Efficient arithmetic on elliptic curves using a mixed Edwards–Montgomery representation

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Abstract. From the viewpoint of x-coordinate-only arithmetic on elliptic curves, switching between the Edwards model and the Montgomery model is quasi cost-free. We use this observation to speed up Montgomery's algorithm, reducing the complexity of a doubling step from $2\mathbf{M} + 2\mathbf{S}$ to $1\mathbf{M} + 3\mathbf{S}$ for suitably chosen curve parameters.

1 Montgomery's algorithm

Aiming for an improved performance of Lenstra's elliptic curve factorization method [6], Montgomery developed a very efficient algorithm to compute in the group associated to an elliptic curve over a non-binary finite field $\mathbb{F}_q$, in which only x-coordinates are involved [8].

The algorithm also proves useful for point compression in elliptic curve cryptography. More precisely, instead of sending a point as part of some cryptographic protocol, one can reduce the communication cost by sending just its x-coordinate. From this, the receiver can compute the x-coordinate of any scalar multiple using Montgomery's method. This idea was first mentioned in [7].

The type of curves Montgomery considered are of the following non-standard Weierstrass type

$$M_{A,B} : \; By^2 = x^3 + Ax^2 + x, \quad A \in \mathbb{F}_q \setminus \{\pm 2\}, \, B \in \mathbb{F}_q \setminus \{0\},$$

which is now generally referred to as a *Montgomery form*. His method works as follows. Let $P = (x_1, y_1, z_1)$ be a point on $\overline{M}_{A,B}$, the projective closure of $M_{A,B}$, and for any $n \in \mathbb{N}$ write $n \cdot P = (x_n, y_n, z_n)$, where the multiple is taken in the algebraic group $\overline{M}_{A,B}, \oplus$ with neutral element $O = (0, 1, 0)$. Then the following recursive relations hold: for any $m, n \in \mathbb{N}$ such that $m \neq n$ we have

$$\begin{aligned} x_{m+n} &= z_{m-n} \left((x_m - z_m)(x_n + z_n) + (x_m + z_m)(x_n - z_n) \right)^2, \\ z_{m+n} &= x_{m-n} \left((x_m - z_m)(x_n + z_n) - (x_m + z_m)(x_n - z_n) \right)^2. \end{aligned} \qquad \text{(ADD)}$$

and

$$
\begin{aligned}
4x_n z_n &= (x_n + z_n)^2 - (x_n - z_n)^2, \\
x_{2n} &= (x_n + z_n)^2 (x_n - z_n)^2, \\
z_{2n} &= 4x_n z_n \left((x_n - z_n)^2 + ((A+2)/4)\,(4x_n z_n)\right)
\end{aligned}
\tag{DOUBLE}
$$

(see also [3]). One can then compute $((x_n, z_n), (x_{n+1}, z_{n+1}))$ from

$$\left((x_{(n \text{ div } 2)}, z_{(n \text{ div } 2)}), (x_{(n \text{ div } 2)+1}, z_{(n \text{ div } 2)+1})\right)$$

by one application of (ADD) and one application of (DOUBLE), the input of the latter depending on n mod 2. Thus approximately $\log_2 n$ applications of (ADD) and (DOUBLE) suffice to recover (x_n, z_n).

Every application of (ADD) has a rough time-cost of $3\mathbf{M} + 2\mathbf{S}$, where $\mathbf{M}$ is the time needed to multiply two general elements of $\mathbb{F}_q$, and $\mathbf{S}$ is the time needed to square a general element (which is typically faster). Here we used that $z_1 = 1$ in practice. Every application of (DOUBLE) needs $2\mathbf{M} + 2\mathbf{S} + 1\mathbf{C}$, where $\mathbf{C}$ is the cost of multiplication of a general element of $\mathbb{F}_q$ with a curve constant. In this case, the constant is $(A+2)/4$ (hence, if A is chosen carefully then $\mathbf{C}$ may be much less than $\mathbf{M}$).

2 Switching to Edwards curves and back

Following recent work of Edwards [4], Bernstein and Lange [2] proved that the elliptic curves

$$E_d : X^2 + Y^2 = 1 + dX^2Y^2 \quad d \in \mathbb{F}_q \setminus \{0, 1\}$$

allow a very esthetic description of the algebraic group law on $\overline{E}_d$, the (desingularized) projective closure of E_d, with $O = (0, 1) \in E_d \subset \overline{E}_d$ as neutral element. Namely, the formula

$$(X_1, Y_1) \oplus (X_2, Y_2) = \left(\frac{X_1Y_2 + Y_1X_2}{1 + dX_1X_2Y_1Y_2}, \frac{Y_1Y_2 - X_1X_2}{1 - dX_1X_2Y_1Y_2}\right)$$

holds at all affine point pairs for which the above denominators are nonzero. The curve E_d is said to be in *Edwards form*. In [1, Theorem 3.2.] it is proven that every Edwards form is birationally equivalent to a Montgomery form via

$$
\begin{aligned}
\varphi &: M_{\frac{2(1+d)}{1-d}, \frac{4}{1-d}} \dashrightarrow E_d : (x, y) \mapsto \left(\tfrac{x}{y}, \tfrac{x-1}{x+1}\right), \\
\psi &: E_d \dashrightarrow M_{\frac{2(1+d)}{1-d}, \frac{4}{1-d}} : (X, Y) \mapsto \left(\tfrac{1+Y}{1-Y}, X\tfrac{1+Y}{1-Y}\right).
\end{aligned}
$$

The dashed arrows indicate that the maps are not defined everywhere. However, the maps can be extended to give an everywhere-defined isomorphism between the respective (desingularized) projective models

$$\overline{M}_{\frac{2(1+d)}{1-d}, \frac{4}{1-d}} \longrightarrow \overline{E}_d$$

that maps the neutral elements O to each other. In particular, wherever φ and ψ are defined, they commute with the group structures on $\overline{M}_{\frac{2(1+d)}{1-d}, \frac{4}{1-d}}$ and $\overline{E}_d$.

