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REVIEW

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# Floquet metamaterials

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**Abstract**

Recent progress in nanophotonics and material science has inspired a strong interest in optically-induced material dynamics, opening new research directions in the distinct fields of Floquet matter and time metamaterials. Floquet phenomena are historically rooted in the condensed matter community, as they exploit periodic temporal drives to unveil novel phases of matter, unavailable in systems at equilibrium. In parallel, the field of metamaterials has been offering a platform for exotic wave phenomena based on tailored materials at the nanoscale, recently enhanced by incorporating time variations and switching as new degrees of freedom. In this Perspective, we connect these research areas and describe the exciting opportunities emerging from their synergy, hinging on giant wave-matter interactions enabled by metamaterials and on the exotic wave dynamics enabled by Floquet and parametric phenomena. We envision Floquet metamaterials in which nontrivial modulation dynamics, and their interplay with tailored material dispersion and nontrivial material properties such as anisotropy, non-Hermiticity and nonreciprocity, introduce a plethora of novel opportunities for wave manipulation and control.

**Keywords:** Floquet, Metamaterials, Time, Switching

## 1 Introduction

The large progress in nanofabrication techniques, combined with a better understanding of electromagnetics and material science at the nanoscale have been at the basis of the recent surge of interest around engineered materials with optical properties not available in nature. The initial interest in metamaterials was sparked by the quest for a negative index of refraction, motivated by the goal of achieving "perfect" lensing [1], but today it encompasses a much broader range of properties emerging from the nanostructure of suitably tailored artificial media with subwavelength features [2]. To date, this field of research has been stretching across various wave platforms, with implementations spanning optics and microwaves, acoustics, elasticity, seismic and water waves, also extending towards the quantum domain. The wealth of exotic wave phenomena enabled by this concept includes negative refraction (Fig. 1a) [1], epsilon-near-zero and other near-zero-index wave phenomena (Fig. 1b) [3, 4], cloaking [5, 6], extreme anisotropic (Fig. 1c) [7], bianisotropic and nonlocal responses [2] and topological phases [8], among several others. Metasurfaces, the planarized version of metamaterials, have enabled smaller footprints, ease of fabrication and lower losses [9, 10]. Furthermore, metasurfaces can be more efficiently pumped externally compared to bulky three-dimensional structures (Fig. 1d) [11], opening interesting opportunities in the realm of time metamaterials: engineered materials whose extreme optical features and light-matter interactions can be controlled and modulated in time and space, leading to new phenomena typically unavailable in time-invariant media, including frequency shifting, active beam steering and nonreciprocity (Fig. 1d) [11, 12], synthetic optical drag [13], Floquet topological effects (Fig. 1h) [13–17], and luminal amplification [18] among several others [19].

The concept of designer matter has also inspired significant research efforts in the condensed matter community, broadening the horizon of known phases of matter: the rise of graphene and other two-dimensional materials, followed by van der Waals stacking [20] and,

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more recently, twisted atomic layers [21], has been opening exciting frontiers for emerging material responses, with opportunities for ultra-low-footprint optoelectronics, moiré physics and conductor-to-insulator transitions [22]. Significant attention has also been raised by topological phases of matter hosting unusually robust edge transport [23]. In the framework of non-Hermitian and open systems, parity-time (PT) symmetric and broken phases have been also exciting both the classical and quantum communities [24]. Finally, many-body-localized phases have gained a spotlight following a long-stretching series of proofs of their existence [25]. Interestingly, several of these material phases, oftentimes envisioned in electronic systems, have been demonstrated in wave settings using suitably engineered metamaterials, by the means of optical and acoustic implementations mapping their tight-binding models into specific architected lattices. Several topological phases of matter, including some that had been predicted but never found in condensed matter systems, have been realized in classical wave settings through metamaterials, highlighting the potential for the crossroad between these two fields to serve as a catalyst for progress and experimental validation, as well as a fast-track for new concepts towards applications.

Out-of-equilibrium systems offer yet more opportunities for exotic condensed matter phenomena. Of particular relevance in this context has been the recent activity in Floquet matter, characterized by Hamiltonians that experience an explicit periodic temporal drive. This field, originally rooted in the study of dynamical systems, landmark examples being the Kapitza pendulum [26] and the kicked rotor model of classical and quantum chaos [27], has recently gained momentum, owing to recent demonstrations of optically pumped material systems with laser beams that can induce sizeable changes in their response properties (Fig. 1e–f), being it atomic lattices or solid-state crystals [29]. Floquet engineering has shown how tailored temporal modulations of materials can produce exotic phenomena, such as non-trivial topological phases (Fig. 1g) [30], also through the engineering of synthetic frequency dimensions (Fig. 1i) [31], exceptional points [32], localization [33] and even superconductivity [34]. The application of these concepts to many-body-localized systems has recently paved the way towards the experimental realization of time-crystals, stable phases of matter characterized by long-range temporal order [35].

As sketched in Fig. 1, the enabling power demonstrated by the metamaterial platform in channeling ideas from condensed matter physics towards wave phenomena suggests that new opportunities arise by bridging Floquet engineering and time metamaterials. In this context, this Perspective discusses opportunities for novel forms of light-matter interactions enabled by Floquet metamaterials. More specifically, Sect. 2 offers a brief introduction to the key concepts underlying Floquet physics. In Sect. 3 we zoom into the temporal variations of a Floquet metamaterial, to showcase the concept of time-domain meta-atoms, discussing some of the exotic phenomena emerging at time-interfaces in dispersive materials, and the major role played by the interplay between dispersion and temporal variations. We demonstrate how, once the timescales of the material response become comparable to the ones of Floquet temporal variations, a wealth of new wave phenomena can be enabled by controlling the modulation scheme and the underlying metamaterial dispersion. The interplay between tailored temporal nonlocality (frequency dispersion) and temporal modulations highlights several directions ahead for exotic wave phenomena induced by temporal drives and pumping of engineered materials, with open challenges and unique opportunities. Finally, in Sect. 4 we look at the confluence between Floquet physics and metamaterials, making the case for a broad spectrum of opportunities opened by blending Floquet engineering with metamaterials and nanophotonics into the rising field of Floquet metamaterials.

