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TOTAL CURVATURES OF COMPACT MANIFOLDS IMMERSED IN EUCLIDEAN SPACE (*)

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1. Introduction.

This paper will be concerned with some kind of total absolute curvatures of compact manifolds X^n of dimension n (without boundary) immersed in euclidean space E^{n+N} of dimension $n+N$ ($N \geqslant 1$). Classical Differential Geometry handled almost exclusively with « local » curvatures for such manifolds X^n (assumed sufficiently smooth) and mainly dealed with the case $N = 1$. The Gauss-Bonnet theorem, extended by Allendoerfer-Weil-Chern to the case $n > 2$ [1], [7], has been for years the most important, and almost the unique, result of a « global » character. In the classical theory of convex manifolds (boundaries of convex sets) in euclidean space, play an important role the Minkowski's « Quermassintegrale » which may be defined globally without any assumption of differentiability and also, for sufficiently smooth convex manifolds, as integrals of the symmetric functions of the principal curvatures. This classical case shows that, in order to define total curvatures of a given X^n (not necessarily convex) immersed in E^{n+N}, one can either give directly a global definition and then try to express it as the integral of certain local curvatures, or give first a local definition (curvature at a point $x \in X^n$) and then computing the total curvature by integrating this local curvature over X^n. The last method makes necessary some assumptions of smoothness for X^n. A noteworthy example of such curvatures are those introduced by H. Weyl in a classical paper on the volume of tubes [28]. These Weyl's curvatures has been used by Chern to get a general kinematic formula in integral geometry for compact submanifolds of E^{n+N} [10]. For more general subsets of E^{n+N} an analogous formula was given by H. Federer [14] whose « curvature measures » are an extension of the Weyl's curvatures.

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In 1957-58 two papers of Chern-Lashof [11], [12] call the attention about « absolute » total curvatures, *i.e.* total curvatures obtained by integrating on X^n the absolute values of certain local curvatures. These papers were followed by a series of papers of several authors, mainly N. H. Kuiper, who related this branch of differential geometry with Morse theory of critical points of real valued functions defined over X^n [17], [18]. A survey and new results about this field is to be found in the lecture notes of D. Ferus [16]. See also T. J. Willmore [29].

In the mark of these studies we have introduced in [24], [25], some total curvatures (absolute) for compact manifolds X^n immersed in E^{n+N}. The main purpose of the present paper is to give a local definition of these curvatures, so that they will appear as the integral over X^n of the absolute value of certain differential forms defined in each point $x \in X^n$. These definitions allow to compare the new curvatures with other curvatures previously introduced in the literature. We will then consider some examples, for instance the case of surfaces X^2 immersed in E^4 which presents some remarkable peculiarities.

NOTE: We will consider throughout that X^n is a compact, C^∞-differentiable manifold of dimension n, without boundary, immersed in some euclidean space. In the non-smooth case, a great deal of difficulties arise. For some questions about total curvature of C^1-manifolds, see W. D. Pepe [20].

By E^r_s, $r < s$, we will indicate a r-plane (linear space of dimension r) in the s-dimensional euclidean space E^s. If the euclidean space E^s in which E^r is immersed is apparent form the context, we will write simply E^r instead of E^r_s.

2. **Preliminaries.**

We will recall the fundamental equations of the differential geometry of a X^n immersed in E^{n+N} and certain know integral-geometric formulae about such manifolds. We use the method of moving frames of Cartan-Chern. See for instance Chern [9] or Willmore [29].

Let $(x; e_1, e_2, \ldots, e_{n+N})$ be a local field of orthonormal frames, such that, restricted to X^n, the vectors $e_1, e_2, \ldots, e_n$ are tangent to X^n and the remaining vectors $e_{n+1}, \ldots, e_{n+N}$ are normal to X^n. The orientation of the unit vectors $e_1, e_2, \ldots, e_{n+N}$ is assumed coherent with that of E^{n+N}. In this section we agree on the following ranges of indices

$$1 < i, j, k, h, \ldots < n , \quad n < \alpha, \beta, \gamma, \ldots < n + N , \quad 1 < A, B, C, \ldots < n + N$$

and the summation convention will be used throughout.

The fundamental equations for the moving frames in E^{n+N} are

$$dx = \omega_A e_A \,, \qquad de_A = \omega_{AB} e_B \tag{2.1}$$

where, because $e_A e_B = \delta_{AB}$,

$$\omega_{AB} + \omega_{BA} = 0 \quad \text{and} \quad \omega_A = e_A \cdot dx \,, \qquad \omega_{AB} = e_B \cdot de_A \,. \tag{2.2}$$

The exterior derivatives satisfy the equations of structure:

$$d\omega_A = \omega_B \wedge \omega_{BA} \,, \qquad d\omega_{AB} = \omega_{AC} \wedge \omega_{CB} \,. \tag{2.3}$$

The assumption that $e_1, \ldots, e_n$ are tangent to X^n gives

$$\omega_\alpha = 0 \tag{2.4}$$

and the condition that X^n has dimension n insures that the forms ω_i are linearly independent. From (2.3) and (2.4) we deduce $\omega_i \wedge \omega_{i\alpha} = 0$ and therefore, according to the so called lemma of Cartan, we have

$$\omega_{i\alpha} = A_{\alpha,ij}\,\omega_j \,, \qquad A_{\alpha,ij} = A_{\alpha,ji} \tag{2.5}$$

where $A_{\alpha,ij}$ are the coefficients of the second fundamental form in the normal direction e_α. Notice that we have represented by ω_A, ω_{AB} the forms in (2.3) corresponding to the space of all frames in E^{n+N} as well as the corresponding forms in the bundle of frames such that e_i are tangent vectors and e_α are normal vectors to X^n at x. We think that this simplification in the notation will not cause confusion.

From (2.3), (2.4) and (2.5) we have

$$d\omega_{ij} = \omega_{ih} \wedge \omega_{hj} + \Omega_{ij} \tag{2.6}$$

where

$$\Omega_{ij} = \omega_{i\alpha} \wedge \omega_{\alpha j} = -A_{\alpha,ih} A_{\alpha,jk}\,\omega_h \wedge \omega_k = \tfrac{1}{2} R_{ijkh}\,\omega_k \wedge \omega_h \tag{2.7}$$

with

$$R_{ijkh} = A_{\alpha,ik} A_{\alpha,jh} - A_{\alpha,ih} A_{\alpha,jk} \,. \tag{2.8}$$

We have also

$$d\omega_{\alpha\beta} = \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} + \Omega_{\alpha\beta} \tag{2.9}$$

where

$$\Omega_{\alpha\beta} = \omega_{\alpha i} \wedge \omega_{i\beta} = -A_{\alpha,ij} A_{\beta,ih}\,\omega_j \wedge \omega_h = \tfrac{1}{2} R_{\alpha\beta hj}\,\omega_h \wedge \omega_j \tag{2.10}$$

with

$$R_{\alpha\beta hj} = A_{\alpha,ih}A_{\beta,ij} - A_{\alpha,ij}A_{\beta,ih}\,. \tag{2.11}$$

Note the relations

$$\begin{aligned} &R_{ijkh} = -R_{ijhk} = -R_{jikh}\,, \qquad R_{ijkh} = R_{khij} \\ &R_{ijkh} + R_{ikhj} + R_{ihjk} = 0 \\ &R_{\alpha\beta kj} = -R_{\beta\alpha kj} = -R_{\alpha\beta jk}\,. \end{aligned} \tag{2.12}$$

The expression

$$(A_{\alpha ij}\omega_i\omega_j)e_\alpha \tag{2.13}$$

is called the second fundamental form of $X^n \subset E^{n+N}$, and

$$\frac{1}{n}(A_{\alpha ii})e_\alpha\,, \tag{2.14}$$

is called the mean curvature vector. X^n is said to be minimal if $A_{\alpha ii} = 0$ for all α.

R_{ijkh} are essentially the components of the Riemann-Christoffel tensor. However, they are not these components. For instance, the Riemannian curvature for the orientation determined by the vectors ξ^i, η^j takes now the form $K(x;\xi^i,\eta^j) = [(R_{ijkh}\xi^i\eta^j\xi^k\eta^h)/(\delta_{ik}\delta_{jh} - \delta_{ih}\delta_{jk})\cdot \cdot\xi^i\eta^j\xi^k\eta^h]$. For $n = 2$, the Gaussian curvature is given by

$$K(x) = R_{1212} \tag{2.16}$$

instead of the classical $K = R_{1212}/g$ when R_{1212} is the component of the Riemann-Christoffel tensor.

3. Densities for linear subspaces and some integral formulae.

We will state some known formulae which will be used in the sequel.

Let E^h_{n+N} denote a h-dimensional linear subspace in E^{n+N}: we will call it, simply, a h-plane. Let $E^h_{n+N}(O)$ denote a h-plane in E^{n+N} through a fixed point O. The set of all oriented $E^h_{n+N}(O)$, with a suitable topology, constitute the Grassman manifold $G_{h,n+N-h}$. We shall represent by $dE^h_{n+N}(O)$ the element of volume in $G_{h,n+N-h}$, which is called the density for oriented h-planes in E^{n+N} through O. The expression of $dE^h_{n+N}(O)$ is well known, but we recall it briefly for completeness (see [22], [23], [10]).

