Dominating Sets of the Cartesian Products of Cycles

by

NAVID ASSADIAN

B.Sc., Amirkabir University of Technology, 2012

M.Sc., Sharif University of Technology, 2014

A Project Submitted in Partial Fulfillment of the

Requirements for the Degree of

MASTER OF SCIENCE

in the Department of Computer Science

Dominating Sets of the Cartesian Products of Cycles

by

NAVID ASSADIAN

B.Sc., Amirkabir University of Technology, 2012

M.Sc., Sharif University of Technology, 2014

Supervisory Committee

Dr. Wendy MYRVOLD: ______________________________

Supervisor

Department of Computer Science

Dr. Frank Ruskey: ______________________________

Departmental Member

Department of Computer Science

Abstract

Dominating Sets of the Cartesian Products of Cycles

A *dominating set* for a graph G is a subset D of $V(G)$ such that every vertex not in D is adjacent to at least one member of D. In this project, we first briefly survey a variety of known results on dominating sets of some families of graphs, especially the Cartesian products of two k-cycles which are our main focus for this project.

Then, we describe the application we developed to facilitate research on dominating sets of the Cartesian products of k-cycles. After that, we obtain linear-time algorithms to generate dominating sets of the Cartesian products of two k-cycles with sizes matching the best known upper bounds. Additionally, for two cases when k is congruent to two or three modulo five, we improve the two known upper bounds.

"Taking a new step, uttering a new word, is what people fear most."

Fyodor Dostoyevsky, Crime and Punishment

Acknowledgements

I would like to render my warmest thanks to my supervisor, Professor Wendy Myrvold, who made this work possible. Her friendly guidance and expert advice have been invaluable throughout all stages of the work.

I would also wish to express my gratitude to all staff of the Department of Computer Science of the University of Victoria for their support during these years.

Special thanks are due to my father, my mother, my sisters Naghmeh and Neda, and my brother Nima, for their continuous and unconditional encouragement, support, and love.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

The definitions for graphs in this document are based on the conventions established by West [22]. An *undirected graph* G is a triple consisting of a *vertex set* $V(G)$, an *edge set* $E(G)$, and a relation that associates with each edge two vertices called its *endpoints*. Each edge corresponds to an *unordered* pair of vertices.

We draw a graph on paper by placing each vertex at a point and representing each edge by a curve joining the location of its endpoints. A *loop* is an edge whose endpoints are equal. *Multiple edges* are edges having the same pair of endpoints. A *simple graph* is a graph having no loops or multiple edges. In this document, all the graphs are undirected simple graphs.

When u and v are the endpoints of an edge, they are *adjacent* and are *neighbors*. The *degree* of a vertex v of a graph G is the number of edges incident to v in G, and it is denoted by $deg(v)$. The maximum degree over all vertices of G is denoted by $\Delta(G)$. Similarly the minimum degree over all vertices of G is denoted by $\delta(G)$.

1.1 Definitions

A *path* P_n is a graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. A *cycle* C_n is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle. A *connected graph* is a graph where there is a path from each vertex to every other vertex. A *tree* is a connected graph with no cycles. A *subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

An *isomorphism* from a simple graph G to a simple graph H is a bijection $f : V(G) \to V(H)$ such that $(u, v) \in E(G)$ if and only if $(f(u), f(v)) \in E(H)$. It is said "G is isomorphic to H", written $G \cong H$, if there is an isomorphism from G to H. An *isomorphism class* of graphs is an equivalence class of graphs under the isomorphism relation. An *automorphism* of G is an isomorphism from G to G.

For sets A and B, the Cartesian product $A \times B$ is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$. The *Cartesian product* of two graphs G_1 and G_2 denoted by $G_1 \square G_2$ is the graph with vertices $V(G_1) \times V(G_2)$ and $u = (u_1, u_2)$ is adjacent with $v = (v_1, v_2)$ where $v_1, u_1 \in V(G_1)$ and $v_2, u_2 \in V(G_2)$ if and only if $u_1 = v_1$ and $(u_2, v_2) \in E(G_2)$ or $u_2 = v_2$ and $(u_1, v_1) \in E(G_1)$.

Example 1.1.1 (Cartesian Product of Graphs). *Assume graph G_1 is defined as $V_1 = \{v_1, v_2, v_3\}$ and $E_1 = \{(v_1, v_2), (v_2, v_3)\}$ and G_2 is defined as $V_2 = \{u_1, u_2, u_3, u_4\}$ and $E_2 = \{(u_1, u_2), (u_2, u_3), (u_1, u_3)\}$. Then the Cartesian product of G_1 and G_2 is graph $G = G_1 \square G_2$. Graphs G_1, G_2 and G are shown in Figure 1.1.*

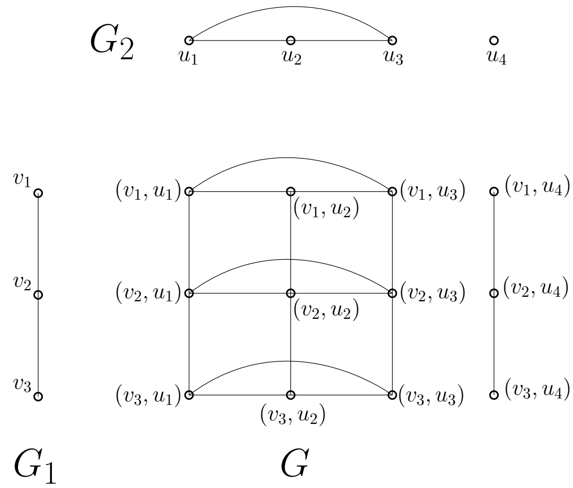


FIGURE 1.1: Graphs for Example 1.1.1

Figure 1.2 shows the Cartesian product of two 3-cycles.

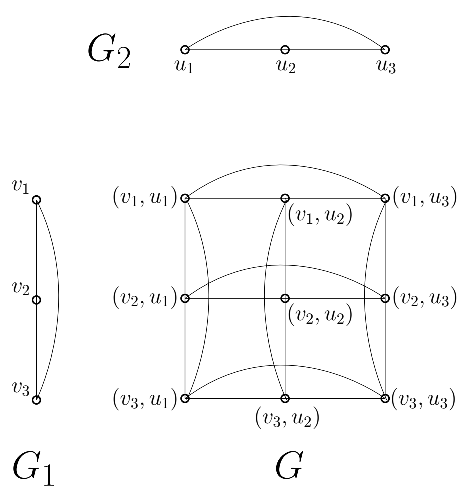


FIGURE 1.2: $G = C_3 \square C_3$

A *dominating set* for a graph G is a subset D of $V(G)$ such that every vertex not in D is adjacent to at least one member of D. The *domination number* $\gamma(G)$ is the number of vertices in a smallest dominating set for G. The set $D \subseteq V(G)$ is called a *γ-set* if it is a dominating set of G and $|D| = \gamma(G)$. A dominating set $S \subseteq V$ is *perfect* if every vertex v in $V - S$ is adjacent to exactly one vertex in S. A vertex v in a dominating set S is *perfect* if none of the vertices dominated by v (including v itself) are dominated by any other vertex.

1.2 Thesis Overview

In this project, we study dominating sets of **Cartesian products of cycles**. Chapter 2 establishes the previous results on domination sets of some classes of graphs, particularly on Cartesian product of graphs. In Chapter 3, we discuss the Java application we developed to facilitate researching on dominating sets of Cartesian product of cycles. Chapter 4 includes some interesting results on dominating sets of Cartesian product of cycles. Finally, Chapter 5 consists of the summary of a results achieved. Additionally, various open problems encountered through this research are summarized in Chapter 5.

Chapter 2

Survey of Recent Results

In this chapter, we explore work related to this project. Section 2.1 describes the history of dominating sets, particularly for some interesting graphs such as *Queen Graphs* and *Triangle Grid Graphs*. In Section 2.2, we summarize previous results on Cartesian product of cycles.

2.1 Dominating Sets

Research on domination of graphs backs to 1901 [15]. In the first published formalization [5], Claude Berge called it *"the coefficient of external stability"*. Later in the 1970's the rate of the research on domination problem has increased significantly. In 1998 Haynes *et al.* [11] published a book on domination which included 1222 references on this area. In 1972, Richard Karp proved the *Set Cover Problem* to be NP-complete by reducing *Vertex Cover Problem* to it [12]. As there are *vertex to set* and *edge to non-disjoint-intersection* bijections between the *Set Cover Problem* and *Dominating Set Problem*, this proved the dominating set problem to be NP-complete as well [1]. The problem remains NP-complete even if restricted to certain classes of graphs such as bipartite graphs. However, for some classes of graphs such as trees, the dominating number can be computed in polynomial time [1].