## 2 Floquet physics

As the temporal analogue of spatial Bloch theory for crystalline solid-state systems, Floquet phenomena emerge in systems characterized by a periodic dependence on time. The evolution of a phenomenon of interest, governed by Schrödinger-like dynamics of the form

$$\frac{\partial}{\partial t}\mathbf{\Psi}(t) = \hat{H}(t)\mathbf{\Psi}(t), \tag{1}$$

where $\mathbf{\Psi}$ is a general (vector or scalar) field and $\hat{H}$ is the Hamiltonian describing the system, is of Floquet-type if we can assume that the Hamiltonian has periodicity $T$, so that:

$$\hat{H}(t) = \hat{H}(t+T) = \sum_m \hat{H}_m e^{-im\Omega t}. \tag{2}$$

where the angular frequency $\Omega = 2\pi/T$ acts as a reciprocal lattice vector along the frequency dimension. In such a scenario, the temporal dependence of the fields can be generally expressed in the form

$$\mathbf{\Psi}(t) = e^{-i\omega t}\mathbf{\Phi}(t), \tag{3}$$

where $-\pi/T < \omega < \pi/T$ is an eigenvalue corresponding to the quasi-frequency, the temporal analogue of crystal momentum in periodic spatial structures, and the Floquet mode obeys

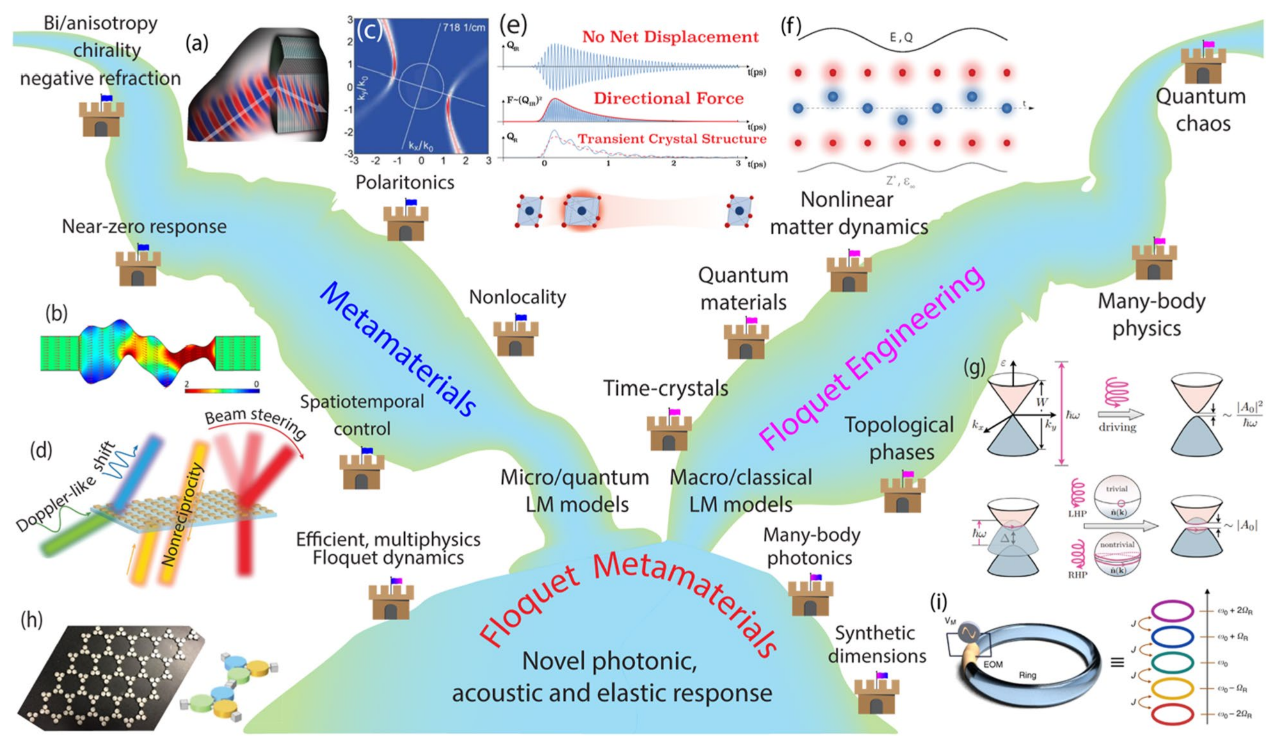


**Fig. 1** Schematic of the key features encompassed within the fields of Floquet engineering and metamaterials, as well as some of the potential outcomes of their combination to engineer novel forms of wave control. Panels: **a** Artistic rendition of negative refraction [36]; **b** Light funneling with epsilon-near-zero materials [4]; **c** Shear-polaritonic dispersion in the non-orthogonal anisotropic material β-Gallium Oxide [7]; **d** Doppler-like frequency shifting, beam steering and nonreciprocity enabled by spatiotemporal metasurfaces [10]; **e** Phonon vibrations induced by an infrared pump beam (top panel) result in a net directional force, responsible for a transient deformation of the crystal structure [29]; **f** An infrared pump beam periodically modulates the effective charge in a crystal lattice, such that the effective permittivity varies at twice the frequency, producing phonon-mediated parametric amplification [28]; **g** Topological transitions enabled by circularly polarized Floquet pumping [30]; **h** Experimental implementation of an elastic Floquet topological insulator [14, 15]; **i** An electro-optically modulated ring resonator hosts a Floquet ladder of states [31]. Insets adapted from the respective references

$$\mathbf{\Phi}(t) = \mathbf{\Phi}(t+T) = \sum_m \phi^{(m)} e^{-im\Omega t}, \tag{4}$$

following the same temporal periodicity of the Hamiltonian $\hat{H}(t)$.

Exploiting the Floquet ansatz, we can derive a time-independent eigenvalue problem as customary in solid-state physics:

$$\sum_m (\hat{H}_{n-m} - m\Omega\delta_{mn})\phi_{\boldsymbol{\alpha}}^{(\mathbf{m})} = \omega_\alpha \phi_{\boldsymbol{\alpha}}^{(\mathbf{n})}, \tag{5}$$

where $\alpha$ is the index of a specific eigenstate with quasi-frequency eigenvalue $\omega_\alpha$, and $m$ and $n$ are indexes denoting the Fourier order of the amplitudes $\phi_\alpha^{(m)}$ and $\phi_\alpha^{(n)}$ of the $\alpha^{\text{th}}$ Floquet mode, which can thus be computed in Fourier basis by direct diagonalization [37].