Let $(O; e_1, \ldots, e_2, \ldots, e_{n+N})$ be an orthonormal frame of origin O. In the space of all orthonormal frames of origin O we define the differential forms

$$(3.1)\qquad \omega_{im} = -\omega_{mi} = e_m de_i = -e_i de_m\,, \qquad (i, m = 1, 2, \ldots, n+N)\,.$$

Assuming $E^h_{n+N}(O)$ spanned by the unit vectors $e_1, \ldots, e_h$, then

$$(3.2)\qquad dE^h_{n+N}(O) = \Lambda\omega_{im}$$

where the right-hand side is the exterior product of the forms ω_{im} over the ranges of indices

$$(3.3)\qquad i = 1, 2, \ldots, h\,, \qquad m = h+1, h+2, \ldots, n+N\,.$$

The $(n+N-h)$-plane $E^{n+N-h}_{n+N}(O)$ orthogonal to $E^h_{n+N}(O)$ is spanned by the unit vectors $e_{h+1}, \ldots, e_{n+N}$ and according to (3.2) we have the « duality » (up to the sign)

$$(3.4)\qquad dE^h_{n+N}(O) = dE^{n+N-h}_{n+N}(O)\,.$$

Notice that the differential forms $dE^h_{n+N}(O)$ and $dE^{n+N-h}_{n+N}(O)$ are of degree $h(n+N-h)$, which is equal to the dimension of the grassmannian $G_{h,n+N-h}$, as it should be.

The density for sets of h-planes E^h, not through O, in E^{n+N} is given by

$$(3.5)\qquad dE^h = dE^h_{n+N}(O)\wedge\omega_{h+1}\wedge\omega_{h+2}\wedge\ldots\wedge\omega_{n+N}$$

where $\omega_{h+1}\wedge\ldots\wedge\omega_{n+N} = (e_{h+1}dx)\wedge(e_{h+2}dx)\wedge\ldots\wedge(e_{n+N}dx)$ is equal to the element of volume in $E^{n+N-h}_{n+N}(O)$ $(= (n+N-h)$-plane spanned by the vectors $e_{h+1}, \ldots, e_{n+N}$ orthogonal to $E^h)$ at the intersection point $E^h \cap E^{n+N-h}(O)$.

The measure of the set of all the oriented $E^h_{n+N}(O)$ $(=$ volume of the Grassman manifold $G_{h,n+N-h})$ may be computed directly from (3.2) or applying the result that it is the quotient space $SO(n+N)/SO(h)\times \times SO(n+N-h)$ (Chern [10]). The result is

$$(3.6)\qquad \int_{G_{h,n+N-h}} dE^h_{n+N}(O) = \frac{O_{n+N-1}O_{n+N-2}\ldots O_{n+N-h}}{O_1 O_2 \ldots O_{h-1}} = \frac{O_h O_{h+1}\ldots O_{n+N-1}}{O_1 O_2 \ldots O_{n+N-h-1}}\,,$$

where O_i is the area of the i-dimensional unit sphere, *i.e.*

$$O_i = \frac{2\pi^{(i+1)/2}}{\Gamma((i+1)/2)} . \tag{3.7}$$

Notice the relation

$$O_1 O_{i-2} = (i-1) O_i . \tag{3.8}$$

For the case $h = 1$, the density of oriented lines through O (assuming that $E^1_{n+N}(O)$ is the line spanned by e_1) writes

$$dE^1_{n+N}(O) = \omega_{12} \wedge \omega_{13} \wedge \ldots \wedge \omega_{1n+N} = (e_2 de_1) \wedge (e_3 de_1) \wedge \ldots \wedge (e_{n+N} de_1) , \tag{3.9}$$

which is equal to the element of volume of the $(n + N - 1)$ dimensional unit sphere at the end point of e_1. By the duality (3.4) this density (3.9) is equal to the density $dE^{n+N-1}_{n+N}(O)$ of hyperplanes through O (in this case (3.9) corresponds to the hyperplane spanned by $e_2, e_3, \ldots, e_{n+N}$).

Later on we shall need the following formula. Let $E^{n+N}(O) \subset \subset E^{n+N+p}(O)$. Given a line $E^1_{n+N+p}(O)$, let $E^{p+1}(O)$ be the $(p+1)$-plane which contains $E^1_{n+N+p}(O)$ and is perpendicular to $E^{n+N}(O)$ and let $dE^1_{p+1}(O)$ be the density of $E^1_{n+N+p}(O)$ as a line of $E^{p+1}(O)$. If $E^1_{n+N+p}(O)$ denotes the projection of the line $E^1_{n+N+p}(O)$ on $E^{n+N}(O)$ and θ denotes the angle between $E^1_{n+N+p}(O)$ and its projection $E^1_{n+N}(O)$, an easy calculation shows that

$$dE^1_{n+N+p}(O) = \sin^{n+N-1}\theta \, dE^1_{n+N}(O) \wedge dE^1_{p+1}(O) . \tag{3.9}$$

For instance, if $p = 1$, we have $dE^1_2(O) = d\theta$ and (3.9) writes

$$dE^1_{n+N+1}(O) = \sin^{n+N-1}\theta \, dE^1_{n+N}(O) \wedge d\theta . \tag{3.10}$$

Projection formulae. The differential geometry of hypersurfaces $X^n \subset E^{n+1}$ is well known. Calling $R_1, R_2, \ldots, R_n$ the principal radii of curvature of X^n at the point x and putting $d\sigma_n(x)$ = area element of X^n at x (given by $d\sigma_n(x) = \omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_n$ according to the notation (2.1)), the total r-th mean curvature of X^n is defined by

$$M_r(X^n) = \frac{1}{\binom{n}{r}} \int_{X^n} \left\{ \frac{1}{R_{i_1}} \frac{1}{R_{i_2}} \cdots \frac{1}{R_{i_r}} \right\} d\sigma_n(x) , \tag{3.11}$$

where $\{\}$ denotes the r-th elementary symmetric function of the principal curvatures $1/R_i$ $(i = 1, 2, \ldots, n)$. For $r = 0$, we have $M_0 =$ area of X^n. For $r = n$, if X^n is the boundary ∂D^{n+1} of a domain D^{n+1} of E^{n+1}, it is known that

$$M_n(X^n) = O_n\,\chi(D^{n+1}) \tag{3.12}$$

where χ denotes the Euler-Poincare characteristic. If n is even, we have $\chi(X^n) = \chi(\partial D^{n+1}) = 2\chi(D^{n+1})$ and (3.12) writes

$$M_n(X^n) = \tfrac{1}{2} O_n\,\chi(X^n), \qquad n \text{ even}. \tag{3.13}$$

If X^n is a topological sphere, we have $\chi(X^n) = 1 + (-1)^n$.

For closed convex hypersurfaces X_c^n (boundaries of convex bodies of E^{n+1}) we must recall the following « projection formulae » (see [23] and Hadwiger [15]): let X_c^{n-r} be the boundary of the orthogonal projection of X_c^n into $E_{n+1}^{n+1-r}(O)$ and let $\mu_{n-r}(X_c^{n-r})$ denote the measure of X_c^{n-r} (with respect to the euclidean metric in E_{n+1}^{n+1-r}). Then we have

$$\int_{G_{n+1-r,r}} \mu_{n-r}(X_c^{n-r})\, dE_{n+1}^{n+1-r}(O) = \tag{3.14}$$
$$= \frac{O_{n-1}O_{n-2}\cdots O_{n-r}}{O_{r-1}O_{r-2}\cdots O_1} M_r(X_c^n) = \frac{O_{n-1}O_{n-2}\cdots O_r}{O_{n-r-1}\cdots O_1} M_r(X_c^n)\,.$$

For $r = n$ we have $\mu_0(X_c^0) = 2$ and (3.14) coincides with (3.12). For $r = 1$, (3.14) gives the total first mean curvature M_1 as the mean value of the measure of the boundaries of the orthogonal projections of X_c^n on all hyperplanes. For instance, for $n = 2$, the total mean curvature of a convex closed surface X_c^2 in E^3 is given by

$$M_1(X_c^2) = \frac{1}{2\pi}\int_{O_2} u\, dO_2\,, \tag{3.15}$$

where $dO_2 = dE_3^1(O)$ denotes the element of surface area on the unit sphere and u denotes the length of the boundary of the projection of X_c^2 into a plane perpendicular to the direction defined by dO_2.

For non-convex hypersurfaces, the formulae (3.14) need to be modified: in the right-hand side appear the total « absolute » mean curvatures which we will consider in a next section.