2.1.1 Queen Graphs

Historically, the first known domination problem is the *queen domination problem*. In 1848, a German chess composer Max Friedrich Wilhelm Bezzel [17] published the *8-queens problem*. The 8-queens puzzle is the problem of placing eight chess queens on an 8×8 chessboard so that no two queens threaten each other. In 1850, Franz Nauck published the first solutions and extended the puzzle to the *n-queens problem* [3], with n queens on a chessboard of $n \times n$ squares. A problem similar to the n-queens problem is the queen domination problem. For a chessboard of size $k \times k$, the *queen domination problem* is to find the minimum number of queens needed on the chessboard such that all the squares are either occupied or can be attacked by a queen. Figure 2.1, shows a board of size 8×8, and Q's indicates the positions of queens.

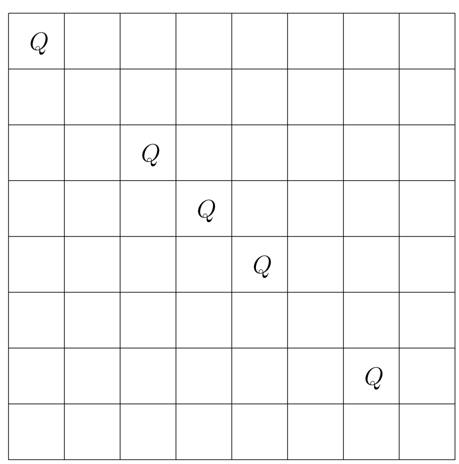


FIGURE 2.1: A Dominating Set of Queens for a Board of Size 8×8

For modeling the queens dominating problem on a graph, $Queen_k$ is the graph that represents a chessboard of size $k \times k$. Each vertex of $Queen_k$ corresponds to a square of the board, and there is an edge between two vertices if and only if their corresponding squares are in a same row, column, diagonal or back-diagonal. $Queen_k$ is called a *Queen Graph*.

A trivial upper-bound for $\gamma(Queen_k)$ is $k-2$. The best known lower-bound for $\gamma(Queen_k)$ is proved to be $\lceil \frac{k}{2} \rceil$ by Finozhenok and Weakley [9] for all the values of k except for $k = 3$ and $k = 11$. Östergård and Weakley established an upper-bound $\gamma(Queen_k) \leq \frac{69k}{133} + O(1)$ [16]. Recently, Bird [7] established $\gamma(Queen_{20}) = 11$, $\gamma(Queen_{22}) = 12$ and $\gamma(Queen_{24}) = 13$. The open case before that was for $Queen_{19}$ which was solved by Kearse and Gibbons [13]. The smallest open case at this time is for $k = 26$ [7]. Due to the elementary fact that $\gamma(Queen_{k+1}) \leq \gamma(Queen_k + 1)$, it is known that $13 \leq \gamma(Queen_{22}) \leq 14$.

2.1.2 Triangular Grid Graphs

A *triangular grid graph* T_l consists of vertices (i, j, k) such that $i, j, k \in \mathbb{Z}^*$ and $i + j + k = l$, and two vertices are adjacent if the total absolute differences in corresponding coordinates is two [22]. Figure 2.2 shows the Triangular Grid Graph T_4.

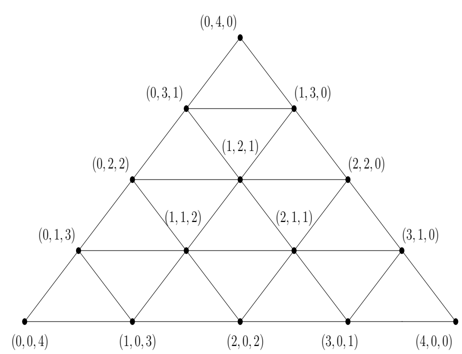


FIGURE 2.2: Triangular Grid Graph of Order 4

Exact values for $\gamma(T_k)$ for $1 \leq k \leq 31$ have been computed by different researchers [21]. Wagon [20] conjectured for $k \geq 14$, the dominating number of T_k is equal to $\lfloor \frac{k^2+7k-23}{14} \rfloor$. The smallest open case at this time is for $k = 32$ [21].

2.1.3 Grid Graphs

A *grid graph* $G_{n,m}$ is a Cartesian product of paths P_n and P_m. A *generalized grid graph* is a Cartesian product of finite number of paths. Figure 2.3 shows $G_{6,5} = P_6 \square P_5$.

In 1992, Chang [8] conjectured for $16 \le m \le n$, $\gamma(P_m \square P_n)$ is equal to $\lfloor \frac{(m+2)(n+2)}{5} \rfloor - 4$. Gonçalves *et al.* [10] proved the conjecture and gave a piecewise formula for $m < 16$.

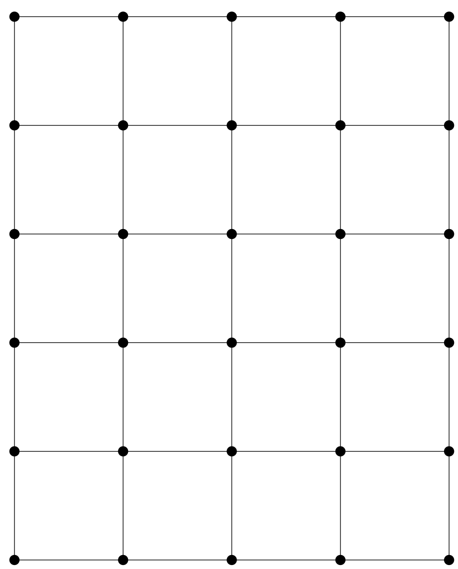


FIGURE 2.3: A Grid Graph $G_{6,5} = P_6 \square P_5$

2.1.4 Hypercube Graphs

The *k*−*hypercube graph*, commonly denoted Q_k, is the graph whose vertices are binary vectors of size k and two vertices are adjacent if and only if they differ in exactly one coordinate. Alternatively, Q_k can be defined as a Cartesian product of k two-paths. As the number of vertices grows exponentially, there has not been much progress on finding the dominating number of Q_k. Moreover, determining $\gamma(Q_k)$ is an intrinsically difficult problem [2]. The smallest open case at this time is for $k = 10$, and $\gamma(Q_9) = 62$ [6]. Moreover, it is know that $107 \leq \gamma(Q_{10}) \leq 120$ [6].

2.2 Dominating Sets of Cartesian Products of Cycle

The relation between the domination number and the Cartesian product of graphs first came to attention in 1963 by V. G. Vizing [19]. Later in 1968 [18], he posed his well-known conjecture.

Conjecture 2.2.1 (Vizing's Conjecture[18])**.** *For two graphs G and H,*

$$\gamma(G \,\square\, H) \geq \gamma(G)\gamma(H).$$

He also proved an upper bound $\gamma(G \square H) \leq \min\{\gamma(G)|V(H)|, \gamma(H)|V(G)|\}$ for the domination number of the Cartesian product of two graphs G and H [19].

Klavžar and Seifter [14] proved some equalities for the domination number of Cartesian product of certain cycles. Theorem 2.2.1 describes them.

Theorem 2.2.1 (Klavžar and Seifter[14])**.** *Let $k \geq 4$. Then,*

$$\gamma(C_3 \,\square\, C_k) = k - \left\lfloor \frac{k}{4} \right\rfloor, \text{ and}$$

$$\gamma(C_4 \,\square\, C_k) = k.$$

Moreover, for $k \geq 5$, if $k \equiv 0 \bmod 5$, then

$$\gamma(C_5 \,\square\, C_k) = k, \text{ and}$$

if $k \equiv 1, 2$ *or* $4 \bmod 5$*, then*

$$\gamma(C_5 \square C_k) = k + 1.$$

Richard Bean [4] stated that the domination number of two k-cycles when k is congruent to zero modulo five is equal to $\frac{k^2}{5}$. Moreover, he conjectured if k is congruent to one modulo five then $\gamma(C_k \square C_k) = \frac{k^2+2k-8}{5}$, and if k is congruent to four modulo five then $\gamma(C_k \square C_k) = \frac{k^2+k}{5}$ [4].

