline{\omega}\bar{\omega} \end{bmatrix}$.

Proof: In the proof we work in a truncated signal space, where for all $T \in \mathbb{R}^+$ the signal $y_T(t) = y(t)$ for $0 \le t \le T$ and $y_T(t) = 0$ for $t \ge T$. Multiplying the inequality in (27) by $\begin{bmatrix} x^T & w^T & u^T \end{bmatrix}$ from the left and by $\begin{bmatrix} x^T & w^T & u^T \end{bmatrix}^T$ from the right gives

$$\frac{d}{dt}(x^T P x) + \beta^2 u^T u + \lambda z^T M z - y^T y + \\ + \Phi_{11}\dot{x}^T Q\dot{x} + \Phi_{12}\dot{x}^T Q x + \Phi_{21} x^T Q \dot{x} + \Phi_{22} x^T Q x < 0. \tag{28}$$

This can be integrated along the state trajectory from $t = 0$ to $t = T$:

$$\begin{aligned} &- x(0)^T P x(0) + \beta^2 \int_0^T u(t)^T u(t) dt + \\ &+ \lambda \int_0^T z(t)^T M z(t) dt - \int_0^T y^T(t) y(t) dt + \\ &+ \int_0^T \left(\Phi_{11}\dot{x}^T Q\dot{x} + \Phi_{12}\dot{x}^T Q x + \Phi_{21} x^T Q\dot{x}\right) dt + \\ &+ \int_0^T \left(\Phi_{22} x^T Q x\right) dt < 0. \end{aligned} \tag{29}$$

It follows from the IQC condition (16) that

$$\begin{aligned} &- x(0)^T P x(0) - \lambda z(0)^T M z(0) + \\ &\beta^2 \int_0^T u(t)^T u(t) dt - \int_0^T y^T(t) y(t) dt + \\ &+ \mathrm{tr}\left[Q \textstyle\int_0^T \left(\Phi_{11}\dot{x}^T \dot{x} + \Phi_{12}\dot{x}^T x\right) dt\right] + \\ &+ \mathrm{tr}\left[Q \textstyle\int_0^T \left(\Phi_{21} x^T \dot{x} + \Phi_{22} x^T x\right) dt\right] < 0. \end{aligned} \tag{30}$$

Note that due to the truncated signal space $x_T(t) = 0\ \forall t > T$, and so the $x^T(T)Px(T)$ term can be omitted. Since $Q \succ 0$ and because we suppose that, the u input signals satisfy condition (10), the tr$[\cdot]$ term is nonnegative, and we have

$$\begin{aligned} &- x(0)^T P x(0) - \lambda z^T(0) M z(0) < \\ &< \int_0^T y^T(t) y(t) dt - \beta^2 \int_0^T u(t)^T u(t) dt \end{aligned} \tag{31}$$

what completes the proof. ■

IV. Numerical Example

Longitudinal control law design for fixed-wing airplanes involves a normal acceleration feedback loop, as it is shown in [18]. Optimizing the aircraft handling qualities requires precise knowledge of achievable transfer capabilities in this loop, even if the available knowledge of the system components is uncertain to some degree. In this example we apply the previous results on an elevator to a_z normal acceleration transfer function, corresponding to a fixed-wing aircraft. The model is taken from [19], and it describes the Aerosonde UAV in a trimmed straight and level flight at 33 m/s. The corresponding state space model is given by

$$G = \left[\begin{array}{cccc|c} -0.68 & 0.07 & -0.46 & -9.81 & -0.14 \\ -0.55 & -2.98 & 33 & -0.14 & 10.13 \\ p_1 & p_2 & -0.66 & 0 & -31.78 \\ 0 & 0 & 1 & 0 & 0 \\ \hline -0.55 & -2.98 & 0 & 0 & 10.13 \end{array}\right], \tag{32}$$

with $x^T = \begin{bmatrix} u & w & q & \theta \end{bmatrix}^T$ corresponding to the longitudinal and vertical speeds in the body frame, the pitch rate and the pitch angle respectively.

A detailed description of how various aerodynamic and structural parameters affect the state space matrices is given in Chapter 5. of [19]. A careful inspection of those equations reveal that the $C_{m\alpha}$ longitudinal static stability derivative affects the p_1, p_2 entries in A. Modeling a $\pm 5\%$ inaccuracy in $C_{m\alpha}$, leads to the $\begin{bmatrix} p_1 & p_2 \end{bmatrix} = \begin{bmatrix} 0.01 & -0.73 \end{bmatrix} + \begin{bmatrix} 0.0005 & -0.0366 \end{bmatrix} \delta$ parametric uncertainty description, by $|\delta| \le 1$.

For describing the effects of unmodelled dynamics, an input multiplicative uncertainty is appended to the system as $G_p(s) = G(s)(1 + \Delta(s) W_m(s))$, with $|\Delta(s)| \le 1$ and

$$W_m(s) = \frac{s + 0.7653}{1.053s + 2.551}. \tag{33}$$

This allows for 30 percent uncertainty at low, and 95 percent uncertainty at high frequency.