Intersection formulae. Let X^n be a closed hypersurface of E^{n+1}, not necessarily convex (recall that we always assume that X^n is of class C^∞). Let E_{n+1}^r be a moving r-plane in E^{n+1} and consider the manifold $X^{r-1} = X^n \cap E^r_{n+1}$. Call $M_i^{(r)}(X^{r-1})$ the total i-th mean curvature of X^{r-1} as a manifold of dimension $r-1$ in E^r. Then the following formula holds (see [22], [23])

$$(3.16)\qquad \int\limits_{X^n \cap E^r \neq \emptyset} M_i^{(r)}(X^n \cap E^r)\, dE^r = \frac{O_{n-1} \ldots O_{n-r+1}}{O_1 O_2 \ldots O_{r-2}} \frac{O_{n-i+1}}{O_{r-i}} M_i(X^n)\,.$$

For $i = r-1$, assuming that $X^{r-1} = X^n \cap E^r$ is the boundary of a domain $D^r \subset E^r$, according to (3.12) we have $M_{r-1}^{(r)}(X^{r-1}) = O_{r-1}\chi(D^r)$ and (3.16) gives

$$(3.17)\qquad \int\limits_{X^n \cap E^r \neq \emptyset} \chi(D^r)\, dE^r = \frac{O_{n-1} \ldots O_{n-r+1}}{O_1 \ldots O_{r-1}} \frac{O_{n-r+2}}{O_1} M_{r-1}(X^n)\,.$$

In particular, if X^n is a closed convex hypersurface X_c^n we have $\chi(D^n) = 1$ and (3.17) gives the total measure of all r-planes which intersect X_c^n,

$$(3.18)\qquad \int\limits_{X_c^n \cap E^r \neq \emptyset} dE^r = \frac{O_{n-1} \ldots O_{n-r+1}}{O_1 \ldots O_{r-1}} \frac{O_{n-r+2}}{O_1} M_{r-1}(X_c^n) = \frac{O_{n-1} \ldots O_{n-r}}{(n-r+1)\, O_{r-1} \ldots O_1} M_{r-1}(X_c^n)\,.$$

If r is odd we have $\chi(D^r) = (\frac{1}{2})\chi(X^r)$ and (3.17) may be written

$$(3.19)\qquad \int\limits_{X^n \cap E^r \neq \emptyset} \chi(X^n \cap E^r)\, dE^r = \frac{2O_{n-1} \ldots O_{n-r+1}}{O_1 \ldots O_{r-1}} \frac{O_{n-r+2}}{O_1} M_{r-1}(X^n) \quad (r \text{ odd}).$$

In order to illustrate the foregoing ideas we will give a typical application. Let $n = 3$, $r = 3$. Then X^3 is a closed hypersurface in E^4; assume that it bounds a domain $D^4 \subset E^4$. According to (3.18) and (3.19) the mean value of $\chi(X^3 \cap E^3)$ is $E(\chi(X^3 \cap E^3)) = 2M_2(X^3)/M_2^*(X^3)$ where M_2^* denotes the 2-th total mean curvature of the convex hull of X^3. If V^* is the volume of the domain bounded by the convex hull, it is known that $M_2^* > (32\pi^6 V^*)^{\frac{1}{4}} > (32\pi^6 V)^{\frac{1}{4}}$ (Hadwiger [15]), where V is the volume of D^4. Thus we have

$$E(\chi(X^3 \cap E^3)) < \frac{M_2(X^3)}{\pi(2\pi^2 V)^{\frac{1}{4}}}\,.$$

The equality sign holds for euclidean spheres.

b) Another known and useful integral formula is the following:

Let X^n be a compact manifold in E^{n+N}. Let $\mu_{r-N}(X^n \cap E^r)$ denote the $(r-N)$-dimensional measure of $X^n \cap E^r$ $(r \geqslant N)$ according to the euclidean metric on E^r. For $r = N$, μ_0 denotes the number of points of the set $X^n \cap E^r$. Then we have

$$\int\limits_{X^n \cap E^r \neq \emptyset} \mu_{r-N}(X^n \cap E^r)\, dE^r = \frac{O_{n+N} O_{n+N-1} \dots O_{n+N-r} O_{r-N}}{O_1 O_2 \dots O_r O_n} \sigma_n(X^n)\,, \tag{3.20}$$

where $\sigma_n(X^n) =$ volume of X^n.

This integral formula holds good for any space of constant curvature, in particular on the $(n+N)$-dimensional sphere, with a suitable definition of dE^r (see [22]).

In all the preceding formulae and in those which will follow, the linear spaces E^r are assumed « oriented ». Otherwise the right-hand side of (3.16), ..., (3.20) should be divided by a factor 2.

4. The total absolute curvatures $K^*_{r,N}(X^n)$.

Let X^n be a compact n-dimensional manifold (without boundary) of class C^∞ in E^{n+N}. To each point $x \in X^n$ we attach the frame $(x; e_1, e_2, \dots, e_{n+N})$ considered in section 2, such that the vectors $e_1, e_2, \dots, e_n$ are tangent to X^n and spann the tangent n-plane $T(x)$. The remaining vectors $e_{n+1}, \dots, e_{n+N}$ are normal to X^n and spann the normal N-plane $N(x)$.

Assuming

$$1 < r < n + N - 1 \tag{4.1}$$

we define the r-th total absolute curvature of X^n as follows (see [24], [25]):

a) Case $1 < r < n$. Let O be a fixed point in E^{n+N} and consider a $(n+N-r)$-plane say $E^{n+N-r}(O)$ through O. Let Γ_r be the set of all r-planes E^r in E^{n+N} which are contained in some $T(x)$ $(x \in X^n)$, pass through x, and are perpendicular to $E^{n+N-r}(O)$. The intersection $\Gamma_r \cap E^{n+N-r}(O)$ is a compact variety in $E^{n+N-r}(O)$ of dimension

$$\delta_1 = n - rN\,, \tag{4.2}$$

Let $\mu_{n-rN}(\Gamma_r \cap E^{n+N-r}(O))$ denote the measure of this variety as subvariety of $E^{n+N-r}(O)$; if $\delta_1 = 0$, then μ_0 means the number of intersection points of Γ_r and $E^{n+N-r}(O)$.

We define the r-th total absolute curvature of X^n immersed in E^{n+N}, as the mean value of the measures μ_{n-rN} over all $E^{n+N-r}(O)$, that is, according to (3.6),

$$(4.3) \qquad K^*_{r,N}(X^n) = \frac{O_1 \ldots O_{n+N-r-1}}{O_r \ldots O_{n+N-1}} \int_{G_{n+N-r,r}} \mu_{n-rN}(\Gamma_r \cap E^{n+N-r}(O))\, dE^{n+N-r}(O).$$

The coefficient on the right-hand side may be replaced by

$$\frac{O_1 O_2 \ldots O_{r-1}}{O_{n+N-r} \ldots O_{n+N-1}}.$$

b) Case $n < r < n + N - 1$. In this case, instead of the set of E^r which «are contained» in some $T(x)$, we consider the set of all $E^r \subset E^{n+N}$ which «contain» some $T(x)$ and are perpendicular to $E^{n+N-r}(O)$. As before, we represent this set by Γ_r. The dimension of the variety $\Gamma_r \cap E^{n+N-r}(O)$ is now

$$(4.4) \qquad \delta_2 = n(r + 1 - n - N)$$

and the r-th total absolute curvature of X^n is defined by the same mean value (4.3) which now writes

$$(4.5) \qquad K^*_{r,N}(X^n) =$$

$$= \frac{O_1 \ldots O_{n+N-r-1}}{O_r \ldots O_{n+N-1}} \int_{G_{n+N-r,r}} \mu_{n(r+1-n-N)}(\Gamma_r \cap E^{n+N-r}(O))\, dE^{n+N-r}(O).$$

The dimensions δ_1, δ_2, given by (4.2), (4.4) have been calculated elsewhere ([24], [25]). From their values, and since $r < n + N - 1$, we deduce

i) The curvatures $K^*_{r,N}$ are only defined for

$$(4.6) \qquad n \geq rN \quad \text{and} \quad r = n + N - 1.$$

ii) If $r < n$ and X^n is immersed in $E^{n+N'}$ with $N' < N$, then $K^*_{r,N}(X^n) = 0$. This result follows from the fact that, if $X^n \subset E^{n+N'}$, all tangent spaces $T(x)$ are also contained in $E^{n+N'}$ and therefore μ_{n-rN} in (4.3) is zero except for the spaces $E^{n+N-r}(O)$ which are perpendicular to $E^{n+N'}$, which form a set of measure zero.

iii) The most interesting cases correspond to $n = rN$ and $r = n + N - 1$, for which the measures μ under the integral signs in (4.3) and (4.5) are non negative integers and therefore the total absolute curvatures are invariant at least under similitudes. We will consider these cases separately in the following sections.

iv) Consider the case $n = N$, $r = 1$. This case has the following geometrical interpretation. Let S^{2n-1} denote the unit $(2n-1)$-dimensional sphere in E^{2n} of center O. Let $E^n(x, O)$ be the n-plane through O parallel to the tangent space $T(x)$. The intersection $S^{2n-1} \cap E^n(x, O)$ is a $(n-1)$-dimensional great circle of S^{2n-1}. If we assume identified the pairs of antipodal points on S^{2n-1} we have the $(2n-1)$-dimensional elliptic space P^{2n-1} and the intersections $S^{2n-1} \cap \cap E^n(x, O)$ define a n-parameter family of $(n-1)$-planes in P^{2n-1}, say C_{n-1}. Let $\nu_{n-1}(y)$ be the number of $(n-1)$-planes of C_{n-1} which contain the point $y \in P^{2n-1}$ and let $\nu_{2n-1}(\eta)$ be the number of $(n-1)$-planes of C_{n-1} which are contained in the hyperplane η in P^{2n-1}. Let $d\sigma_{2n-1}(y)$ denote the volume element in P^{2n-1} at y, and let $dE^{2n-1}(\eta)$ denote the density of the hyperplanes of P^{2n-1} at η. Then, the curvatures (4.3) and (4.5) are clearly equal to

$$K^*_{1,n}(X^n) = \frac{2}{O_{2n-1}} \int_{P^{2n-1}} \nu_{n-1}(y)\, d\sigma_{2n-1}(y) \,, \tag{4.7}$$

$$K^*_{2n-1,n}(X^n) = \frac{2}{O_{2n-1}} \int_{E^{2n-1} \subset P^{2n-1}} \nu_{2n-1}(\eta)\, dE^{2n-1}(\eta) \,. \tag{4.8}$$

For $n = 2$, $N = 2$, $r = 1$ we have a congruence of lines C_1 in P^3 and, in a certain sense, the foregoing curvatures are the mean « order » and the mean « class » of the congruence C_1. This relation between compact surfaces of E^4 and congurences of lines in the elliptic space P^3 seems to deserve further attention.