Chapter 3

DomGrid App

Finding a smallest dominating set is a difficult and time consuming task even for small graphs. There are two main issues with doing this task by hand. Choosing a good vertex to include in the dominating set is a main issue and determining the dominated vertices is another one. There are several families of graphs that can be shown on a grid. Two obvious ones are *Grid Graphs* and *Queen Graphs*. In this context, for exploring dominating sets we developed an application called *DomGrid App* specifically for visualization and playing with dominating sets of Cartesian product of cycles. However, we can easily modify it for other graphs that can be represented by a grid just by changing the adjacency information.

3.1 Visualization

The *DomGrid App* can take one or two input arguments. The first input argument is the number k indicating that the graph is $C_k \,\square\, C_k$. The optional second argument is a file that contains a sequence of partial or complete dominating sets.

3.1.1 Running DomGrid with One Input Argument

Figure 3.1 shows an example of running the application with input $k = 6$. The grid corresponds to the graph $C_6 \,\square\, C_6$ in Figure 3.2. The indices that are shown in Figure 3.1 are vertex numbers. For this grid representation of the graph, there is a bijection from $[0, k^2 - 1]$ to $[0, k - 1] \times [0, k - 1]$ where x maps to (r, c) where $r = \lfloor \frac{x}{k} \rfloor$ and $c = x \bmod k$. The coordinates (r, c) map to $x = r * k + c$.

In this document we refer to each copy of C_k in $C_k \,\square\, C_k$ by a *row* or a *column* in the grid representation. Rows and columns are numbered starting with 0.

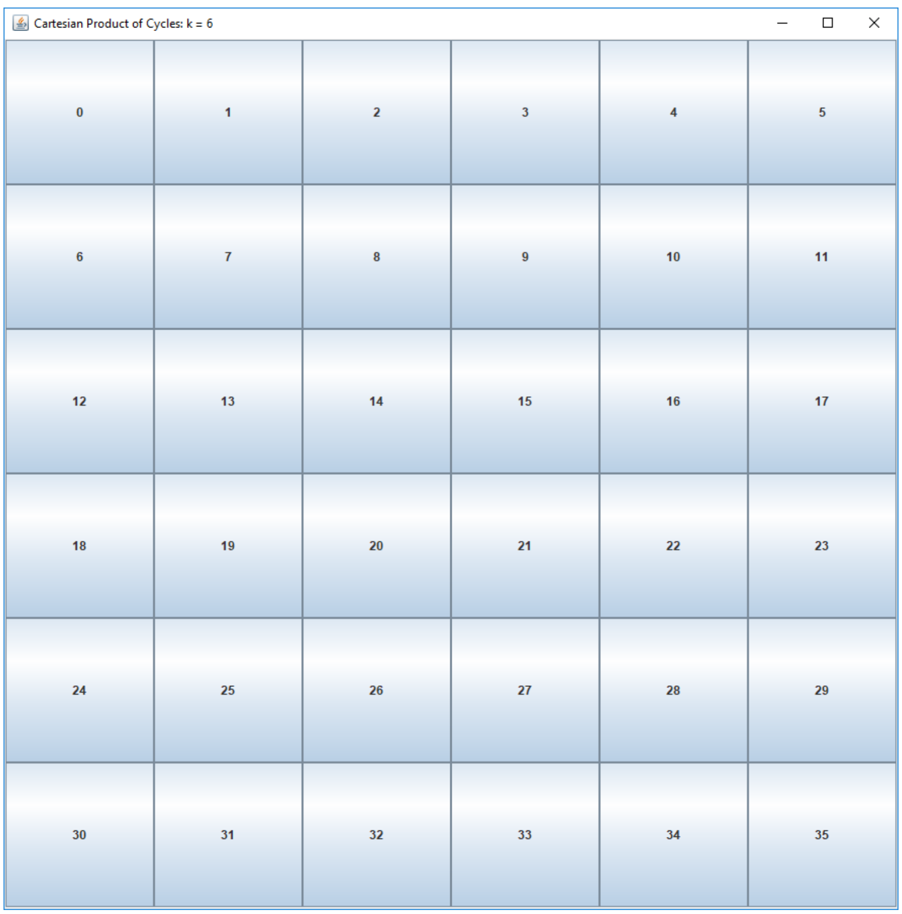


FIGURE 3.1: The constructed grid for $k = 6$

Each box represents a vertex in the graph. Clicking on a box for vertex v adds v to the partial dominating set if it is not already included, otherwise v

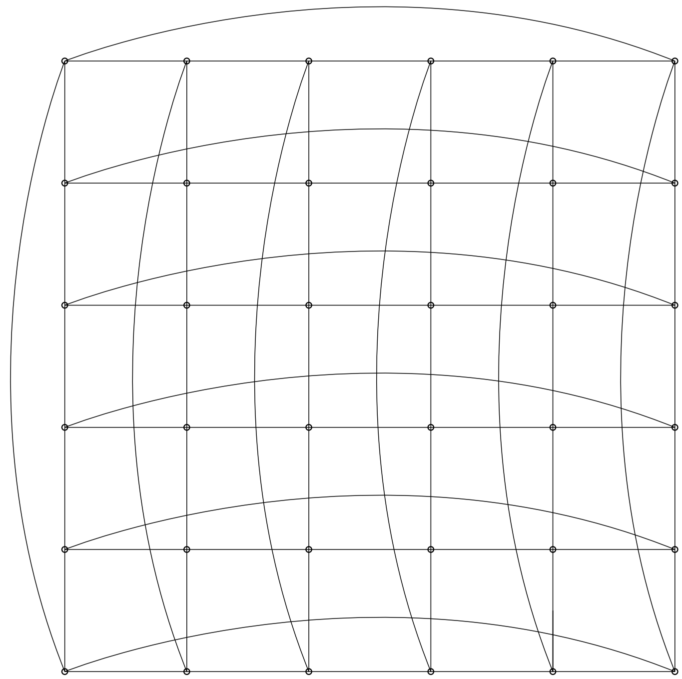


FIGURE 3.2: The Graph of $C_6 \square C_6$

is removed from the partial dominating set. As shown in Figure 3.3, after the first click on some box, each box gets a color and an integer label. If the vertex v is not selected to be in the dominating set, the label of v is the number of vertices that become dominated when v is added. If v is in the dominating set, the label indicates the number of vertices that would become undominated if v is removed from the dominating set.

Example 3.1.1. *Note that vertex numbers for the graph in Figure 3.3 are shown in Figure 3.1. In Figure 3.3, the red vertex (vertex 14) is in the dominating set. If it is removed, five vertices will become undominated. Moreover, the top green vertex (vertex 2) is not in the dominating set and by selecting it four more vertices will become dominated; vertices 2, 1, 3 and 32. Vertex 8 will be dominated as well, but because it is already dominated by 14, it is not included in the count that labels vertex 2.*

For each value of k, the *DomGrid App* has a hard-coded value for the desired dominating set size. For $1 \leq k \leq 22$, the optimal dominating set sizes are known