An effective way of interpreting Floquet systems is to consider how a periodic temporal drive introduces an effective ladder of states, which introduces a "synthetic" dimension to the problem. Each incommensurate frequency contributing to the temporal drive results in an additional dimension in the Hamiltonian. This is illustrated in Fig. 2a: the $\hat{H}_0$ term is responsible for hopping within a single frequency "sheet", as it may be induced by the crystal structure, external magnetic fields, etc., whereas the $\hat{H}_{n-m}$ terms are responsible for hopping between different frequency sheets [37]. This approach has been exploited in Floquet engineering to produce topological phenomena across a number of dimensions larger than those of the underlying spatial crystal [38]. One instance where the Floquet picture can be illuminating is its use in revealing the connection between Thouless pumping[1] in a 1D chain and the quantum Hall effect [39] in a 2D lattice, where one of the two dimensions is

[1] Thouless pumping refers to the quantized charge transport that can be induced in a crystal by periodically modulating hopping and on-site energy in e.g. the Rice-Mele model of a 1D topological consisting of intracell and intercell hopping, plus a staggered on-site potential [37].

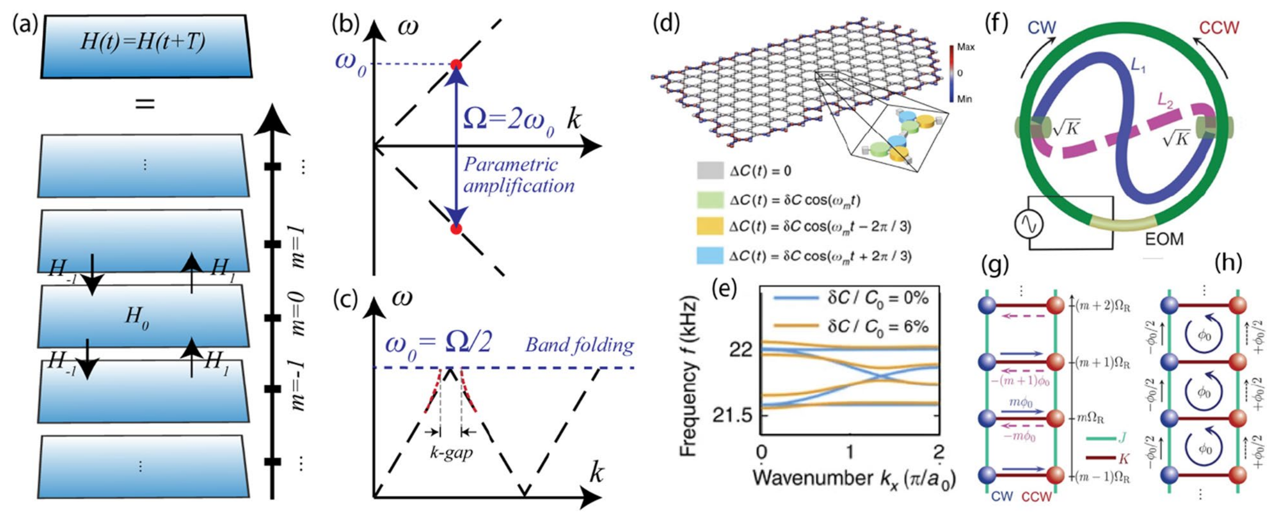


**Fig. 2** **a** A periodically modulated system can be equivalently modelled as a lattice of modes, with the non-zero frequency components of the Hamiltonian providing the coupling between the different frequency "sheets". **b** Periodically pumped systems exhibit parametric instabilities at those states which are separated from their time-reversal partner by a modulation frequency $\Omega = 2\omega_0$, where $\omega_0$ is the frequency of the amplified input wave. **c** These instabilities can be equivalently regarded as the result of the opening of k-gaps, hosting imaginary states which grow or decay exponentially over time. **d** Acoustic Floquet topological insulator formed by a Kagome lattice featuring a phased time-modulation of the acoustic capacitance $C$ at its three corner sites, producing an angular bias. **e** The introduction of the capacitance modulation results in the opening of a topological band-gap. An instance of edge state hosted within the topological gap is shown in panel (d). Adapted from [14]. **f** A ring resonator is temporally modulated by an electro-optic modulator (EOM), and its clockwise (CW) and counterclockwise (CCW) states are coupled by a pair of non-intersecting waveguides. **g**, **h** This setup produces a system with two synthetic dimensions, the Floquet ladder and the angular momentum basis, within a single ring-resonator. Adapted from [38]

replaced by the Floquet ladder. Another effect induced by a periodic pump in the same setup is the formation of Wannier-Stark ladders and Bloch oscillation[2] [40], which arises from the effective electric field produced by the temporal modulation in the high-modulation-frequency regime [37].

A periodic drive can generate new harmonics, and can also distort the band structure at the fundamental frequency. The generation of new harmonics can at the same time support parametric amplification (Fig. 2b), as a result of the resonant constructive interference between waves sharing the same momentum but with opposite (positive/negative) frequency. An analogous way of understanding parametric gain in the spirit of solid-state theory is the formation of a horizontal bandgap (also called a "k-gap") near the edge of the temporal Brillouin zone (Fig. 2c), where the bands "fold" back towards lower energies and higher momenta, mirroring their spatial counterpart. As opposed to the case of spatial periodicities, in the temporal case the evanescent states in the gap do not necessarily decay in space, but they can grow exponentially in time, yielding parametric amplification. Importantly, the resulting formation of a bandgap from the coupling between positive and negative frequencies in a Floquet system also implies significant alteration of the system response near these temporal high-symmetry points. In this direction, important progress has been recently reported on several fronts across nonlinear optics and photonics [29], whereby strong optical or infrared pulses are used to induce parametric gain [28], as well as to induce phase transitions, for instance between trivial and superconductive phases [34], between topologically inequivalent phases [41] and between phases characterized by widely different optical nonlinearities [42].

Recent implementations of Floquet systems have found fertile ground in acoustics, with the realization of Floquet topological insulators in Kagome lattices (Fig. 2d) [14] of acoustic resonators featuring a phased spatio-temporal modulation in the acoustic capacitance of their constituents (Fig. 2e). Furthermore, photonics has also been shown to offer great potential for the realization of synthetic frequency dimensions, with several works deploying electro-optical modulators to generate Floquet ladders in ring resonators [17, 43], also in conjunction with multiple synthetic degrees of freedom, such as angular momentum, as depicted in Fig. 2f–h [38].

[2] Bloch oscillations are periodic temporal oscillations that arise when Bloch waves in a crystal are subject to a static potential gradient, such a static electric field applied to electrons in a periodic potential [40].