5. A reproductive formula.

Let $X^n \subset E^{n+N}$. Consider the intersection $X^{s-N} = X^n \cap E^s$, $N < s < n + N$, and assume that X^{s-N} is a compact differentiable manifold of dimension $s - N$. Let $K^{*(s)}_{r,N}(X^{s-N})$, $r < s - N$, denote the total absolute curvature of X^{s-N} as a manifold immersed in E^s. We wish to prove the following « reproductive formula »

$$\int_{E^s \subset E^{n+N}} K^{*(s)}_{r,N}(X^{s-N})\, dE^s = \frac{O_{n+N-s} \ldots O_{n+N-1} O_{n+N-r} O_{s-N-rN}}{O_1 \ldots O_{s-1} O_{s-r} O_{n-rN}} K^*_{r,N}(X^n) \,, \tag{5.1}$$

where

$$n > s > (r+1)N . \tag{5.2}$$

Consider first the orthogonal linear spaces $E^s(O)$, $E^{n+N-r}(O)$ through a fixed point O and the intersection $E^{s-r}(O) = E^s(O) \cap E^{n+N-r}(O)$, $s > r$. Let $(O; e_1, e_2, \ldots, e_{n+N})$ be an orthonormal frame and suppose that $E^s(O)$ is spanned by the unit vectors $\{e_1, \ldots, e_s\}$, $E^{s-r}(O)$ is spanned by $\{e_{r+1}, \ldots, e_s\}$ and $E^{n+N-r}(O)$ is spanned by $\{e_{r+1}, \ldots, e_s, e_{s+1}, \ldots, e_{n+N}\}$. The density of $E^s(O)$ in E^{n+N} is

$$\begin{aligned} dE^s_{n+N}(O) = {} & (\omega_{1.s+1} \wedge \omega_{1.s+2} \wedge \cdots \wedge \omega_{1.n+N}) \\ & \wedge (\omega_{2.s+1} \wedge \omega_{2.s+2} \wedge \cdots \wedge \omega_{2.n+N}) \\ & \cdots\cdots\cdots\cdots\cdots\cdots \\ & \wedge (\omega_{r.s+1} \wedge \omega_{r.s+2} \wedge \cdots \wedge \omega_{r.n+N}) \\ & \cdots\cdots\cdots\cdots\cdots\cdots \\ & \wedge (\omega_{s.s+1} \wedge \omega_{s.s+2} \wedge \cdots \wedge \omega_{s.n+N}) . \end{aligned} \tag{5.3}$$

The density of $E^{s-r}(O)$ as subspaces of $E^s(O)$ is

$$\begin{aligned} dE^{s-r}_s(O) = {} & (\omega_{r+1.1} \wedge \omega_{r+1.2} \wedge \cdots \wedge \omega_{r+1.r}) \\ & \wedge (\omega_{r+2.1} \wedge \omega_{r+2.2} \wedge \cdots \wedge \omega_{r+2.r}) \\ & \cdots\cdots\cdots\cdots\cdots\cdots \\ & \wedge (\omega_{s.1} \wedge \omega_{s.2} \wedge \cdots \wedge \omega_{s.r}) \end{aligned} \tag{5.4}$$

and as subspace of E^{n+N-r}

$$\begin{aligned} dE^{s-r}_{n+N-r}(O) = {} & (\omega_{r+1.s+1} \wedge \cdots \wedge \omega_{r+1.n+N}) \\ & \wedge (\omega_{r+2.s+1} \wedge \cdots \wedge \omega_{r+2.n+N}) \\ & \cdots\cdots\cdots\cdots\cdots\cdots \\ & \wedge (\omega_{s.s+1} \wedge \cdots \wedge \omega_{s.n+N}) . \end{aligned} \tag{5.5}$$

Finally, the density of $E^{n+N-r}(O)$ in E^{n+N} is

$$\begin{aligned} dE^{n+N-r}_{n+N}(O) = {} & (\omega_{r+1.1} \wedge \omega_{r+1.2} \wedge \cdots \wedge \omega_{r+1.r}) \\ & \wedge (\omega_{r+2.1} \wedge \omega_{r+2.2} \wedge \cdots \wedge \omega_{r+2.r}) \\ & \cdots\cdots\cdots\cdots\cdots\cdots \\ & \wedge (\omega_{n+N.1} \wedge \omega_{n+N.2} \wedge \cdots \wedge \omega_{n+N.r}) . \end{aligned} \tag{5.6}$$

Since we are only interested in the absolute value of the densities, we make no question on the order in the exterior products.

From (5.3) to (5.6) we deduce the identity

$$dE_s^{s-r}(O)\wedge dE_{n+N}^s(O) = dE_{n+N-r}^{s-r}(O)\wedge dE_{n+N}^{n+N-r}(O)\,. \tag{5.7}$$

According to the definition (4.3) we have

$$K_{r,N}^{*(s)}(X^{s-N}) = \frac{O_1\dots O_{s-r-1}}{O_r\dots O_{s-1}}\int\limits_{G_{s-r,r}}\mu_{s-N-rN}\,dE_s^{s-r}(O)\,, \tag{5.8}$$

where μ_{s-N-rN} denotes the measure of the $(s-N-rN)$-dimensional variety in $E_s^{s-r}(O)$ generated by the intersection points of $E_s^{s-r}(O)$ with the r-planes in E^s which are perpendicular to $E_s^{s-r}(O)$ and are contained in some tangent space of X^{s-N}. From (3.5) and (5.7) we have

$$\int\limits_{E^s\cap X^n\neq\emptyset} K_{r,N}^{*(s)}(X^{s-N})\,dE_{n+N}^s = \tag{5.9}$$

$$= \frac{O_1\dots O_{s-r-1}}{O_r\dots O_{s-1}}\int\mu_{s-N-rN}\,dE_{n+N-r}^{s-r}(O)\wedge dE_{n+N}^{n+N-r}(O)\wedge\omega_{s+1}\wedge\dots\wedge\omega_{n+N}\,.$$

The form $\omega_{s+1}\wedge\dots\wedge\omega_{n+N}$ is equal to the element of volume in E^{n+N} orthogonal to E^s, which is also equal to the element of volume in E^{n+N-r} orthogonal to E^{s-r} and therefore we have

$$dE_{n+N-r}^{s-r}(O)\wedge\omega_{s+1}\wedge\dots\wedge\omega_{n+N} = dE_{n+N-r}^{s-r} \tag{5.10}$$

(= density of $(s-r)$-planes, not necessarily through O, in $E^{n+N-r}(O)$) and (5.9) gives

$$\int\limits_{E_{n+N}^s\cap X^n\neq\emptyset} K_{r,N}^{*(s)}(X^{s-n})\,dE_{n+N}^s = \tag{5.11}$$

$$= \frac{O_1\dots O_{s-r-1}}{O_r\dots O_{s-1}}\int\mu_{s-N-rN}\,dE_{n+N-r}^{s-r}\wedge dE_{n+N}^{n+N-r}(O)\,.$$

Applying (3.20) to the $(n-rN)$-dimensional variety Y^{n-rN} in $E_{n+N}^{n+N-r}(O)$ generated by the intersection points of $E_{n+N}^{n+N-r}(O)$ with the linear r-spaces of E^{n+N} which are perpendicular to $E_{n+N}^{n+N-r}(O)$ and are contained in some tangent space of X^n and to the $(s-r)$-planes of

$E^{n+N-r}_{n+N}(O)$ which intersect Y^{n-rN} we have

$$(5.12) \qquad \int \mu_{s-N-rN}\, dE^{s-r}_{n+N-r} = \frac{O_{n+N-r} \dots O_{n+N-s} O_{s-N-rN}}{O_1 \dots O_{s-r} O_{n-rN}} \mu_{n-rN} ,$$

where μ_{n-rN} denotes the measure of Y^{n-rN}.

Thus (5.11) writes

$$\int_{E^s_{n+N} \cap X^n \neq \emptyset} K^{*(s)}_{r,N}(X^{s-N})\, dE^s_{n+N} =$$

$$= \frac{O_1 \dots O_{s-r-1}\, O_{n+N-r} \dots O_{n+N-s}\, O_{s-N-rN}}{O_r \dots O_{s-1} O_1 \dots O_{s-r} O_{n-rN}} \int_{G_{n+N-r,r}} \mu_{n-rN}\, dE^{n+N-r}_{n+N}(O) .$$

This formula and the definition (4.3), give the desired formula (5.1).