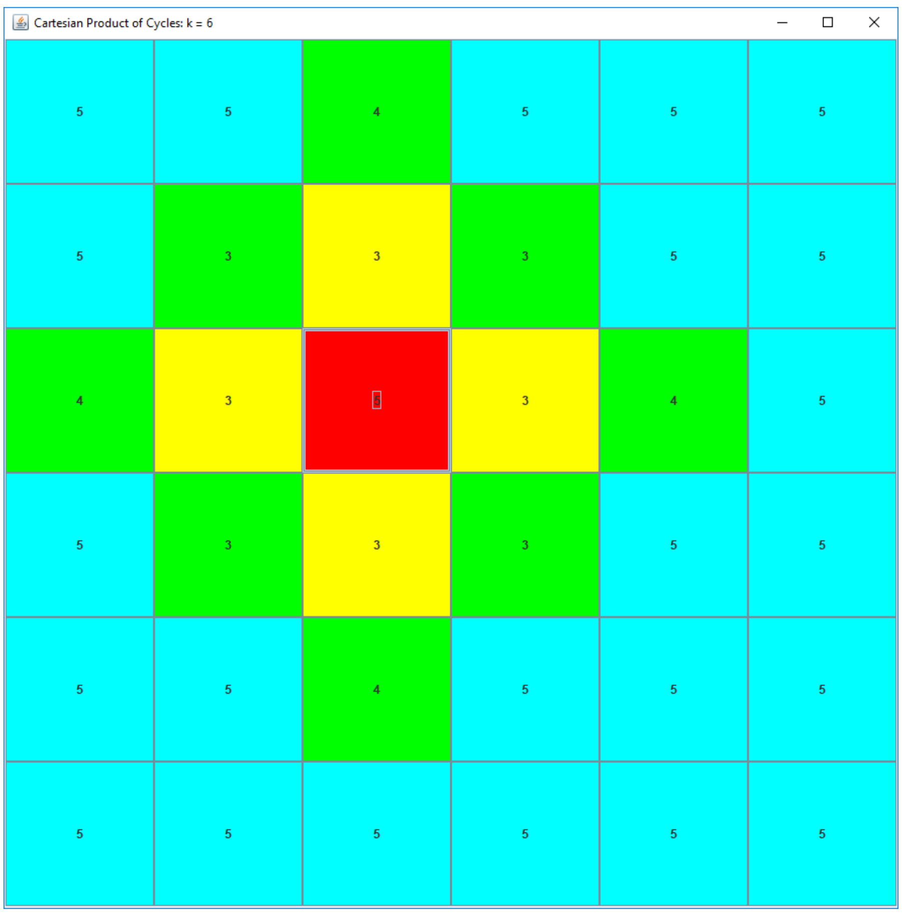


FIGURE 3.3: The Grid for $C_6 \,\square\, C_6$ after clicking on vertex 14

[4], and the program uses these values for the desired size. The optimal dominating set sizes for $1 \leq k \leq 22$ are listed in Table 3.1.

k	$\gamma(C_k \square C_k)$
1	1
2	2
3	3
4	4
5	5
6	8
7	12
8	16
9	18
10	20
11	27
12	32
13	38
14	42
15	45
16	56
17	64
18	71
19	76
20	80
21	95
22	104

TABLE 3.1: Known Optimal Values for $\gamma(C_k \square C_k)$ for $1 \leq k \leq 22$

As it is stated in [14], an obvious upper bound for $\gamma(C_k \square C_k)$ is $\frac{k^2}{4}$. In Chapter 4 we improve this bound to $k \times \lceil \frac{k}{5} \rceil$. Moreover, when k is congruent to zero modulo five, it is proved that $\gamma(C_k \square C_k) = \frac{k^2}{5}$ [4]. The program uses the following equalities for the desired number for $\gamma(C_k \square C_k)$ where $d(C_k \square C_k)$ refers to the desired dominating number and $k > 22$. At the moment, these are the best known upper-bounds.

- $k \equiv 0 \bmod 5 \rightarrow d(C_k \square C_k) = \frac{k^2}{5}$

- $k \equiv 1 \bmod 5 \rightarrow d(C_k \square C_k) = \frac{k^2+2k-8}{5}$ (Conjectured by Bean [4] and proved as an upper bound in this project)
- $k \equiv 2 \bmod 5 \rightarrow d(C_k \square C_k) = \frac{k^2+2k-8}{5}$ (Upper bound proved in this project)
- $k \equiv 3 \bmod 5 \rightarrow d(C_k \square C_k) = \frac{k^2+2k}{5}$ (Upper bound proved in this project)
- $k \equiv 4 \bmod 5 \rightarrow d(C_k \square C_k) = \frac{k^2+k}{5}$ (Conjectured by Bean [4] and proved as an upper bound in this project)

If a user wants to try different desired dominating set sizes, the values which are hard-coded in the program need to be changed.

The color of the boxes help researchers to understand the nature of existing partial or complete dominating sets. The colors are defined in terms of the desired dominating set size.

For each k, the desired dominating set size is denoted by d, and at each step of using the *DomSet App*, the size of the selected dominating set is denoted by t. Therefore, $r = d - t$ is the number of vertices still required. We partition the vertices into 4 different categories

- **Selected Vertices**: These vertices are already in the partial dominating set.
- **Good Vertices**: If the label x is on this box and there are u vertices that are not dominated so far, then $x \geq \lceil \frac{u}{r} \rceil$. They are good candidates to be selected for the dominating set, because if the current partial dominating set can be extended to one of the desired size, then at least one *good* vertex must be used.
- **Bad Vertices**: Sort the labels on cells which are not currently selected in the partial dominating set in decreasing order, and let S be the summation of the first $r - 1$ values. Let x be the label on a box. If $S + x < u$, then the box

corresponds to a bad vertex. If a bad vertex is added, then the resulting partial dominating set provably cannot be extended to one of the desired size.

- **Maybe Good Vertices**: They are not *selected*, *good* or *bad* vertices.

Based on the properties described earlier and their domination set status, we define the colors of each box.

- **Selected Vertices:** These vertices are already in the partial dominating set.
  - **Red**: The vertex is a perfect vertex. In Figure 3.4, vertices 8, 16 and 19 belong to this type.
  - **Pink**: The vertex dominates at least one vertex that is dominated by more than one vertex in the dominating set. In Figure 3.4, vertices 27 and 29 belong to this type.
- **Dominated Vertices:** These vertices are dominated by another vertex and they are not in the partial dominating set.
  - **Gray**: This vertex is dominated by more than one vertex. It can have properties of other non-selected vertices, but it is shown as *gray*. In Figure 3.4, vertex 28 is the only vertex with this property.

  For vertices that are not gray, the colors are:

  - **Orange**: This vertex is a *good* vertex.
  - **Yellow**: This vertex is a *maybe good* vertex. In Figure 3.4, vertex 2 belongs to this type.
  - **White**: This vertex is a *bad* vertex. In Figure 3.4, vertex 7 belongs to this type.

- **Undominated Vertices:** These vertices are not dominated by any vertex.
  - **Cyan**: This vertex is a *good* vertex. In Figure 3.4, vertex 0 belongs to this type.
  - **Green**: This vertex is a *maybe good* vertex. In Figure 3.4, vertex 1 belongs to this type.
  - **Blue**: This vertex is a *bad* vertex. In Figure 3.4, vertex 3 belongs to this type.