In particular, Floquet topological insulators feature unique properties compared to their static counterparts, which highlight the powerful opportunities enabled by Floquet concepts: their quasi-energy eigenspectrum is no longer defined along the entire energy (frequency) axis, but on a torus. As a consequence, the existence of topologically protected edge states, typical of topological insulators, is no longer tightly linked to the topological index of the respective bands: edge modes can be found in Floquet systems even in the absence of a difference between topological invariants of two bulk bands [30]. Due to the difficulties of implementing efficient time-modulation schemes, the first implementations of Floquet topological insulators have been reproduced by replacing the temporal dimension with a spatial dimension. For instance, helically modulated waveguide arrays have been used to implement photonic Floquet topological insulators. Although these systems remain reciprocal due to their time-invariance, the chiral evanescent coupling between the waveguides leads to a splitting in degeneracy between states with opposite pseudo-spin, thereby enabling a degree of topological protection [44]. More recently, this concept has been extended to include chiral arrangements of interstitial sites, realizing single photonic systems capable of hosting distinct topologically nontrivial phases [45]. In addition, following the demonstration of topological lasers [46], theoretical proposals have been put forward to realize Floquet topological lasing [47], opening a promising path for further advances. Finally, blending Floquet physics with nonlinear optics has recently led to the first observation of topological solitons in twisted waveguide arrays [48]. It is worth remarking however that temporal and spatial degrees of freedom are not interchangeable, due to the different causality relations between impinging and scattered waves underlying the respective scenarios. More specifically, forward and backward scattered waves at a spatial interface do not interfere with each other, whilst temporally scattered forward and backward waves do interfere, an effect which underpins for instance parametric amplification and modulation instabilities (see e.g. Sect. 3, details are also discussed in [19]), phenomena that cannot emerge in spatial-analogues of Floquet systems like the ones mentioned above.

In acoustics, space-Floquet photonic systems have been deployed to demonstrate a bounty of topological effects. Notably, exploiting the helical waveguide concept, corner states and higher-order topological phases were recently demonstrated experimentally [49]. An additional methodology to realize Floquet-topological phases without time-modulation exists, which relies on coupled resonator optical waveguides (CROWs), originally proposed in [50]. This concept exploits arrays of resonators coupled via multiple channels, such as the coupling via the different channels introduces a different phase, thereby producing a synthetic gauge field. This concept has been deployed in a periodic fashion to experimentally realize $Z_2$ topological phases for sound [16].

The framework of metamaterials indeed offers a rich host of opportunities for implementations and further explorations of Floquet engineering. Their subwavelength constitutive elements, the meta-atoms, support engineered scattering features ideally suited to support the exotic targeted response, playing a paramount role in defining emergent responses, such as dispersive [51], chiral [52], non-Hermitian [53] and nonreciprocal properties [11, 54]. In the next section we argue that, in analogy with spatial metamaterials, the form of temporal switching imposed on a medium can be treated as a temporal meta-atom, offering a wealth of new opportunities for engineering wave phenomena.

## 3 Time-interfaces as temporal meta-atoms

In the spatial domain, meta-atoms with a spatial extent smaller than the wavelength can be engineered to possess tailored features, such as electric and magnetic responses, spatial and temporal dispersion, chirality, gyromagnetism, and others. These scattering signatures dictate and determine the exotic response of the resulting metamaterial once collections of these meta-atoms are properly arranged in a lattice. The time-dependence in Floquet engineering can be mapped into the same framework: tailored temporal modifications at timescales shorter than the periodic variations of a signal in time can act as the analogue of meta-atoms in the time domain, supporting emergent temporal scattering responses. Following this analogy, tailored collections of these temporal interfaces can therefore introduce exotic wave phenomena, which can be leveraged in time or space–time metamaterials. In this paradigm, time acts truly as an additional dimension for wave engineering [55]. For instance, photonic time crystals can leverage light scattering and interference from multiple time-interfaces, as the parameters of a medium are switched faster than one cycle of the signal they are manipulating [13]. In this Section, we discuss the unusual wave phenomena at these time-interfaces and their combinations, and discuss an outlook on how they can be leveraged to form Floquet metamaterials.

### 3.1 Macroscopic description of temporal scattering

The macroscopic electromagnetic properties of isotropic, non-dispersive optical materials can be described by the conventional constitutive relations $\mathbf{D} = \varepsilon_0\varepsilon\mathbf{E}$ and $\mathbf{B} = \mu_0\mu\mathbf{H}$, where $\varepsilon$ and $\mu$ are the relative permittivity and permeability, respectively. The corresponding refractive

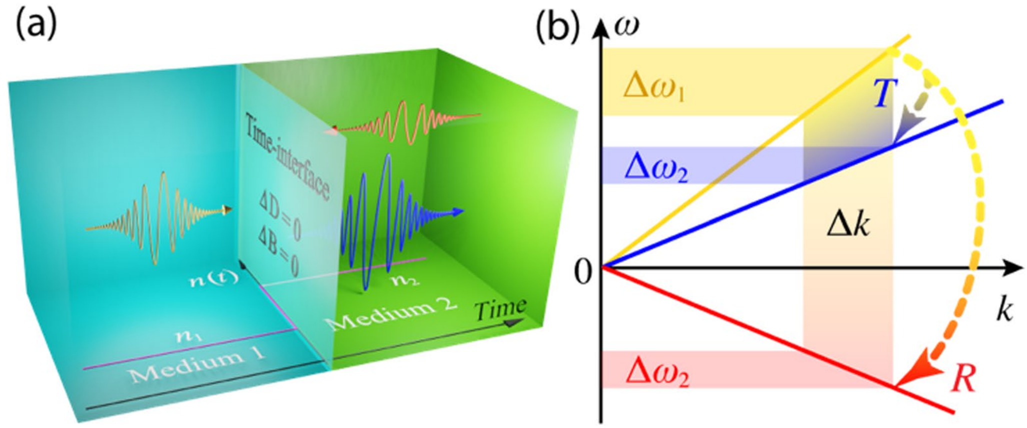


**Fig. 3** A macroscopic time-interface and the associated effect on the wave dispersion of time-scattering. **a** Field variations at a point in an unbounded homogeneous medium experiencing a time-interface created by a switching refractive index $n(t)$. The boundary conditions are the continuity of both electric displacement D and magnetic flux density B. An arbitrary wave packet (in yellow) splits into a forward (in blue) and a backward (in red) wave with the same envelope but oscillate at a different frequency. **b** The arrows in the dispersion diagram show the conservation of the spatial spectrum $\Delta k$ and the frequency and bandwidth conversion from $\Delta\omega_1$ to $\Delta\omega_2$