6. The case $K^*_{n+N-1,N}(X^n)$: curvature of Chern-Lashof.

The case $r = n + N - 1$ gives rise to the curvature defined by Chern and Lashof [11], [12]. The identity of both curvatures will be apparent from the analytical expression of K^*_{n+N-1}, which will be given in a subsequent section. For the moment, we wish to show how the geometrical definition above allows to obtain directly some known properties of the Chern-Lashof curvature.

a) Notice that $\mu_0(\Gamma_{n+N-1} \cap E^{n+N-1}(O))$ in (4.5) is equal to the number ν of hyperplanes E^{n+N-1} which are perpendicular to a given line $E^1(O)$ and contain some tangent space $T(x)$ of X^n. This number is surely ≥ 2, since there are at least the two support hyperplanes of X^n which are perpendicular to $E^1(O)$. Therefore we have $K^*_{n+N-1,N}(X^n) \geq 2$ (theorem 1 of Chern-Lashof [11]).

For an oriented surface X^2 (compact) the number of hyperplanes of support which are perpendicular to a direction $E^1(O)$ is $\geq 2(1+g)$, where g is the genus of X^2, related to the Euler characteristic by $\chi(X^2) = 2(1-g)$. Thus we have

$$(6.1) \qquad K^*_{N+1,N}(X^2) \geq 2(1+g) = 4 - \chi(X^2) .$$

b) The inequality $K_{n+N-1,N}(X^n) < 3$, means that there exists a set of directions $E^1(O)$ (with positive measure) such that the number of hyperplanes in E^{n+N} which contain some $T(x)$ and are perpendicular to $E^1(O)$ is exactly 2, a condition which suffices for X^n to be homeomorphic to a n-dimensional sphere (theorem 2 of Chern-Lashof [11]).

c) Assume that $X^n \subset E^{n+N}(O) \subset E^{n+N+1}(O)$. To each hyperplane E^{n+N} in $E^{n+N+1}(O)$ which is perpendicular to the line $E^1_{n+N+1}(O)$ and contains some $T(x)$ corresponds the hyperplane $E^{n+N-1} = E^{n+N} \cap E^{n+N}(O)$ of E^{n+N} which is perpendicular to the projection $E^1_{n+N}(O)$ of $E^1_{n+N+1}(O)$ into E^{n+N}. According to (3.10) we have

$$K^*_{n+N,N+1}(X^n) = \frac{1}{O_{n+N}} \int_{G_{1,n+N}} \nu \, dE^1_{n+N+1}(O) =$$

$$= \frac{1}{O_{n+N}} \int_{G_{1,n+N-1}} \nu \sin^{n+N-1}\theta \, dE^1_{n+N}(O) \wedge d\theta = \frac{1}{O_{n+N-1}} \int_{G_{1,n+N-1}} \nu \, dE^1_{n+N}(O) =$$

$$= K^*_{n+N-1,N}(X^n) \, .$$

By induction on N, we get that the total absolute curvature $K^*_{n+N-1,N}(X^n)$ of $X^n \subset E^{n+N}$ does not change if we consider X^n as an immersed manifold in $E^{n+N+\nu} \supset E^{n+N}$ (Lemma 1 of Chern-Lashof [12]).

7. The case $n = rN$. Local representation of the curvatures $K^*_{r,N}(X^n)$.

Let x be a point of the manifold X^n immersed in E^{n+N} and consider the frame $(x; e_1, e_2, \ldots, e_{n+N})$ of Sect. 2. The density for r-planes through x is given by (3.2) which we will now write

$$dE^r_{n+N}(x) = \Lambda \omega_{im} \quad (i = 1, 2, \ldots, r;\ m = r+1, r+2, \ldots, n+N) \tag{7.1}$$

where $r < n$. The density for r-planes $E^r_n(x)$ in the tangent space $T(x)$ spanned by $e_1, e_2, \ldots, e_n$ is

$$dE^r_n(x) = \Lambda \omega_{im} \quad (i = 1, 2, \ldots, r;\ m = r+1, \ldots, n) \, . \tag{7.2}$$

The densities (7.1), (7.2) refers to the r-space spanned by $(e_1, e_2, \ldots, e_r)$. It is important to note that if $r = n$, $N = 1$, the density (7.2) is not defined. Since we have in this case only one $E^n_n (= T(x))$ its average is the same space, so that in this case we must cancel dE^n_n (and the corresponding integrations) in all the formulae in which it appears. On the other hand, this case corresponds to the well known case of hypersurfaces X^n in E^{n+1} and the curvature here defined is the absolute value of the classical Gauss-Kronecker curvature.

The element of volume of X^n at x is

$$d\sigma_n(x) = \omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_n \, . \tag{7.3}$$

Assuming $n = rN$, the differential forms $dE^r_{n+N}(x)$ and $dE^r_n(x) \wedge \wedge d\sigma_n(x)$ have the same degree, so that we can define a function $G(x, E^r_n(x))$ by the equation (as noted in section 2, the differential forms in this equality must be considered as forms in the bundle of frames trangent to X^n; for details, see [11] or [29])

$$dE^r_{n+N}(x) = G(x, E^r_n(x))\, dE^r_n(x) \wedge d\sigma_n(x) . \tag{7.4}$$

Calling $\nu = \nu(E^r_{n+N})$ the number of r-planes E^r_{n+N} which are parallel to $E^r_{n+N}(x)$ and belong to some tangent space $T(x)$ of X^n, (7.4) gives

$$\int\limits_{G_{r,n+N-r}} \nu\, dE^r_{n+N}(x) = \int\limits_{X^n} \Big(\int\limits_{G_{r,n-r}} |G(x, E^r_n(x))| dE^r_n(x) \Big) \wedge d\sigma_n(x) . \tag{7.5}$$

Thus, setting

$$K^*_{r,N}(X^n) = \int\limits_{X^n} Q^*_{r,N}(x)\, d\sigma_n \tag{7.6}$$

according to (4.3) and (7.5), (having into account (3.4)), we have

$$Q^*_{r,N}(x) = \frac{O_1 \dots O_{n+N-r-1}}{O_r \dots O_{n+N-1}} \int\limits_{G_{r,n-r}} |G(x, E^r_n(x))|\, dE^r_n(x) . \tag{7.7}$$

From (7.4) and (7.1), (7.2), (7.3) we can obtain the expression for the « local » sectional curvature $G(x, E^r_n)$ corresponding to the point x and the section $E^r_n(x)$ (spanned by the unit vectors $e_1, e_2, \dots, e_r$). We get

$$G(x, E^r_n)\, d\sigma_n = \Lambda \omega_{im} \tag{7.8}$$

$$(i = 1, 2, \dots, r;\ m = n+1, n+2, \dots, n+N) .$$

Using (2.5) we get

$$G(x, E^r_n(x)) = \begin{vmatrix} A_{n+1,11} & A_{n+1,12} & \dots & A_{n+1,1n} \\ \dots & \dots & \dots & \dots \\ A_{n+N,11} & A_{n+N,12} & \dots & A_{n+N,1n} \\ A_{n+1,21} & A_{n+1,22} & \dots & A_{n+1,2n} \\ \dots & \dots & \dots & \dots \\ A_{n+N,21} & A_{n+N,22} & \dots & A_{n+N,2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ A_{n+1,r1} & A_{n+1,r2} & \dots & A_{n+1,rn} \\ \dots & \dots & \dots & \dots \\ A_{n+N,r1} & A_{n+N,r2} & \dots & A_{n+Nrn} \end{vmatrix} \tag{7.9}$$

the determinant being of order n because $n = rN$.

This formula corresponds to the r-plane spanned by $e_1, e_2, \ldots, e_r$. For a general r-plane in $T(x)$ spanned by the set of orthogonal vectors $e'_1, e'_2, \ldots, e'_r$ of the frame $e'_i = \gamma_{ih} e_h$ $(i, h = 1, 2, \ldots, n)$ defined by the orthogonal matrix (γ_{ih}), the elements $A_{\alpha,ij}$ in (7.9) must be substituted by $A'_{\alpha,ij} = \gamma_{ih}\gamma_{jm}A_{\alpha,hm}$ (h, m summed over the ranges $h, m = 1, 2, \ldots, n$) as it follows easily from (2.5) and (2.2).

In order to evaluate $Q^*_{r,N}(x)$ we must compute the mean value of $|G(x, E^r_n(x))|$ over all $E^r_n(x)$ (*i.e.* over the Grassmann manifold $G_{r,n-r}$). Actual evaluation of this mean value seems to be difficult. We will only consider some particular cases in the following sections. As follows either from the geometrical definition or from (7.8), if $r = n$, $N = 1$, $G(x, E^n_n(x)) = G$ is the classical Gauss-Kroneker curvature of X^n at the point x, and we have

$$Q^*_{n,N} = \frac{1}{O_n}|G|. \tag{7.10}$$

8. Local representation of $K^*_{n+N-1,N}(X^n)$.

The hyperplanes in E^{n+N} which contain some tangent space $T(x)$ of X^n, may be determined by its normal vector $E^1_N(x)$ in the normal space to X^n at x, *i.e.* in the N-space spanned by the vectors e_{n+1}, $e_{n+2}, \ldots, e_{n+N}$. Then, instead of the equation (7.4) we consider

$$dE^1_{n+N}(x) = \bar{G}(x, E^1_N(x))\, dE^1_N(x) \wedge d\sigma](x) \tag{8.1}$$

and $K^*_{n+N-1}(X^n)$ may be written

$$K^*_{n+N-1,N}(X^n) = \int_{X^n} Q^*_{n+N-1,N}(x)\, d\sigma_n(x) \tag{8.2}$$

where

$$Q^*_{n+N-1,N}(x) = \frac{1}{O_{n+N-1}} \int_{G_{1,N-1}} |\bar{G}(x, E^1_N(x))|\, dE^1_N(x). \tag{8.3}$$

(8.1), (8.2) and (8.3) show that the absolute total curvature $K^*_{n+N-1,N}(X^n)$ coincides with the Chern-Lashof curvature [11], [12] as stated in section 6.