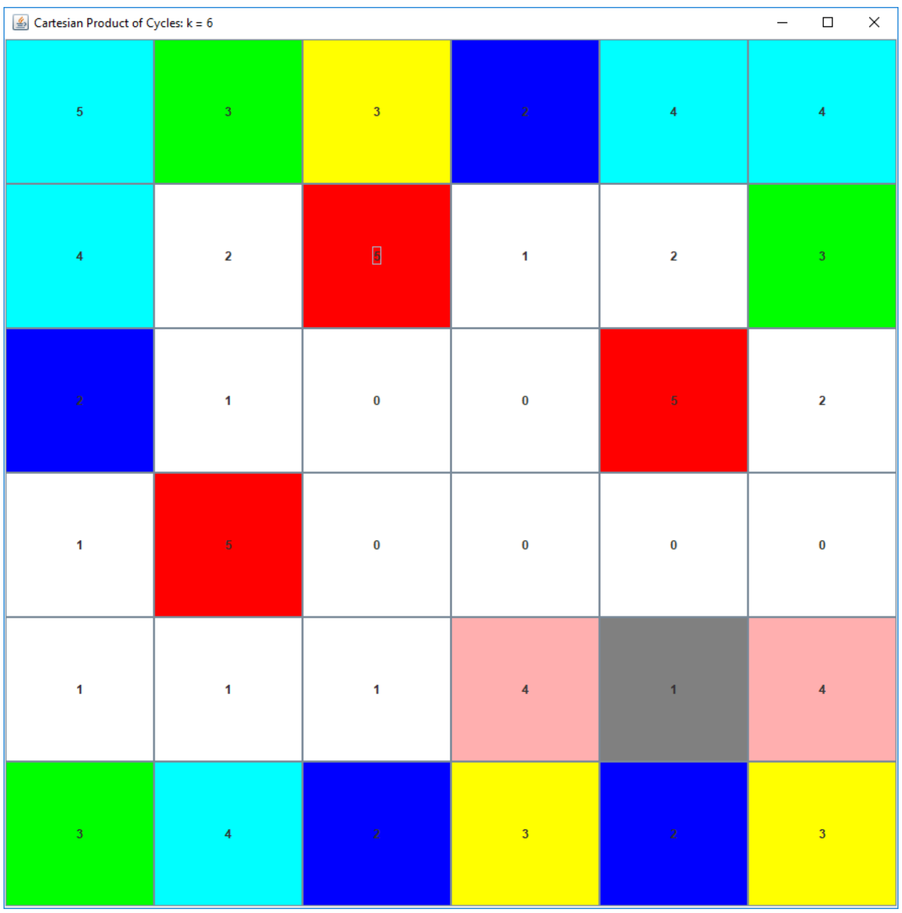
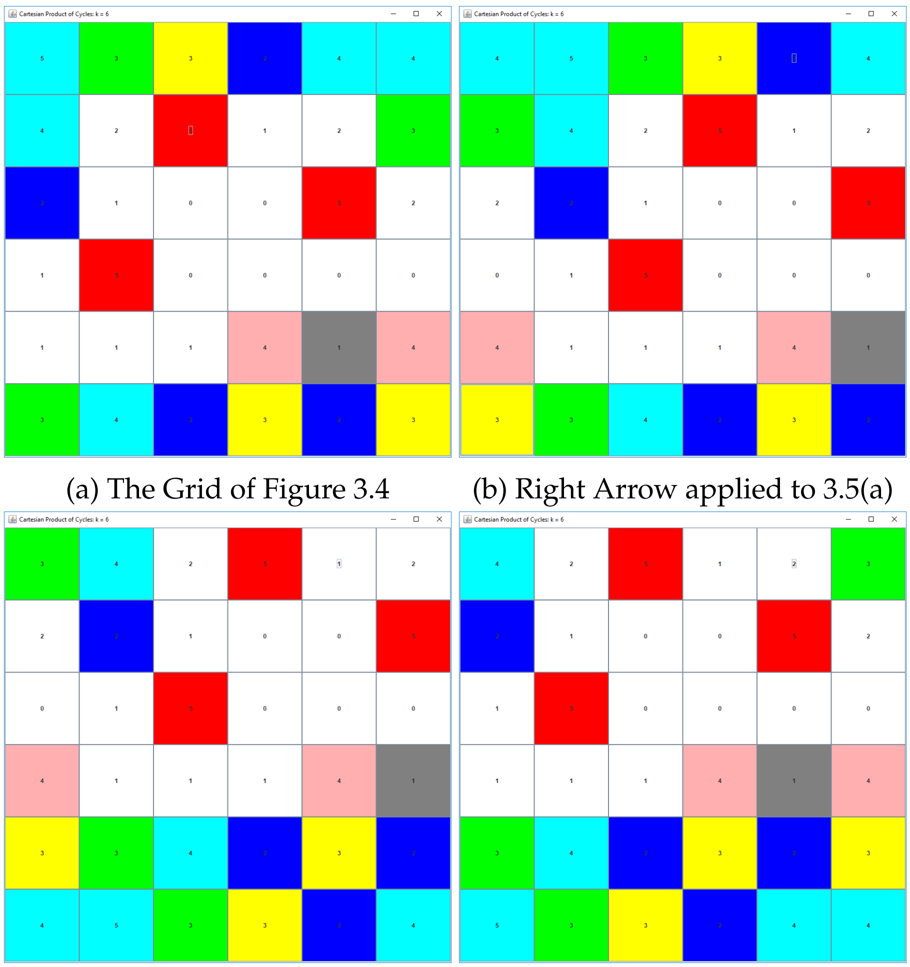


FIGURE 3.4: The Grid for $C_6 \square C_6$ after clicking on vertices 8, 16, 19, 27 and 29

DomGrid App has another visualization feature as well. As cyclically shifting all vertices up, down, left or right gives an automorphism, by using *arrow keys* on the keyboard, all the vertices can be shifted in the desired direction. Figure 3.5 illustrates this feature using a series of actions. These actions can be applied to dominating sets to recenter them so that the symmetries of a solution are more visually apparent on the grid. Changing the position of the border of the grid can facilitate playing with the vertices on the edges.



(a) The Grid of Figure 3.4

(b) Right Arrow applied to 3.5(a)

(c) Up Arrow applied to 3.5(b)

(d) Left Arrow applied to 3.5(c)

FIGURE 3.5: *DomGrid:* Using Arrow Keys

3.1.2 Seeding the Program With Partial Dominating Sets

When used with two input arguments, *DomGrid App* supports all the properties described when it has one argument. As mentioned before, the second argument is a file name. The file consists of a sequence of partial or complete dominating sets. The file starts with an integer s denotes the number of different sets which are included in the file. Each set starts with a number m that is the number of vertices in this set followed by m different indices in the range $[0, k^2 - 1]$. Using the *Page Up/Page Down* keys on the keyboard, a user can switch between the different partial dominating sets in the file. The program starts with no vertices selected in the partial dominating set. *Page Down* moves to the first dominating set in the file and *Page Up* moves to the last one. By pressing the *Page Down* key the program goes to the next partial dominating set, and by pressing the *Page Up* key it goes to the previous dominating set. If it reaches the last set and *Page Down* is pressed or if it is on the first set and *Page Up* is pressed, the program goes to its initial state, i.e. no vertices are selected for the partial dominating set.

Example 3.1.2. *Assume DomGrid App is executed with $k = 6$, and a file with following numbers.*

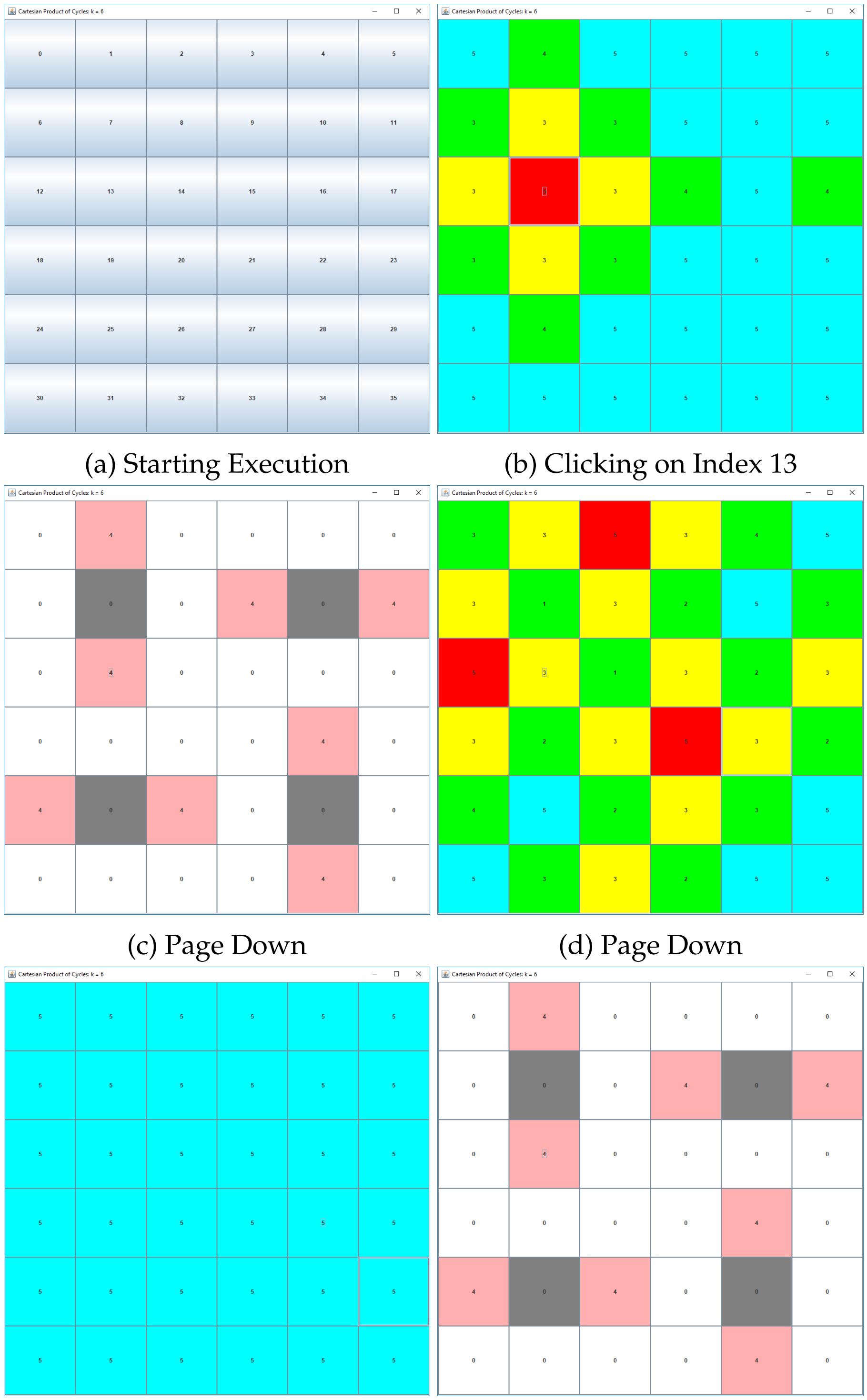
INPUT FILE:

8 1 9 11 13 22 24 26 34

3 2 12 21

Figure 3.6 shows the output of DomGrid App with this input and the series of actions that are mentioned in Figure 3.6.