index is $n = \sqrt{\varepsilon\mu}$ and the wave impedance is $\eta = \sqrt{\mu/\varepsilon}$. A spatial boundary generally forms at a discontinuity of $n$ and/or $\eta$ in space. For example, an interface at $z = 0$ can be represented by $n(z) = n_1\Theta(-z) + n_2\Theta(z)$, where $\Theta(z)$ is the Heaviside step function. As shown in Fig. 3(a), the temporal analogue is a time-interface, which emerges if we consider a discontinuity of $n$ (and/or $\eta$) in time, e.g. $n(t) = n_1\Theta(-t) + n_2\Theta(t)$, occurring uniformly across space. Dual to a spatial boundary, which preserves frequency and energy, such a temporal interface conserves the wavelength and the electromagnetic momentum. Light scattering at such a time-interface was first investigated in [56], and it has been revisited among different communities over the years, including signal processing [57], microwave engineering [58], plasma physics [61, 62] and more recently optics [60]. Recently, several applications of time interfaces in wave engineering have emerged, ranging from time-refraction [59, 63–65], impedance transformations in time [66, 67], inverse prism phenomena [68], temporal aiming [69], temporal Brewster angle [70], and even non-Hermitian physics, such as the temporal analogue of PT-symmetry [53] and time-metamaterials with gain and loss [71]. For an extensive review, the interested reader may refer to Ref. [19].

Mathematically, temporal scattering is an initial value problem of the wave equation defined by temporal boundary conditions. If we assume the continuity of electric and magnetic charges at the time interface, the electric displacement $D$ and magnetic flux density $B$ must be continuous [72]. By applying the Laplace transform and the initial-value theorem, one can obtain these same initial conditions [73]. A monochromatic plane wave scattered by a time-interface at $t = 0$, whose flux density fields are written as $D_x(t<0) = D_0 e^{-i\omega_1 t} e^{ik_1 z}$ and $B_y(t<0) = \eta_1 D_0 e^{-i\omega_1 t} e^{ik_1 z}$, undergoing a refractive index change from $n_1 = \sqrt{\varepsilon_1\mu_1}$ to $n_2 = \sqrt{\varepsilon_2\mu_2}$ at $t = 0$, and a wave impedance change from $\eta_1 = \sqrt{\mu_1/\varepsilon_1}$ to $\eta_2 = \sqrt{\mu_2/\varepsilon_2}$, generates forward and backward waves with coefficients

$$T = \frac{\eta_2 + \eta_1}{2\eta_2}, \text{ and } R = \frac{\eta_2 - \eta_1}{2\eta_2}, \tag{6}$$

respectively.

In an actual experiment, the involved waves have a finite frequency bandwidth $\Delta\omega_1$ and momentum bandwidth, as depicted by the yellow curves in Fig. 3a, whose temporal and spatial spectrum is shaded in yellow in the dispersion diagram in Fig. 3b. Intuitively, the scattered waves should be simply the linear superposition of the individual plane wave components of the input: we expect the spatial spectrum $\Delta k$ to be preserved, whereas the frequency spectrum is reshaped to $\Delta\omega_2$, as shown by the transition in Fig. 3b from the yellow shadow to the blue (forward wave T) and the red (backward wave R) shadows. More rigorously, we can expand the incident wave at $t = t_0$ in momentum space:

$$D_{inc}(z, t_0) = \int_{-\infty}^{+\infty} D_0(k, t_0) e^{ikz} dk, \tag{7}$$

where the integrand is the plane wave component with wavenumber $k$ and associated frequency $\omega_1 = ck/n_1$. Because of linearity, the total field after the time-interface at $t = t_0$ then equals the linear superposition of all scattered plane waves:

$$D_{tot}(z, t > t_0) = \int_{-\infty}^{+\infty} M(k, t; t_0) D_0(k, t_0) e^{ikz} dk, \tag{8}$$

where $M(k, t; t_0) = Te^{-i\omega_2(t-t_0)} + Re^{i\omega_2(t-t_0)}$ is the transfer function in momentum space. Under the assumption of no material dispersion, $\omega_2 = ck/n_2$, as illustrated by the blue line in Fig. 3b, and the scattering coefficients in Eq. (6) are independent of both $k$ and $\omega$. Hence, we can explicitly write the expression in real space by applying an inverse Fourier transform, leading to

$$D_{tot}(z, t > t_0) = TD_{inc}\left(z - \frac{c}{n_2}(t - t_0), t_0\right) + RD_{inc}\left(z + \frac{c}{n_2}(t - t_0), t_0\right). \tag{9}$$

This result indicates that the total field after the time-interface consists of two counter-propagating waves sharing the same envelope as the incident wave. Such waveform preservation is guaranteed by the non-dispersive nature of our material, while in the presence of frequency dispersion some distortion would be expected. Mathematically speaking, the absence of dispersion guarantees not only k-independence in T and R, but more importantly that the Fourier transform of the transfer function $M(k, t; t_0)$ consists of the sum of two Dirac delta functions, associated with the instantaneous optical response of a non-dispersive medium.

### 3.2 Time-interfaces in the presence of material dispersion

If material dispersion cannot be neglected, i.e., when the material response time is comparable with the temporal variations of the involved signals, Eq. (8) no longer admits an explicit solution, and the waveform of the incident wave will be distorted in time. Such spectral distortion is physically caused by nonlocal phenomena of the material in time, corresponding to a convolution of the applied electric field $E$ with the temporal response of the electric susceptibility $\chi$ of the medium, which induces the macroscopic electric polarization density:

$$P(\mathbf{r}, t) = \varepsilon_0 \int d\mathbf{r}' \int dt' \chi\left(\mathbf{r}, t; \mathbf{r}', t'\right) E\left(\mathbf{r} - \mathbf{r}', t - t'\right), \tag{10}$$

where the susceptibility $\chi$ is an analytic function subject to causality. Extensive research efforts have been dedicated to the investigation of scattering problems in such dispersive media. The relation between the electric and the polarization fields in nonlocal materials appears as an additional differential equation, raising the order of the scattering problem and therefore introducing additional eigenstates. Therefore, additional boundary conditions (ABCs) are required at the interface to determine how an incident wave couples to these eigenmodes in temporally nonlocal media. Quite interestingly, analogous problems have been analyzed in the context of spatial interfaces involving spatially nonlocal media, particularly the context of plasmonics and metamaterials, for which spatial nonlocalities often cannot be neglected. In solid state physics, the first proposal of a spatially dispersive model with ABCs was discussed by Pekar, who introduced it in the 1950s in order to model excitons in semiconductors [74], initiating a tide of different arguments for finding ABCs based on various phenomenological arguments [75] and a scrutiny on their need [76, 77].