Taking $E^1_N(x)$ to be the line of the unit vector e_{n+N} and writting $\bar{G}(x, e_{n+N})$ instead of $\bar{G}(x, E^1_N(x))$, from (8.1) we deduce

$$\bar{G}(x, e_{n+N})\, d\sigma_n(x) = \omega_{n+N,1} \wedge \omega_{n+N,2} \wedge \cdots \wedge \omega_{n+N,n} \tag{8.4}$$

or, by virtue of (2.5),

$$(8.5) \qquad \bar{G}(x, e_{n+N}) = (-1)^n \det (A_{n+N,ij})$$

with $i, j = 1, 2, \ldots, n$.

If, instead of e_{n+N} we consider the general normal vector $e = \cos\theta_s e_{n+s}$ $(s = 1, 2, \ldots, N)$, we get

$$(8.6) \qquad \bar{G}(x, e) = (-1)^n \det (\cos\theta_s A_{n+s,ij})$$

and to get $Q^*_{n+N-1,N}(x)$ (= absolute curvature at x = Chern-Lashof curvature at x) we must evaluate the mean value of $|\bar{G}(x, e)|$ over the $(N-1)$-dimensional unit sphere (*i.e.* over $\cos^2\theta_1 + \cos^2\theta_2 + \ldots + \cos^2\theta_N = 1$). Only in some simple cases, this calculation has been carried out.

9. Total (no absolute) curvatures $K_{n+N-1,N}(X^n)$.

The total absolute curvatures $K^*_{r,N}(X^n)$ are easily defined geometrically by (4.3) or (4.5), but their actual evaluation seems to be difficult, mainly due to the absolute values under the integral sign in (7.7) and (8.3). From the analytical point of view, it is much more natural to consider the curvatures « defined » by the same formulae (7.7), (8.3) and then (7.6) and (8.2) without the absolute value under the integral sign. We shall denote these no absolute curvatures by $Q_{r,N}(x)$ and $K_{r,N}(X^n)$ (or $Q_{n+N-1,N}(x)$ and $K_{n+N-1,N}(X^n)$) respectively. One can handle analytically with these curvatures more easily than with the absolute curvatures, but for a geometrical interpretation like (4.3) or (4.5) it is necessary to provide an orientation (or a sign) to the manifolds $\Gamma_r \cap E^{n+N-1}(O)$ and some difficulties arise.

We will first consider the case $K_{n+N-1,N}(X^n)$. We define

$$(9.1) \qquad Q_{n+N-1,N}(x) = \frac{1}{O_{n+N-1}} \int_{G_{1,N-1}} \bar{G}(x, E^1_N(x))\, dE^1_N(x)$$

where $\bar{G}(x, E^1_N(x))$ is defined by (8.5), (8.4) if $E^1_N(x)$ is the line spanned by the vector e_{n+N} or by (8.6) if $E^1_N(x)$ is the line spanned by the vector e. From (9.1) we define

$$(9.2) \qquad K_{n+N-1,N}(X^n) = \int_{X^n} Q_{n+N-1,N}(x)\, d\sigma_n(x).$$

To calculate the mean value (9.1) we consider the unit vector e on the line $E^1_N(x)$, say $e = \cos\theta_s e_{n+s}$ ($s = 1, 2, \ldots, N$; $\cos^2\theta_1 + \cos^2\theta_2 + \ldots + \cos^2\theta_N = 1$). We have

$$(9.3) \qquad \bar{G}(x, e)\, d\sigma_n(x) = (-1)^n (e\, de_1) \wedge (e\, de_2) \wedge \ldots \wedge (e\, de_n)$$
$$= \Lambda(\cos\theta_1 \omega_{n+1,i} + \cos\theta_2 \omega_{n+2,i} + \ldots + \cos\theta_N \omega_{n+N,i})$$

where in the exterior product on the right-hand side we have $i = 1, 2, \ldots, n$.

The forms $\omega_{n+s,i}$ do not depend on θ_s. Thus, in order to compute (9.1) we must calculate the mean value of monomials $\cos^{\lambda_1}\theta_1 \cos^{\lambda_2}\theta_2 \ldots \cos^{\lambda_N}\theta_N$ with $\lambda_1 + \lambda_2 + \ldots + \lambda_N = n$ over the N-sphere $\cos^2\theta_1 + \cos^2\theta + \ldots + \cos^2\theta_N = 1$. These mean values are known: they are zero unless all exponents λ_i are even, and in the later case their values are

$$(9.4) \qquad E(\cos^{\lambda_1}\theta_1 \ldots \cos^{\lambda_N}\theta_N) = \frac{\lambda_1)\, \lambda_2) \ldots \lambda_N)}{N(N+2)\ldots(N+n-2)}$$

where λ_i even, $\lambda_1 + \ldots + \lambda_N = n$ and $\lambda) = 1.3 \ldots (\lambda - 1)$. From these mean values, expanding the exterior product (9.3) and using (2.5) and (2.8), by some invariant-theoretic arguments dues to H. Weyl [28], one can deduce the following explicit form of the curvature $Q_{n+N-1,N}(x)$ (n even)

$$(9.5) \qquad Q_{n+N-1,N}(x) = \frac{1}{2^n (2\pi)^{n/2} (n/2)!}\, \delta^{i_1 i_2 \ldots i_n}_{j_1 j_2 \ldots j_n} R_{i_1 i_2 j_1 j_2} R_{i_3 i_4 j_3 j_4} \ldots R_{i_{n-1} i_n j_{n-1} j_n}$$

where $\delta^{i_1 i_2 \ldots i_n}_{j_1 j_2 \ldots j_n}$ is equal to $+1$ or -1 according as $(i_1 i_2 \ldots i_n)$ is an even or odd permutation of $(j_1 j_2 \ldots j_n)$ and is otherwise zero and the summation is over all $i_1, i_2, \ldots, i_n$ and $j_1, j_2, \ldots, j_n$ independently from 1 to n. If n is odd, $Q_{n+N-1,N}(x) = 0$. Notice that $Q_{n+N-1,N}$ does not depend upon N.

This curvature (9.5) is called the curvature of Lipschitz-Killing (Chern-Lashof [11], Thorpe [27]). It appears in the work of H. Weyl on the volume of tubes [28] and in several papers of Chern ([7], [9], [10]) and others. The total curvature $K_{n+N-1,N}(X^n)$ (n even) gives the Euler-Poincaré characteristic of X^n, according to the formula of Gauss-Bonnet:

$$(9.6) \qquad K_{n+N-1,N}(X^n) = \chi(X^n)\,.$$

The case $n=2$. For surfaces $X^2 \subset E^{2+N}$, we have $Q_{N+1,N} = (1/2\pi) R_{1212} = K/2\pi$, where K is the Gaussian curvature (2.16). The expression of $\bar{G}(x, e)$ (9.3) is a quadratic form in the variables $\cos\theta_i$. Under the hypothesis that this quadratic form is everywhere positive or negative definite, we have

$$Q^*_{N+1,N} = Q_{N+1,N} = K/2\pi \qquad \text{if } K > 0$$

$$Q^*_{N+1,N} = -Q_{N+1,N} = -K/2\pi \qquad \text{if } K < 0 .$$

Hence we have

$$K^*_{N+1,N} = (1/2\pi)\Big(\int_U K\,d\sigma - \int_V K\,d\sigma\Big)$$

where $U = \{x \in X^2;\ K(x) > 0\}$, $V = \{x \in X^2;\ K(x) < 0\}$.

The inequality (6.1) and the Gauss-Bonnet theorem give

$$\int_U K\,d\sigma - \int_V K\,d\sigma \geq 4\pi(1+g)\,, \qquad \int_U K\,d\sigma + \int_V K\,d\sigma = 2\pi\chi(X^2) = 4\pi(1-g)\,.$$

Thus, we have: If the quadratic form $\bar{G}(x, e)$ (9.3) is everywhere definite (positive or negative) on the surface $X^2 \subset E^{2+N}$, then the following inequalities hold

$$\int_U K\,d\sigma \geq 4\pi\,, \qquad \int_V K\,d\sigma \leq -4\pi g\,. \tag{9.7}$$

These inequalities are due to B. Y. Chen [3].

10. The case $n = N$, $r = 1$.

If $r = 1$ and e_1 is the unit vector on the line $E^1_n(x)$, equation (7.8) writes

$$G(x, e_1)\,d\sigma_n(x) = \omega_{1.n+1} \wedge \omega_{1.n+2} \wedge \dots \wedge \omega_{1.n+N}\,. \tag{10.1}$$

For the general tangent vector $e = \cos\theta_i e_i$ $(i = 1, 2, \dots, n)$ we have

$$G(x, e_1) = \Lambda(\cos\theta_1\omega_{1.n+s} + \cos\theta_2\omega_{2.n+s} + \dots + \cos\theta_n\omega_{n.n+s}) \tag{10.2}$$

where $s = 1, 2, \dots, N$.