Using input files facilitates visualizing possible dominating sets and validating the correctness of algorithms for dominating sets. They also facilitate a search for possible patterns that can be extended for large values of k.



(a) Starting Execution (b) Clicking on Index 13

(c) Page Down (d) Page Down

(e) Page Down (f) Page Down

FIGURE 3.6: *DomGrid:* Using a File and Series of Actions

3.2 More Features

As mentioned earlier in this chapter, the *DomGrid App* provides many visualization aids to help users to choose better vertices as candidates for the dominating set and to understand the properties of their partial dominating sets better. In this section, we describe more information which is displayed by the application on the console output.

At first, the console displays the following descriptions.

- A brief description of the colors and their properties.
- The desired size d of a dominating set.

After clicking on a vertex, the user sees following information on the console:

- A text-based representation of the current situation of the grid. It shows the grid in k lines. If vertex (r, c) is added into the dominating set, character X is the $c + 1$th character on the $r + 1$ row, otherwise the character _ is placed.
- Indices of selected vertices.
- The number of unselected vertices.
- The number of selected vertices t.
- The number of required vertices $r = d - t$.
- The first r unselected labels after sorting them in decreasing order.
- The number of undominated vertices u.
- If all the vertices are dominated, it shows a message whether the selected dominating set is a good or bad set based on the predefined desired dominating set size.

Chapter 4

Improvement on Known Results for Dominating Sets of the Cartesian Products of Cycles

As mentioned earlier in Chapter 3, *Richard Bean* [4] has conjectured that for $k \equiv 1 \bmod 5$, the size of an optimal dominating set of $C_k \,\square\, C_k$ is $\frac{k^2+2k-8}{5}$. Also, for $k \equiv 4 \bmod 5$, he has conjectured that the optimal size is $\frac{k^2+k}{5}$. In both cases, there is no suggested construction for such dominating sets.

In this chapter we describe three interesting constructions for dominating sets of the Cartesian product of cycles. In Section 4.1, we describe and prove the necessary and sufficient conditions for a perfect dominating set of $C_k \,\square\, C_k$ when $k \equiv 0 \bmod 5$. Using that, we develop a construction for dominating sets of $C_k \,\square\, C_k$.

In Section 4.2, we describe a construction for a dominating set of the conjectured size for $C_k \,\square\, C_k$ when k is congruent to one modulo five.

Finally, in Section 4.3 we describe another construction for a dominating set of $C_k \,\square\, C_k$ when k is congruent to two modulo five. This construction improves the upper bound of $\gamma(C_k \,\square\, C_k)$. Moreover, in this Section we conjecture that the exact value of $\gamma(C_k \square C_k)$ when k is congruent to two modulo five is $\frac{k^2+2k-8}{5}$.

4.1 General Construction For Dominating Sets of $C_k \square C_k$

This section provides a general construction for a perfect dominating set for $C_k \square C_k$. This construction gives the perfect dominating when k is congruent to zero modulo five. Richard Bean [4] stated that for this case $\gamma(C_k \square C_k) = \frac{k^2}{5}$. Figure 4.1 illustrates a perfect dominating set for $C_{10} \square C_{10}$.

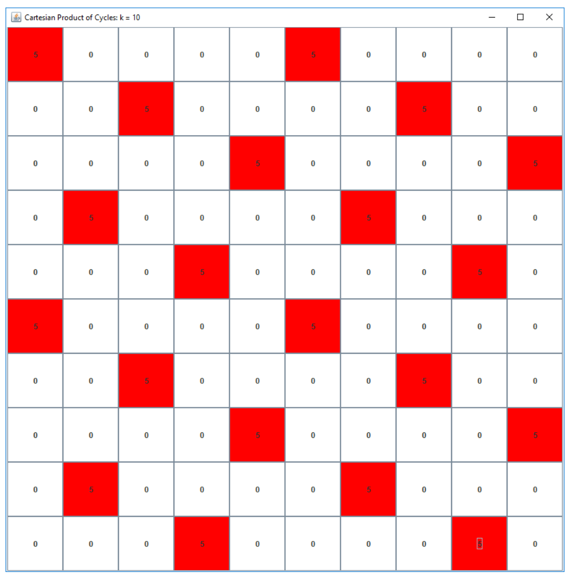
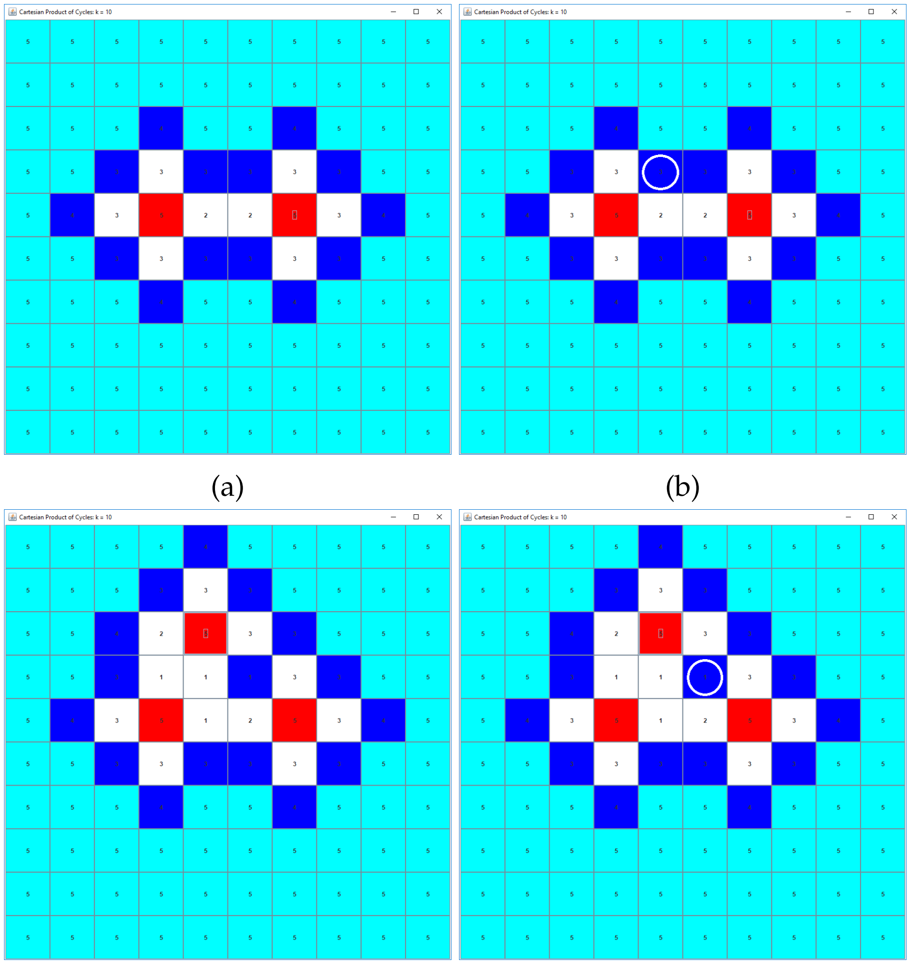


FIGURE 4.1: A perfect dominating set of $C_{10} \square C_{10}$

Theorem 4.1.1. *In a perfect dominating set of $C_k \square C_k$ for $k \geq 5$, selected consecutive vertices for the dominating set of one cycle (row/column) must have a distance of five and they follow the staircase pattern in Figure 4.1.*

Proof. Clearly, two vertices in a perfect dominating set cannot have distance one or two. Assume they are in the same row or column and have distance three. Without loss of generality, assume they are as shown in Figure 4.2(a). To dominate the vertex circled in white in Figure 4.2(b) perfectly, there is just one option



(a) (b)

(c) (d)

FIGURE 4.2: Two vertices in the dominating set with distance three

and we need to pick the vertex above it for the dominating set as it shown in Figure 4.2(c). Now for dominating the vertex circled in white in Figure 4.2(d), there is no option that leads to a perfect dominating set. Therefore, two vertices in the dominating set cannot have distance three and be in the same row or column.