These lead to a blockdiagonal uncertainty structure in the form of $\mathbf{\Delta} = \left\{\mathrm{diag}\left\{\delta,\ \Delta\right\} \in \mathbb{C}^{n_w \times n_v}, \delta \in \mathbb{R}, \Delta \in \mathbb{C}\right\}$. The next question is the appropriate filter selection for the various type of uncertainties.

For the case when Δ is a dynamic LTI uncertainty, with $||\Delta(s)||_\infty < 1$, [3] proposes an IQC multiplier in the form

$$\Pi_d(j\omega) = \begin{bmatrix} x(j\omega)I & 0 \\ 0 & -x(j\omega)I) \end{bmatrix}. \tag{34}$$

A hard factorization of (34), was used for the analysis problem, which is given as

$$M_d = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ \Psi_d(s) = \begin{bmatrix} \frac{s+10.2}{s+5.102} & 0 \\ 0 & \frac{s+10.2}{s+5.102} \end{bmatrix}. \tag{35}$$

For $|\delta| \le 1$ parametric uncertainties [3] suggests a multiplier in the form

$$\Pi_p(j\omega) = \begin{bmatrix} X(j\omega) & Y(j\omega) \\ Y(j\omega)^* & -X(j\omega) \end{bmatrix}, \tag{36}$$

where $X(j\omega) = X(j\omega)^* \ge 0$ and $Y(j\omega) = -Y(j\omega)^*$

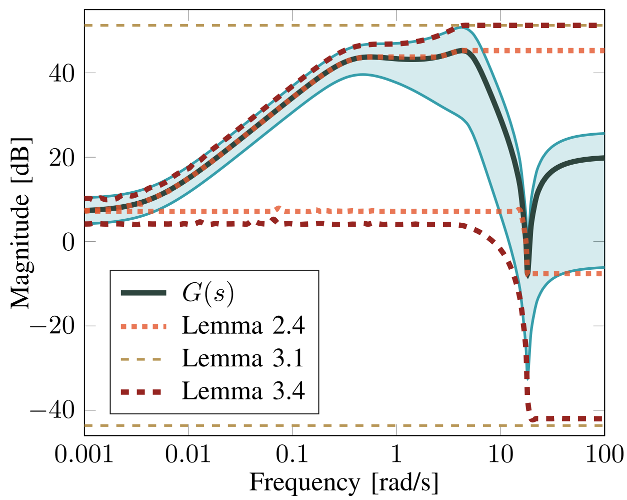

Fig. 4: Robust sensitivity analysis example

are bounded and measurable matrix functions. A hard factorization of Π_p is provided by its J-spectral factorization, described in the appendix B of [5]. By selecting $\Pi_p(j\omega)$ as

$$\Pi_p(s) = \begin{bmatrix} \frac{51.02}{s+51.02} & \frac{s}{s+0.04017} \\ \frac{s}{s-0.04017} & \frac{-51.02}{s+51.02} \end{bmatrix}, \tag{37}$$

it's J-spectral factorization leads to

$$\begin{aligned} M_p &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \\ \Psi_p(s) &= \begin{bmatrix} \frac{0.70711(s+10.57)}{s+5.102} & \frac{0.70711(s+2.759)}{(s+5.102)} \\ \frac{-70711(s+2.759)}{(s+5.102)} & \frac{0.70711(s+10.57)}{(s+5.102)} \end{bmatrix}. \end{aligned} \tag{38}$$

In the forthcoming sensitivity analysis (35) and (38) were used for describing the model uncertainties.

The nominal $G(s)$ system is shown in Figure 4, along with a shaded area where the $G_p(s)$ perturbed plant can take its values. The upper bound of this area was found by the *wcgain*, worst case gain computing MATLAB function. The theoretical lower bound was found by the worst case gain of the $G_p^{-1}(s)$ inverse system.

Lemma 2.4 is used to calculate the minimum and maximum sensitivities of the $G(s)$ nominal system over finite frequency ranges. Dotted lines show their calculated values when the investigated frequency range was increased from $[0\ 10^{-3}]$ to $[0\ 10^{2}]$ rad/s in 100 steps.

Theorem 3.4 allows the calculation of $||G||_{\Delta-}$ and $||G||_{\Delta\infty}$ over a finite frequency range. The upper bound of the frequency range was again increased from 10^{-3} to 10^{2} rad/s in 100 steps.

Theorem 3.1 and Theorem 3.4 gives the same result for the $[0 \quad \infty]$ frequency range. However if $G(s)$ would be just proper (with zero gain at high frequency), then only Theorem 3.4 could be applied over a finite frequency range to calculate the minimum sensitivity.

V. Conclusion

The paper presented a convex, robust minimum sensitivity analysis approach relying on Integral Quadratic Constraints. It was shown that the method is a direct extension of the Minimum Gain Lemma to systems containing uncertain elements. By applying the GKYP lemma it was possible to further generalize the results to analyze the minimum sensitivity over a certain frequency range. A simple numerical example was presented to show the potential of the proposed approach. This example involved a simple dynamic LTI uncertainty. However by selecting suitable IQC multipliers other types of uncertainties or nonlinearities might be incorporated into the analysis as well.

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