In the field of metamaterials, sub-wavelength meta-atoms often imply the emergence of nontrivial spatial nonlocal phenomena [78, 79], since their response may be nontrivially determined not only by the local electric and magnetic fields, but also by their derivatives in space. For instance, metamaterials consisting of planar multilayers or nanowire-arrays (Fig. 4a and b respectively) are both well-known to support a peculiar hyperbolic dispersion of their isofrequency contours [80]. Hence, when homogenized they can support a similar set of eigenmodes and associated nonlocal response, but the required boundary conditions at their interface can be completely different. Consider for instance the nanowire metamaterials in Fig. 4b, whose homogenized permittivity response for electric fields polarized along the wires is nonlocal, with a dependence on the longitudinal wave number $k_x$ given by $\varepsilon_{xx} = 1 + k_p^2/\left(k_h^2 - k_x^2\right)$, where $k_p$ is the plasma wavenumber, and $k_h$ is the wavenumber of the host medium [81]. At an interface, the required ABCs can have different macroscopic forms: if the metal nanowires are not connected to an exterior perfect electric conductor (PEC), then the normal component of conduction current at the spatial interface between the metamaterial and the upper homogeneous dielectric should vanish [81, 82]: $J_n(x = 0) = 0$. On the other hand, if the nanowires are grounded, i.e., their bottom ends are connected to a PEC, then the normal conduction current at the boundary follows the Neumann boundary condition $\partial J_n/\partial n = 0$ [83]. More generally, if the nanowire metamaterial is terminated by an imperfect conductor, the ABC is a linear combination of the previous two. In each scenario, the scattering at the interface will have different responses as a function of these ABCs. Similar arguments can be applied to excitons and polaritons in semiconductors [75], where

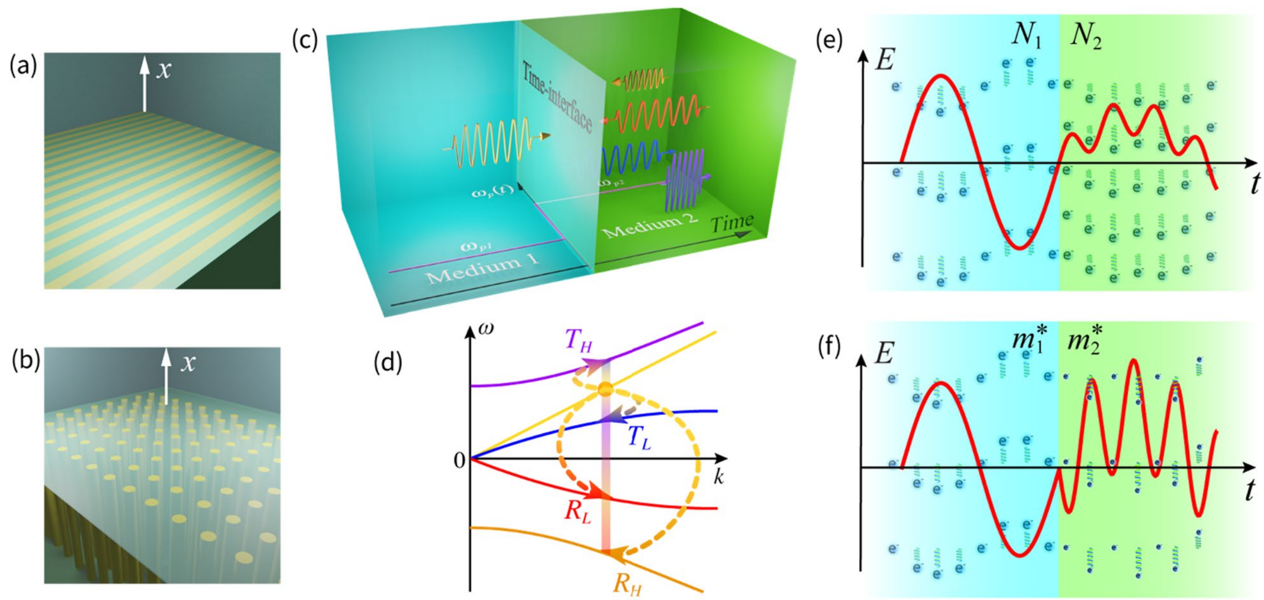


**Fig. 4** Time-interfaces with temporal nonlocality arising from different microscopic dynamics, analogous to spatially nonlocal metamaterials. (a-b) Nonlocal metamaterials in the form of **a** multilayer, and **b** nanowires. **c** A time-interface between two temporally nonlocal (dispersive) media. The plasma frequency $\omega_p$ is assumed to be switched from $\omega_{p1}$ to $\omega_{p2}$. The incident wave is denoted by the yellow wavepacket, while the four scattered waves, two forward and two backward, are drawn in different colors. **d** Dispersion diagram of a non-dispersive medium (yellow line) and the one of a medium with Lorentz-type dispersion (the other four curves). The dashed arrows illustrate the temporal scattering process. **e** and **f** show two different protocols of switching $\omega_p$: the density of carriers is suddenly increased from $N_1$ to $N_2$ in **e** where the number of symbolic oscillators increases; the effective mass becomes heavier from $m_1^*$ to $m_2^*$ in **f** where the individual electron becomes smaller visually. The red curves show the temporal evolution of the electric field under two different switching schemes

the conduction current is replaced by the excitonic or polaritonic polarization density.

By duality, time-interfaces involving temporally nonlocal (dispersive) media are subject to similar features. Time interfaces involving frequency dispersion have been originally studied in [58], and later in the context of rapidly growing plasmas [61]. After time-interfaces have become of interest in the optics community [60], the topic has been revamped [84]. Recently, additional boundary conditions at time-interfaces have been explored for the Drude-Lorentz model, accounting for a time-switched density of oscillators [85] and for more general dynamic models based on the balance of distributions [86]. A time-interface between two dispersive media obeying the Lorentz model is illustrated in Fig. 4c. As an example, here we consider a monochromatic incident wave (yellow curve) and a time interface for which the second medium features a single resonance at $\omega_0 = \Delta E/\hbar$, where $\Delta E$ is the energy gap of the two-level system describing the material. In the frequency domain, the relative permittivity reads

$$\varepsilon(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\omega\gamma}, \tag{11}$$

where the plasma frequency $\omega_p = \sqrt{Ne^2/(m^*\varepsilon_0)}$, $N$ and $m^*$ are the volume carrier density and their effective mass, respectively, and $\gamma$ is the collision frequency responsible for absorption. Across the time-interface, momentum conservation requires

$$\omega_1\sqrt{\varepsilon_1} = \omega_2\sqrt{1 + \frac{\omega_{p2}^2}{\omega_0^2 - \omega_2^2 - i\omega_2\gamma}}. \tag{12}$$