Now assume two vertices are in the same row or column and have distance four. This case is illustrated in Figure 4.3(a). For dominating the circled vertex in Figure 4.3(b) perfectly there are two options; the vertex above it or below it. Without loss of generality, we pick the vertex above it. This causes a problem for dominating the vertices marked with the yellow box in Figure 4.3(d).

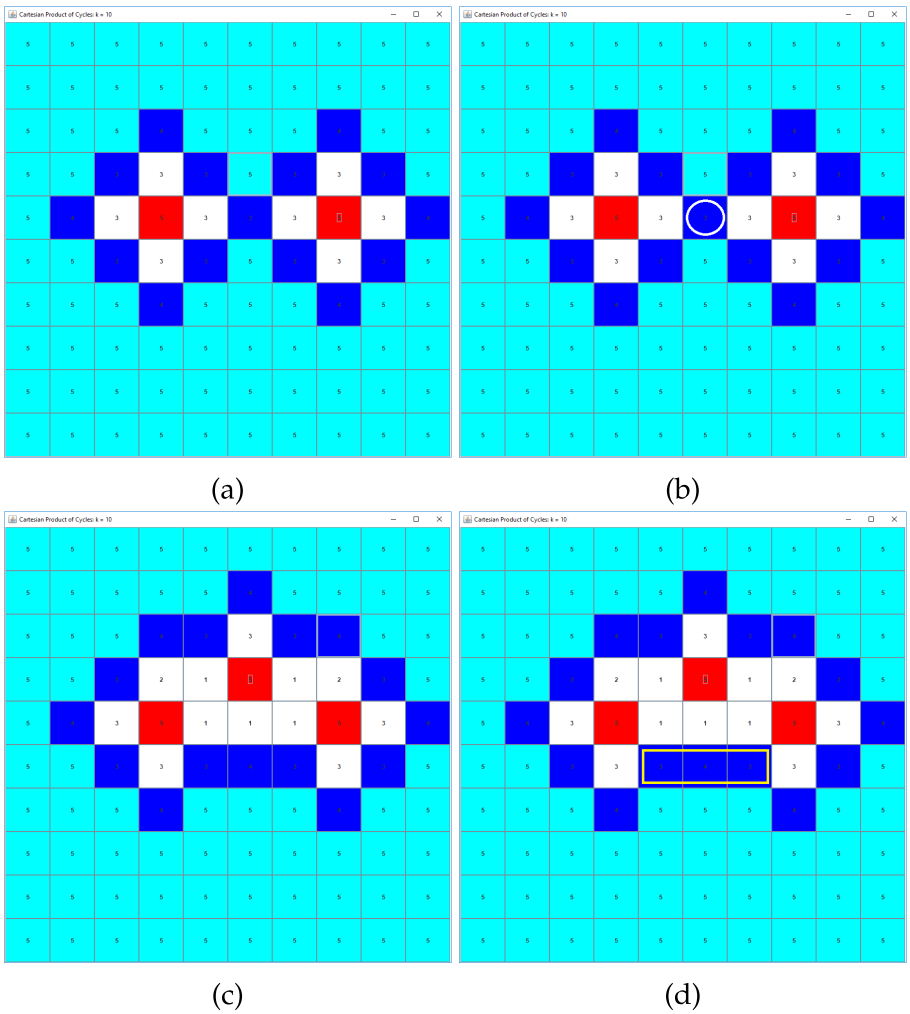


FIGURE 4.3: Two vertices in the dominating set with distance four

Therefore, the dominating vertices in a same row or column cannot have distance less than five. Assume their distance is either 6, 7, 8 or 9. Figure 4.4 illustrates the case when they are at distance seven. For dominating the vertices marked by a yellow box, and using the *Pigeon Hole Principle*, at least one of its adjacent rows has vertices in the dominating set with distance less than five.

So they must be at distance 5 which makes a pattern similar to 4.1.

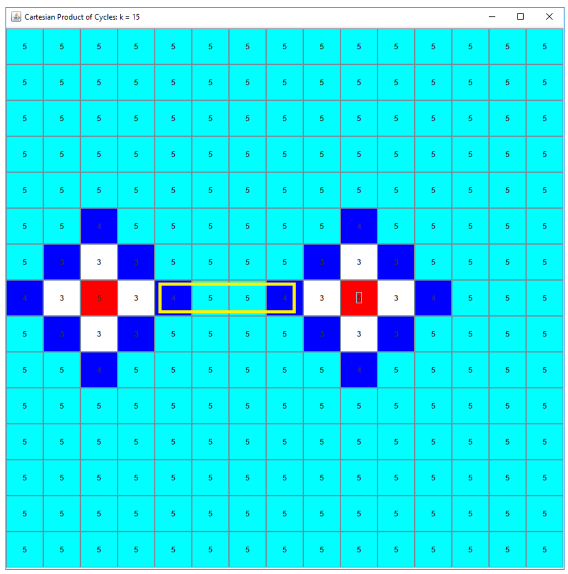


FIGURE 4.4: Two vertices in the dominating set with distance seven

□

The graph $C_k \square C_k$ has a perfect dominating set for k congruent to zero modulo five because the dominating set vertices can be selected in such a way that in each row and column they are at distance five.

The following algorithm is a construction for creating a perfect dominating set for $C_k \square C_k$ where $k \equiv 0 \mod 5$. Note that because of the symmetric property of $C_k \square C_k$, several different perfect dominating sets exist, but we construct one of them and the construction for the rest are similar.

Algorithm 4.1.1 Construction Algorithm for a perfect dominating set for $C_k \square C_k$

```
procedure DOMINATING SET(k)
    i ← 0
    j ← 0
    for j ← 0 to k − 1 do                                   ▷ For each column:
        for t ← 0 to ⌊(k − 1)/5⌋ do
            Add vertex [(t × 5 + i) mod k, j] to S
        end for
        i ← (i + 2) mod k
    end for
    return S
end procedure
```

The resulting dominating set of Algorithm 4.1.1 for $k = 15$ is shown in Figure 4.5.

Theorem 4.1.2. *Algorithm 4.1.1 generates a perfect dominating set of $C_k \square C_k$ where $k \equiv 0 \bmod 5$.*

Proof. Based on the fact that k is congruent to zero modulo five, the distance between two selected vertices in a same row or column is multiple of 5. As the algorithm constructs the pattern described in Theorem 4.1.1, so Algorithm 4.1.1 generates a perfect dominating set.

□

Algorithm 4.1.1 generates a dominating set of size $\frac{k^2+3k}{5}$ when k is congruent to two modulo five. This improves the previous upper bound of $\frac{k^2}{4}$ [14]. Figure 4.6 illustrates the resulting dominating set of Algorithm 4.1.1 for $C_{27} \square C_{27}$.

Theorem 4.1.3. *Algorithm 4.1.1 generates a dominating set of size $\frac{k^2+3k}{5}$ for $C_k \square C_k$ when $k \equiv 2 \bmod 5$.*

Proof. There are k columns, and for each column $p = \frac{k+3}{5}$ vertices are added to the dominating set. Therefore, the size of the resulting dominating set of