According to (7.7) we have now

$$Q_{1,N}(x) = \frac{1}{O_{n+N-1}} \int_{G_{1,n-1}} G(x, e)\,dE^1_n(x) \tag{10.3}$$

i.e. $Q_{1,N}(x)$ is the mean value of $G(x, e)$ over the unit sphere $\cos^2\theta_1 + \cos^2\theta_2 + \ldots + \cos^2\theta_n = 1$, which may be evaluated by the same method of H. Weyl of the preceding section. The result is $Q_{1,N}(x) = 0$ if $n = N$ is odd and

$$(10.4) \qquad Q_{1,N}(x) = \frac{1}{2^n(2\pi)^{n/2}(n/2)\}} \delta^{i_1 i_2 \ldots i_n}_{j_1 j_2 \ldots j_n} R_{\alpha_1\alpha_2 j_1 j_2} R_{\alpha_3\alpha_4 j_3 j_4} \ldots R_{\alpha_{n-1}\alpha_n j_{n-1} j_n}$$

where $\alpha_h = n + i_h$ $(h = 1, 2, \ldots, n)$ if $n = N$ is even.

Notice that $Q_{1,N}(X)$ depends on the immersion.

EXAMPLE: For $n = N = 2$, $r = 1$, having into account the properties of symmetry (2.12) we get

$$(10.5) \qquad Q_{1,2}(x) = \frac{1}{2\pi} R_{3412}\,.$$

11. The cases $n + N \leq 6$.

In the following sections we wish to consider some particular cases. For $n + N \leq 6$, the conditions $n = rN$ and $r = n + N - 1$ give the following possibilities:

a) $n = 2$, $N = 1$, $r = 2$. Corresponds to the classical case of surfaces $X^2 \subset E^3$. We have $Q_{2,1}(x) = (1/2\pi)K$, $K =$ Gaussian curvature. Consideration of $Q^*_{2,1}$ and $K^*_{2,1}$ gives rise to interesting problems (Kuiper [17], Willmore [29]).

b) $n = 2$, $N = 2$, $r = 1$ and $n = 2$, $N = 2$, $r = 3$. These cases correspond to $X^2 \subset E^4$ and will be considered with detail in the next section.

c) $n = 3$, $N = 3$, $r = 1$: $X^3 \subset E^6$. Particular case of the case considered in sections 7 and 10. Since $n = N = 3$, is odd, we have $Q_{1,3}(x) = 0$.

d) $n = 3$, $N = 2$, $r = 4$: $X^3 \subset E^5$. Particular case of the case considered in section 9. Since $n = 3$ is odd, we have $Q_{4,2}(x) = 0$.

e) $n = 3$, $N = 1$, $r = 3$: $X^3 \subset E^4$. Hypersurfaces in E^4. $Q_{3,1}(x) = (2\pi^2)^{-1}K$ ($K =$ Gauss-Kronecker curvature).

f) $n = 3$, $N = 3$, $r = 5$: $X^3 \subset E^6$. Particular case of the case considered in section 9. $Q_{5,3}(x) = 0$.

g) $n = 4$, $N = 1$, $r = 4$: $X^4 \subset E^5$. Hypersurfaces in E^5. $Q_{4,1}(x) = (8\pi^2/3)^{-1}K$ ($K =$ Gauss-Kronecker curvature).

h) $n = 4, N = 2, r = 2$: $X^4 \subset E^8$. This is a noteworthy case which will be discussed in section 13.

i) $n = 4, N = 2, r = 5$: $X^4 \subset E^8$. Particular case of the case considered in section 9.

j) $n = 5, N = 1, r = 5$: $X^5 \subset E^8$. Particular case of the case considered in section 9.

12. Surfaces in E^4.

We will consider separately the cases *a*) $n = 2, N = 2, r = 3$, and *b*) $n = 2, N = 2, r = 1$.

a) *The case* $n = 2, N = 2, r = 3$. Putting $\theta_1 = \theta$, $\theta_2 = (\pi/2) - \theta$ into (9.3) we have

$$(12.1) \qquad \bar{G}(x, e)\omega_1 \wedge \omega_2 = \cos^2\theta\omega_{31} \wedge \omega_{32} + \sin^2\theta\omega_{41} \wedge \omega_{42} + \\ + \sin\theta\cos\theta(\omega_{31} \wedge \omega_{42} + \omega_{41} \wedge \omega_{32}) .$$

The density for lines about a point in E^2 is $dE_2^1(x) = d\theta$ and thus

$$(12.2) \qquad \int_0^{2\pi} \bar{G}(x, e)\omega_1 \wedge \omega_2 \wedge d\theta = \pi(\omega_{31} \wedge \omega_{32} + \omega_{41} \wedge \omega_{42}) .$$

Therefore we have

$$(12.3) \qquad Q_{3,2}(x)\omega_1 \wedge \omega_2 = \frac{1}{2\pi}(\omega_{31} \wedge \omega_{32} + \omega_{41} \wedge \omega_{42}) .$$

The Gaussian curvature of X^2 at x is defined by $d\omega_{12} = -K(x)\omega_1 \wedge \omega_2$. Thus, according to (2.3) and (12.3) we get

$$(12.4) \qquad Q_{3,2}(x) = \frac{1}{2\pi} K(x) .$$

Integration over X^2 and application of the Gauss-Bonnet formula for surfaces, gives $K_{3,2}(X^2) = \chi(X^2)$, in accordance with (9.6).

We will now consider the absolute curvature $Q^*_{3,2}$. To this end it is convenient to introduce the normal curvatures of Otsuki [19]. Notice that the form $\omega_{31} \wedge \omega_{42} + \omega_{41} \wedge \omega_{32}$ remains invariant under rotations $e_1 \to \cos\alpha e_1 + \sin\alpha e_2$, $e_2 \to -\sin\alpha e_1 + \cos\alpha e_2$ on the tangent plane, but it can be annihilated by a suitable rotation on the normal plane e_3, e_4. Hence, choosing a suitable pair e_3, e_4 of normal unit vectors one can get

$$\omega_{31}\wedge\omega_{42}+\omega_{41}\wedge\omega_{32}=0\,. \tag{12.5}$$

Then, assuming that the forms ω_{ij} refer to the new frame, we define the normal curvatures λ_n, μ_n (Otsuki's curvatures) by

$$\omega_{31}\wedge\omega_{32}=\lambda_n\omega_1\wedge\omega_2\,,\qquad \omega_{41}\wedge\omega_{42}=\mu_n\omega_1\wedge\omega_2 \tag{12.6}$$

so that according to (12.3) and (12.4) we have

$$\lambda_n+\mu_n=K=\text{Gauss curvature}\,. \tag{12.7}$$

Having into account (12.5), equation (12.1) writes

$$\bar{G}(x, e)=\cos^2\theta\lambda_n+\sin^2\theta\mu_n \tag{12.8}$$

where we may assume

$$\lambda_n>\mu_n\,. \tag{12.9}$$

If $\lambda_n\mu_n>0$, the absolute curvature at x is

$$Q^*_{2,2}(x)=\frac{1}{2\pi^2}\int_0^{2\pi}|G(x, e)|\,d\theta=\frac{1}{2\pi}|\lambda_n+\mu_n|=\frac{1}{2\pi}|K|\,. \tag{12.10}$$

If $\lambda_n\mu_n<0$ we notice that

$$\lambda_n\cos^2\theta+\mu_n\sin^2\theta>0 \qquad \text{if} \qquad |\theta|<\arctan\sqrt{-\lambda_n/\mu_n}\,,$$

$$\lambda_n\cos^2\theta+\mu_n\sin^2\theta<0 \qquad \text{if} \qquad \arctan\sqrt{-\lambda_n/\mu_n}<|\theta|<\pi/2$$

and

$$\int_0^{2\pi}|G(x, e)|\,d\theta=4\int_0^{\pi/2}|\lambda_n\cos^2\theta+\mu_n\sin^2\theta|\,d\theta=$$

$$=4\left\{\sqrt{-\lambda_n\mu_n}+(\lambda_n+\mu_n)\left(\arctan\sqrt{-\lambda_n/\mu_n}-\pi/4\right)\right\}.$$

Therefore we have

$$Q^*_{2,2}(x)=\frac{2}{\pi^2}\left\{\sqrt{-\lambda_n\mu_n}+K\left(\arctan\sqrt{-\lambda_n/\mu_n}-\pi/4\right)\right\}. \tag{12.11}$$

We shall do two simple applications of the preceding results.

i) If X^2 is orientable and $K = \lambda_n + \mu_n = 0$ (flat torus), we have

$$K^*_{3,2}(X^2) = \frac{2}{\pi^2}\int_{X^2} \lambda_n \, d\sigma_2 \, .$$

Applying the inequality (6.1), having into account that $K = 0$ implies $g = 1$, we get the following inequality of Otsuki [19]:

$$\int_{X^2} \lambda_n \, d\sigma_2 \geq 2\pi^2 \, .$$

ii) If $\mu_n > 0$, $\lambda_n > 0$, we have $Q^*_{3,2} = K/2\pi$ and the Gauss-Bonnet theorem gives

$$K^*_{3,2} = \int_{X^2} Q^*_{3,2} \, d\sigma_2 = \frac{1}{2\pi}\int_{X^2} K \, d\sigma_2 = \chi(X^2) \, .$$