In general, the incoming wave can couple to four scattered waves (wavepackets in four colors in Fig. 4c), corresponding to the four solutions for $\omega_2$ of the biquadratic equation obtained by squaring Eq. (12). Phenomenologically, temporal wave scattering from a non-dispersive medium to a Lorentz-type dispersive medium is shown in the dispersion diagram of Fig. 4d: a mode on the yellow straight line (the incident monochromatic plane wave) couples to the four branches of the new dispersion curves, while retaining its momentum. Their amplitudes $T_H$, $R_H$, $T_L$ and $R_L$ in Fig. 4d can be obtained by applying the two temporal boundary conditions discussed in Sect. 3.1, coupled to two ABCs determined by the microscopic dynamics of the time-interface, i.e., the actual phenomena involved in the temporal scattering process. Crucially, these microscopic temporal dynamics will determine the coefficients of the four waves.

In order to highlight the role of the microscopic picture in the temporal scattering process, we can consider a Drude-Lorentz material, in which the time-domain polarization vector follows the dynamic equation [85]

$$\frac{d^2P}{dt^2} + \gamma\frac{dP}{dt} + \omega_0^2 P = \varepsilon_0\omega_p^2 E. \tag{13}$$

If we assume that the time interface is formed by switching $\omega_p$ from $\omega_{p1}$ to $\omega_{p2}$ at $t = 0$, there are different options in which such an event can be realized: for instance, we can create or annihilate carriers in the material, such that the volume carrier density $N$ suddenly jumps from $N_1$ to $N_2$ [85, 87], as shown in Fig. 4e, or we may consider alter the effective mass of the electrons from $m_1^*$ to $m_2^*$, as shown in Fig. 4f, by somehow modifying the band structure. Both events result in the same macroscopic effect on the permittivity of the material. In the first switching scheme, as shown in Ref. [85], the electric polarization density and its first derivative in time must be continuous across the time-interface:

$$P\left(t = 0^+\right) = P\left(t = 0^-\right). \tag{14}$$

$$\frac{dP}{dt}\left(t = 0^+\right) = \frac{dP}{dt}\left(t = 0^-\right). \tag{15}$$

However, if the effective mass is switched in time, the complete form of Newton's second law should be considered, allowing for a time-dependent mass $F = d(mv)/dt$. Accordingly, in this second scenario the Lorentz dispersion is recast as [85]

$$\frac{d}{dt}\left(m^*\frac{dP}{dt}\right) + m^*\gamma\frac{dP}{dt} + m^*\omega_0^2 P = Ne^2E, \tag{16}$$

different from Eq. (13). Inferring conservation of carrier momentum, Eq. (15) may be modified as

$$m_1^*\frac{dP}{dt}\left(t = 0^+\right) = m_2^*\frac{dP}{dt}\left(t = 0^-\right) \tag{17}$$

Hence, the same macroscopic model of dispersion at a time interface may result in different ABCs, which in turn produce a different scattering response. A numerical example comparing the electric field produced in a time-scattering where the electron number $N$ and the effective mass $m^*$ are switched is shown in Fig. 4e and f respectively. In both cases, the resulting plasma frequency changes from $\omega_{p1} = 0.03\omega_{inc}$ to $\omega_{p2} = 3\omega_{inc}$ with $\omega_0 = 2\omega_{inc}$ and $\gamma = 0$, $\omega_{inc}$ being the frequency of the incident wave. In Fig. 4e we assume that the carrier density suddenly increases, and four scattered waves (with frequencies approximately $\pm 0.54\omega_{inc}$,$\pm 3.70\omega_{inc}$) are generated. Importantly, in the case where $N$ is switched, both the electric field (red curve) and its first derivative are continuous, as required by the ABCs in Eqs. (14) and (15). In contrast, when the effective mass is switched to accomplish the same change in plasma frequency, the scattered electric field oscillates at the same frequency components, but the relative coefficients are significantly different. Hence, the resulting fields that emerge from the time interface experience a completely different evolution in time. In the second scenario, the first derivative of the electric field becomes discontinuous, as a result of the discontinuity in $dP/dt$ required to balance the change in effective carrier mass [Eq. (17)].

Modifications of the effective electron mass of a material are also at the basis of the large nonlinearities recently reported in epsilon-near-zero (ENZ) materials [88], connecting these problems to exciting developments in the field of metamaterials. Similarly, we can expect interesting phenomena to emerge when considering time-interfaces in polaritonic systems. Polaritons emerge when light is strongly coupled to resonant material responses, such that light and matter oscillations cannot be considered decoupled [89]. For instance, when the intersubband transitions of multiple quantum wells are aligned and strongly coupled with optical modes of nanoresonators, quantum-well polaritons emerge, which can support ultrafast optical switching and nonlinearities [90]. The enhanced nonlinearities based on intersubband transitions in multiple quantum wells have also been explored to achieve electrically tunable polaritonic metasurfaces, featuring efficient intensity modulation and beam manipulation of harmonically generated light [91]. It is intriguing to explore how multiple engineered time-interfaces, i.e., a tailored sequence of time-switching events, can manipulate and engage polaritons. Such tailored (space-)time polaritons may form a new category of meta-atoms, unveiling abundant physics and applications ready to be exploited in the context of nontrivial topologies in the synthetic frequency dimension [31] as discussed in Sect. 2, ultrafast frequency conversion [92], and efficient phase conjugation [93, 94], to name a few.

## 4 Floquet metamaterials

Time-interfaces as meta-atoms in the temporal domain host rich opportunities for the engineering of temporal and spatiotemporal meta-structures. We now stand at the confluence of the fields of metamaterials and Floquet physics, opening exciting directions in the emerging area of Floquet metamaterials, as we illustrated in the lower part of Fig. 1. The opportunities opened by this confluence are several and diverse.