Now the Y-coordinate of $\varphi(x, y)$ only depends on x, and conversely the x-coordinate of $\psi(X, Y)$ only depends on Y. In projective coordinates this correspondence becomes remarkably simple:

$$\varphi : (x, z) \mapsto (x - z, x + z) \quad \text{and} \quad \psi : (Y, Z) \mapsto (Z + Y, Z - Y).$$

Therefore, from the x/Y-coordinate-only viewpoint, switching between Edwards curves and Montgomery curves is quasi cost-free. As a consequence, one is free to pick the best from either world. In the next section we show that it is worth considering the (DOUBLE) step in the Edwards setting.

3 Y-coordinate-only doubling on Edwards curves

A general affine point (X, Y) on E_d doubles to a point whose second coordinate equals

$$\frac{Y^2 - X^2}{1 - dX^2Y^2} = \frac{Y^2(1 - dY^2) - (1 - Y^2)}{(1 - dY^2) - dY^2(1 - Y^2)} = \frac{-1 + 2Y^2 - dY^4}{1 - 2dY^2 + dY^4}.$$

Here we used the curve equation $X^2+Y^2 = 1+dX^2Y^2$. Therefore the (DOUBLE) analog becomes

$$\begin{aligned} Y_{2n} &= -Z_n^4 + 2Y_n^2Z_n^2 - dY_n^4 = -(Z_n^4 + dY_n^4) + 2Y_n^2Z_n^2, \\ Z_{2n} &= Z_n^4 - 2dY_n^2Z_n^2 + dY_n^4 \;= (Z_n^4 + dY_n^4) - 2dY_n^2Z_n^2. \end{aligned}$$

Suppose that d has a square root $\sqrt{d}$ in $\mathbb{F}_q$. Then the above step can be done using $1\mathbf{M} + 3\mathbf{S} + 3\mathbf{C}$ by computing

$$Y_n^2, \quad Z_n^2, \quad Y_n^2Z_n^2, \quad \sqrt{d}Y_n^2, \quad \sqrt{d}Y_n^2Z_n^2, \quad dY_n^2Z_n^2, \quad (Z_n^2 + \sqrt{d}Y_n^2)^2$$

and then recovering $Z_n^4 + dY_n^4$ as $(Z_n^2 + \sqrt{d}Y_n^2)^2 - 2\sqrt{d}Y_n^2Z_n^2$. If d is nonsquare, one easily verifies that a time cost of $5\mathbf{S} + 2\mathbf{C}$ can be achieved.

4 Conclusion and additional remarks

To sum up, our proposal is to work with a Montgomery curve of the type $M_{\frac{2(1+d)}{1-d}, \frac{4}{1-d}}$, and to replace (DOUBLE) by

$$\begin{aligned} Y_n &= x_n - z_n \\ Z_n &= x_n + z_n \\ Y_{2n} &= -(Z_n^4 + dY_n^4) + 2Y_n^2Z_n^2 \\ Z_{2n} &= (Z_n^4 + dY_n^4) - 2dY_n^2Z_n^2 \\ x_{2n} &= Z_{2n} + Y_{2n} \\ z_{2n} &= Z_{2n} - Y_{2n}. \end{aligned}$$

These formulas are complete, in the sense that for *every* input (x_n, z_n) they give the correct output (x_{2n}, z_{2n}). This is in contrast with the switching maps φ and ψ

and with the Edwards doubling formulas. But under the above composition, the incompleteness disappears: this can be checked by directly expressing (x_{2n}, z_{2n}) in terms of (x_n, z_n) and verifying that – up to scalar multiplication by $-2d+2$ – it matches with classical Montgomery doubling.

If the curve constant d is a square such that multiplication by $\sqrt{d}$ is cheap, then the above method improves upon Montgomery doubling by roughly $\mathbf{M}-\mathbf{S}$, i.e. it replaces a multiplication by a squaring. Therefore, our simple ideas can serve in constructing slightly improved ECC protocols for devices with limited computational power and memory. We remark that an even better speed-up of $\mathbf{2M}-\mathbf{2S}$ has been independently[1] obtained by Gaudry and Lubicz [5], who work however on a Kummer line instead of directly on a Montgomery form.

Not every Montgomery form is birationally equivalent to an Edwards curve, but this is resolved by extending to the class of so-called *twisted* Edwards forms $aX^2+Y^2 = 1+dX^2Y^2$ $(a \neq d)$, as was pointed out in [1]. For this class, exactly the same ideas apply, resulting in a doubling algorithm using $1\mathbf{M}+3\mathbf{S}+6\mathbf{C}$ if ad is a square, and $5\mathbf{S}+4\mathbf{C}$ in general.

We end by recalling that the Edwards-Montgomery setting only covers non-binary fields. Over binary fields there is less need for arithmetic directly on compressed representations, since a received point can be typically decompressed by solving a quadratic equation, which is easy in characteristic two. The transmission of an extra bit then allows the decompressor to decide upon the correct solution.

References

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[1] This is an euphemistic rephrasing of our ignorance about Gaudry and Lubicz' result, which is somewhat hidden in a different framework. Its existence was pointed out to us by Dan Bernstein and Tanja Lange.