Inequality (6.1) gives then $\chi(X^2) \geq 2$ and we have the following theorem of Chen [4]: *if* $\mu_n \geq 0$, $\lambda_n > 0$, *then* X^2 *is homeomorphic to a* 2-*sphere*.

b) *The case* $n = 2$, $N = 2$, $r = 1$. This is a particular case of that considered in section 10. Putting $e = \cos\theta e_1 + \sin\theta_2 e_2$, (10.2) becomes

$$\begin{aligned} (12.12) \qquad G(x, e)\omega_1 \wedge \omega_2 &= (\cos\theta\omega_{13} + \sin\theta\omega_{23}) \\ &\quad \wedge (\cos\theta\omega_{14} + \sin\theta\omega_{24}) \\ &= \cos^2\theta\omega_{13}\wedge\omega_{14} + \sin^2\theta\omega_{23}\wedge\omega_{24} + \sin\theta\cos\theta(\omega_{13}\wedge\omega_{24} + \omega_{23}\wedge\omega_{14}) \, . \end{aligned}$$

The form $\omega_{13}\wedge\omega_{24} + \omega_{23}\wedge\omega_{14}$ remains invariant under changes of frames in the normal plane, but by a suitable rotation $e_1 \to \cos\alpha e_1 + \sin\alpha e_2$, $e_2 \to -\sin\alpha e_1 + \cos\alpha e_2$ in the tangent plane, we may attain that

$$(12.13) \qquad \omega_{13}\wedge\omega_{24} + \omega_{23}\wedge\omega_{14} = 0 \, .$$

Assuming the frame $(x; e_1, e_2, e_3, e_4)$ chosen in such a way that (12.13) holds, we put

$$(12.14) \qquad \omega_{13}\wedge\omega_{14} = \lambda_t \omega_1\wedge\omega_2 \, , \qquad \omega_{23}\wedge\omega_{24} = \mu_t \omega_1\wedge\omega_2$$

where λ_t, μ_t are the tangent curvatures of X^2 at x.

The curvature $Q_{1,2}(x)$ is then

$$Q_{1,2}(x) = \frac{1}{2\pi^2}\int_0^{2\pi}(\lambda_t \cos^2\theta + \mu_t \sin^2\theta)\,d\theta = \frac{1}{2\pi}(\lambda_t + \mu_t) \tag{12.15}$$

and the absolute curvature takes the values

$$Q^*_{1,2}(x) = \frac{1}{2\pi}|\lambda_t + \mu_t| \quad \text{if} \quad \lambda_t\mu_t \geq 0 \tag{12.16}$$

and

$$Q^*_{1,2}(x) = \frac{2}{\pi^2}\{\sqrt{-\lambda_t\mu_t} + (\lambda_t + \mu_t)(\operatorname{arc\,tan}\sqrt{-\lambda_t/\mu_t} - \pi/4)\} \tag{12.17}$$

if $\lambda_t\mu_t < 0$.

If we compare with the preceding case $Q^*_{2,2}(x)$ we observe that, instead of the Gaussian curvature K, we now have the invariant $I = \lambda_t + \mu_t$, such that

$$I\omega_1\wedge\omega_2 = (\lambda_t + \mu_t)\omega_1\wedge\omega_2 = \omega_{13}\wedge\omega_{14} + \omega_{23}\wedge\omega_{24}\,. \tag{12.18}$$

Notice that $d\omega_{34} = -I\omega_1\wedge\omega_2$ and therefore, since every orientable $X^2 \subset E^4$ has a continuous field of normal vectors (Seifert [26]), from the Stokes theorem follows that

$$\int_{X^2} d\omega_{34} = -\int_{X^2} I\omega_1\wedge\omega_2 = 0 \tag{12.19}$$

i.e. the invariant $I(x)$ does not give any non trivial invariant by integration over X^2.

The curvatures $\lambda_t, \mu_t, \lambda_n, \mu_n$ are not independent. From their definition follows easily that

$$\lambda_n\mu_n = \lambda_t\mu_t\,. \tag{12.20}$$

The invariant I has been introduced by Blaschke [2] and, from a more topological point of view, it has been considered by Chern-Spanier [13]. It is easy to see that I (like K) remains invariant under changes of frames (e_1, e_2) on the tangent plane, and also under changes of frames (e_3, e_4) on the normal plane. From (12.18), using the equations (2.5) one gets

$$I = \begin{vmatrix} A_{3,11} & A_{3,12} \\ A_{4,11} & A_{4,12} \end{vmatrix} + \begin{vmatrix} A_{3,21} & A_{3,22} \\ A_{4,21} & A_{4,22} \end{vmatrix} = R_{3412}\,. \tag{12.21}$$

13. Manifolds of dimension 4 immersed in E^6.

We will now consider the case

$$n = 4\,, \quad N = 2\,, \quad r = 2\,.$$

According to (7.8), if $E_4^2(x)$ is the 2-plane spanned by e_1, e_2 we have

$$G(x, \{e_1, e_2\})\, d\sigma_4 = \omega_{15}\wedge\omega_{16}\wedge\omega_{25}\wedge\omega_{26}\,. \tag{13.1}$$

For the general 2-space $E_4^2(x)$ spanned by the vectors $e_1' = \gamma_{1h}e_h$, $e_2' = \gamma_{2h}e_h$ $(h = 1, 2, 3, 4)$, we have

$$G(x, \{e_1', e_2'\})\, d\sigma_4 = \gamma_{1h_1}\gamma_{1h_2}\gamma_{2h_3}\gamma_{2h_4}\omega_{h_15}\wedge\omega_{h_26}\wedge\omega_{h_35}\wedge\omega_{h_46} =$$
$$= \frac{1}{4}\begin{vmatrix}\gamma_{1h_2} & \gamma_{1h_4}\\ \gamma_{2h_2} & \gamma_{2h_4}\end{vmatrix}\begin{vmatrix}\gamma_{1h_1} & \gamma_{1h_3}\\ \gamma_{2h_1} & \gamma_{2h_3}\end{vmatrix}\omega_{h_15}\wedge\omega_{h_26}\wedge\omega_{h_35}\wedge\omega_{h_46}\,.$$

Instead of evaluating the integral at the right side over $G_{2,2}$ it is easier to observe that for any frame $\{e_1', e_2', e_3', e_4'\}$ the sum

$$S' = \sum_{(i,j)} \omega_{i5}'\wedge\omega_{i6}'\wedge\omega_{j5}'\wedge\omega_{j6}' \tag{13.2}$$

where the summation is over all permutations of i, j from 1 to 4, does not depend on the orthogonal frame $\{e_1', e_2', e_3', e_4'\}$. Indeed, setting $e_i' = \gamma_{ih}e_h$ in (13.2), we have

$$S' = \frac{1}{4}\sum_{(i,j)}\begin{vmatrix}\gamma_{ih_2} & \gamma_{ih_4}\\ \gamma_{jh_2} & \gamma_{jh_4}\end{vmatrix}\begin{vmatrix}\gamma_{ih_1} & \gamma_{ih_3}\\ \gamma_{jh_1} & \gamma_{jh_3}\end{vmatrix}\omega_{h_15}\wedge\omega_{h_26}\wedge\omega_{h_35}\wedge\omega_{h_46}$$

where the dummy indices h_i take the values 1, 2, 3, 4. Having into account a well known theorem on orthogonal matrices which states that any minor is equal to its complementary, and since $\det(\gamma_{ij}) = 1$, we get $S' = S = \sum_{(i,j)}\omega_{i5}\wedge\omega_{i6}\wedge\omega_{j5}\wedge\omega_{j6}$.

Consequently S is equal to its mean value over $G_{2,2}$ and according to (3.6) we have

$$Q_{2,2}(x)\, d\sigma_4 = \frac{O_3}{6O_4O_5}S = \frac{1}{8\pi^3}S\,. \tag{13.3}$$

In terms of the invariants R_{ijkh} an easy calculation gives

$$(13.4) \qquad Q_{2,2}(x) = \frac{1}{8\pi^2}\sum_{(i,j)} \begin{vmatrix} A_{5i1} & A_{5i2} & A_{5i3} & A_{5i4} \\ A_{5j1} & A_{5j2} & A_{5j3} & A_{5j4} \\ A_{6i1} & A_{6i2} & A_{6i3} & A_{6i4} \\ A_{6j1} & A_{6j2} & A_{6j3} & A_{6j4} \end{vmatrix} =$$

$$= \frac{1}{8\pi^2}\sum_{(i,j)} (R_{ij12}R_{ij34} + R_{ij13}R_{ij24} + R_{ij14}R_{ij23}) .$$

It is noteworthy that this invariant does not depend on the immersion of X^4 into E^6. The total curvature $K_{2,2}(X^4)$ coincides, up to a constant factor, with a topological invariant introduced by Chern [8]. For a topological sphere we have $K_{2,2}(X^4) = 0$ (as follows from ii) of section 4). Samelson [21] has given examples of manifolds for which $K_{2,2}(X^4) \neq 0$. It can be seen that the differential form (13.1) defines the Pontrjagin class p_1 of X^4 (see Chern [9]).

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