From the experimental viewpoint, the mature field of metamaterials consists of a highly multidisciplinary community, where new ideas find prompt experimental validation across a wide range of wave realms: from optics to radio waves, acoustics and elasticity, water and seismic waves, the metamaterials framework offers an effective pathway towards not only experimental implementations, but also technological impact, particularly due to the opportunity for fundamentally novel concepts to be readily deployed in addressing challenges across multiple wave-related fields. For instance, water waves have already proven a fertile ground for the realization of time-reversal, dynamical localization and other effects induced by switching at rates faster than the wave frequency (see e.g. Refs. [95, 96]). Other strategies have been developed in acoustics using both piezoelectric components [97], and more recently digitally activated meta-atoms which can reproduce an arbitrary time-varying response at ultrafast speeds [98]. In low-frequency electromagnetics, switching by means of varactor diodes and nonlinear inductors at rates faster than the period of the propagating waves still remains an open challenge, particularly due to the need for realizing pump-circuits with a sufficiently low time-constant. Such an implementation would undoubtedly constitute a groundbreaking result across the electromagnetics community, opening new technological avenues in microwave science, and a new playground for Floquet metamaterials.

Pushing Floquet metamaterials towards the infrared and optical domain currently constitutes a formidable challenge, promising however groundbreaking rewards for next-generation light-matter control. Highly nonlinear material responses and long-lived resonances capable of achieving large field-enhancements and at the same time slow down light propagation appear to hold the key towards the implementation of Floquet metamaterials at higher frequencies. In order to enable this vision, as briefly mentioned in Sect. 3.2, polaritonic materials have recently proven to be promising candidates.

When material resonances are sufficiently strong, ENZ phenomena arise, which enable large relative changes in dielectric permittivity within a low loss platform, ideally suited for Floquet metamaterials, as recently demonstrated with the realization of time-refraction [64], efficient harmonic generation [99], and negative refraction [100]. In a similar context, vibrational modes strongly coupled with light in phonon polaritons have opened various opportunities for Floquet matter: being longer-lived than plasmons, phonon polaritons offer an ideal trade-off between field confinement and quality factors, in addition to their high directionality, and associated exotic dispersion relations [7, 101]. Recent efforts have successfully demonstrated pumping of phonon modes in SiC for parametric amplification in a pump-probe setting [28] based on Floquet phenomena [102, 103]. Along a similar direction, pump-induced switching of the dispersion of surface polaritons has recently been reported [104]. This avenue is further broadened by the opportunity to structure these polaritonic media, in the spirit of the metamaterial concept. Structuring polaritonic media in the plane of propagation, realizing phonon-polaritonic metasurfaces, or by stacking them in thin layers, with the option of introducing a finite twist angle between the optical axes of the different layers, offers unique opportunities to tailor their dispersion in extreme ways, which may be then exploited in pump-probe experiments to demonstrate Floquet metamaterials. In particular, the high directionality achievable in polaritonic media enables anisotropic gain, which may be exploited, for instance, to achieve novel non-Hermitian functionalities by exploiting systems with balanced gain and loss in different directions.

From the theoretical viewpoint, the synergy between metamaterials and Floquet physics offers the opportunity to realize new wave responses hinging on both the spatial and the temporal structure of a system, which can couple non-trivially to their mutual geometric and material dispersion [44, 84–86]. While Floquet engineering can be used to design artificial spatially nonlocal effects [105], temporal nonlocality, as introduced in e.g. an electromagnetic system via coupling to a (natural or engineered) resonance of the host medium, can supply an additional timescale to a Floquet system, enabling room to design the interplay between wave frequency, modulation frequency, and resonance frequency of the material, effectively opening a potential opportunity for the design of polaritons in the time-domain. In addition, these new degrees of freedom, when coupled to spatial ones such as in spatiotemporally modulated systems [11], can further broaden the spectrum of opportunities for Floquet metamaterials.

From a fundamental viewpoint, one mismatch in this confluence appears to be the fact that much of the many-body physics constituting a substantial branch of Floquet engineering appears out of reach for the metamaterials world, since classical waves are typically equivalent to "single-particle" problems in quantum mechanics. Rather than an obstacle, however, this challenge ultimately constitutes an unprecedented opportunity for much of the nonlinear wave physics developed in classical systems to be approached via fundamentally new angles. For instance, while photons do not interact directly with each other, they can do so via material nonlinearities (and similar arguments hold for elastic and acoustic media) [106]. Preliminary efforts in this direction have been made to map $\chi_3$ nonlinearities onto effective Bose-Hubbard models for photons interacting along the synthetic frequency dimension [107, 108]. Under this light, Floquet metamaterials may offer a much more accessible route to investigate many-body phenomena in classical wave settings through material nonlinearities, potentially enabling new insights to be reached into the physics of both nonlinear waves and interacting condensed matter systems.

## 5 Conclusions

In this Perspective, we have connected the two flourishing and so far mostly disconnected areas of Floquet engineering and metamaterials, as illustrated in Fig. 1. In particular, we envision a wealth of opportunities available in the rising context of Floquet metamaterials. Rather than illuminating a blueprint, we have stepped inside of the building blocks of Floquet metamaterials, the time-interfaces constituting the temporal analogues of meta-atoms, and discussed some of their key features in both a macroscopic and microscopic picture, in the context of the interplay between temporal inhomogeneities and material dispersion. In this framework, we highlighted how the microscopic mechanism behind material dispersion plays a key role in determining the resulting wave dynamics at each switching event, and, transitively, within a Floquet metamaterial.

Crucially, the simple consideration of dispersion in Sect. 3 and the outlook in Sect. 4 stand as an entrée before the wealth of additional physics which can be unleashed once additional features such as nonlinearity, anisotropy and chirality are taken into account at time-interfaces, highlighting that forthcoming opportunities with Floquet metamaterials stretch way beyond our present discussion here. From an experimental perspective, non-ideal time-interfaces with finite rise-time not only challenge existing theories, but also suggests another degree of freedom in designing sub-periodic temporal unit cells, hinting at a potentially new concept of time-polariton, which could be formed when the rise-time, the response time of a material, and the period of the involved wave coexist over comparable timescales. Besides, the modulation efficiencies of the optical properties of a material are typically weak. Facing this challenge may inspire us to enhance the scattering at purely temporal interfaces, by creating temporal interfaces with strong spatial field confinement in resonant modes. Finally, inspired by the artificial intelligence (AI)-aided inverse design of photonic systems, similar techniques could be transferred to the design of temporal meta-atoms, taking more degrees of freedom into account simultaneously.

In conclusion, we believe that the confluence of Floquet engineering and metamaterials opens an unprecedented opportunity to design novel forms of wave-matter interactions, with sub-period temporal structures promising to play a key role in establishing Floquet metamaterials as a new research horizon for both condensed matter physics and photonics.

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## Declarations

**Competing interests**
The authors declare that they have no competing interests.

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