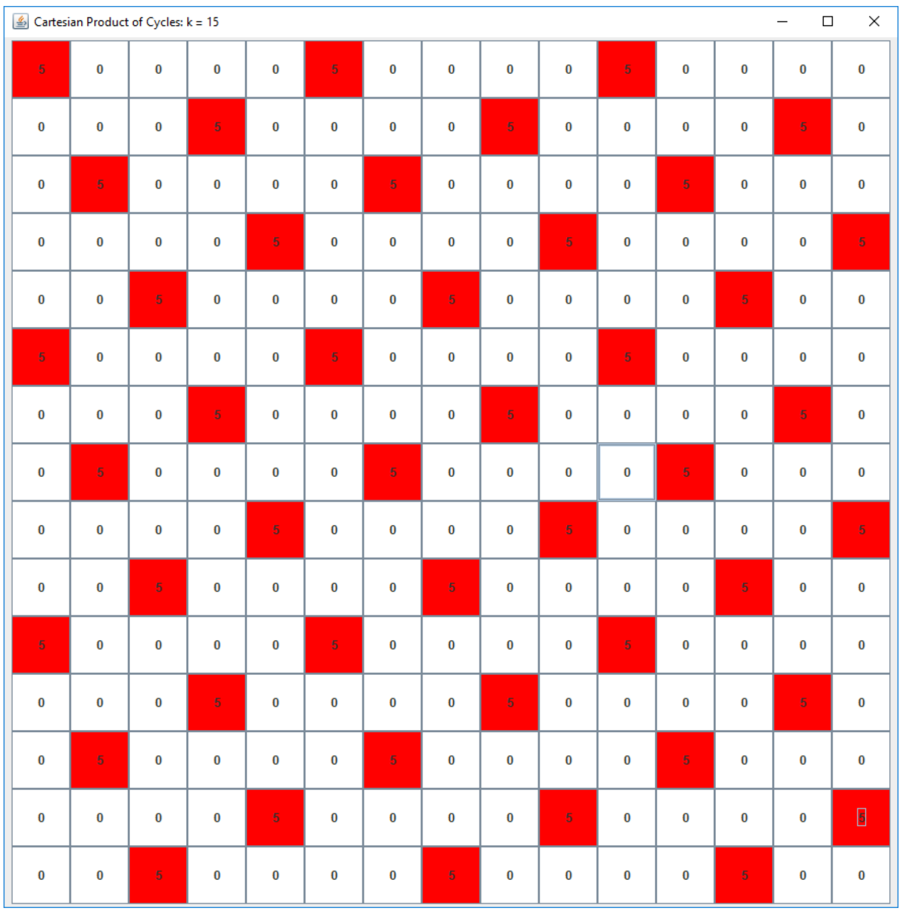


FIGURE 4.5: Output of Algorithm 4.1.1 for $k = 15$

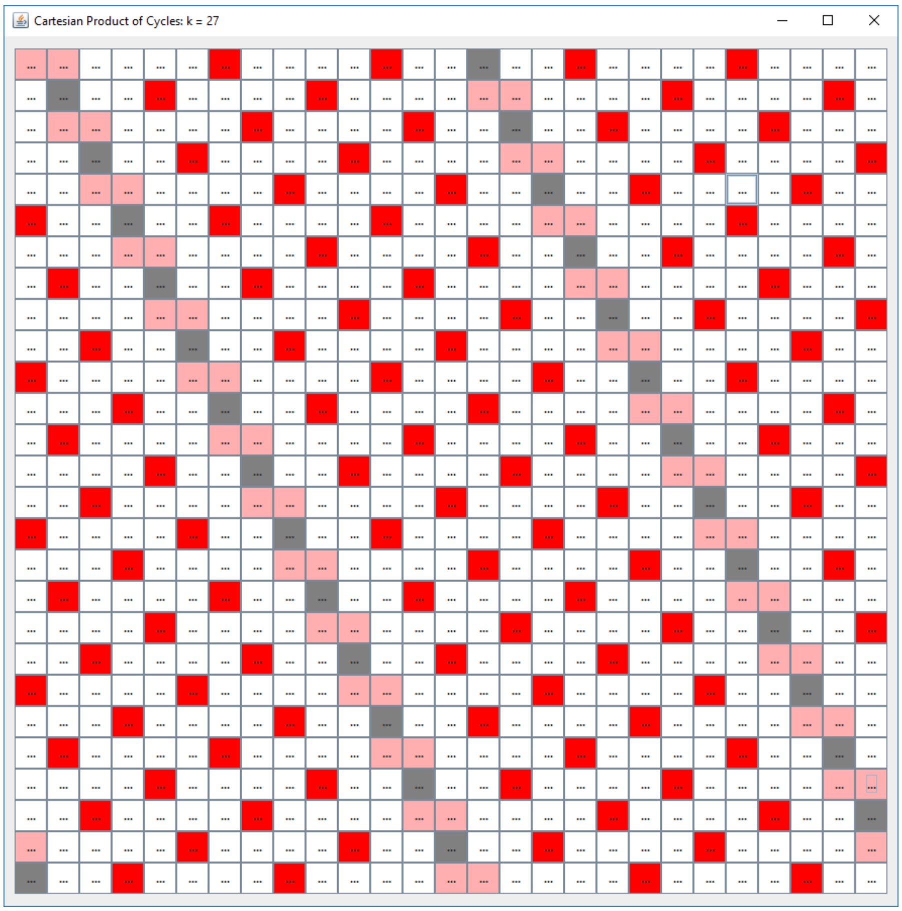


FIGURE 4.6: Dominating set generated by Algorithm 4.1.1 for $k = 27$.

Algorithm 4.1.1 is $k \times \frac{k+3}{5} = \frac{k^2+3k}{5}$, which improves the upper bound of $\frac{k^2}{4}$ for $C_k \,\Box\, C_k$ proved by Klavžar and Seifter [14]. $\Box$

Moreover, Algorithm 4.1.1 generates a dominating set of size $\frac{k^2+2k}{5}$ when k is congruent to three modulo five. Figure 4.7 illustrates the resulting dominating set of Algorithm 4.1.1 for $C_{23} \,\Box\, C_{23}$.

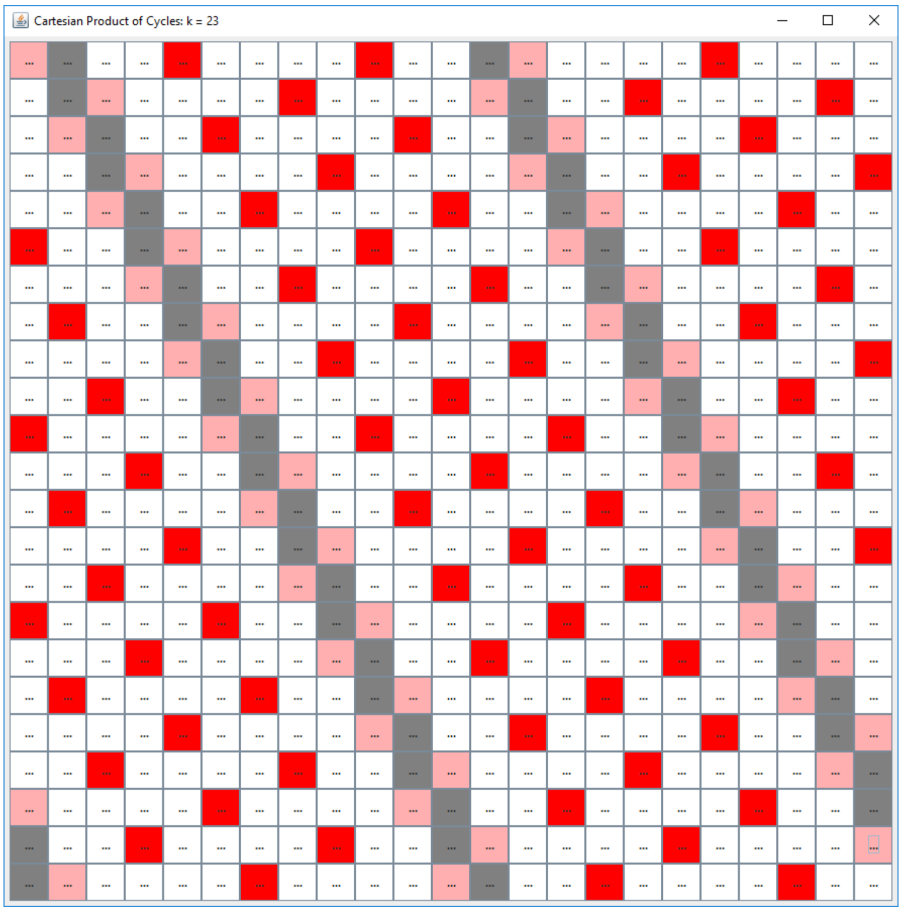


FIGURE 4.7: Dominating set generated by Algorithm 4.1.1 for $k = 23$.

Theorem 4.1.4. *Algorithm 4.1.1 generates a dominating set of size $\frac{k^2+2k}{5}$ for $C_k \,\Box\, C_k$ when $k \equiv 3 \bmod 5$.*

Proof. There are k columns, and for each column $p = \frac{k+2}{5}$ vertices are added to the dominating set. Therefore, the size of the resulting dominating set of Algorithm 4.1.1 is $\frac{k^2+2k}{5}$, which improves the upper bound of $\frac{k^2}{4}$ for $C_k \square C_k$ proved by Klavžar and Seifter [14]. $\square$

Bean [4] conjectured for cases when k is congruent to four modulo five the dominating number is $\frac{k^2+k}{5}$. Algorithm 4.1.1 generates a dominating set of size $\frac{k^2+k}{5}$ when k is congruent to four modulo five. Figure 4.8 illustrates the resulting dominating set of Algorithm 4.1.1 for $C_{19} \square C_{19}$.

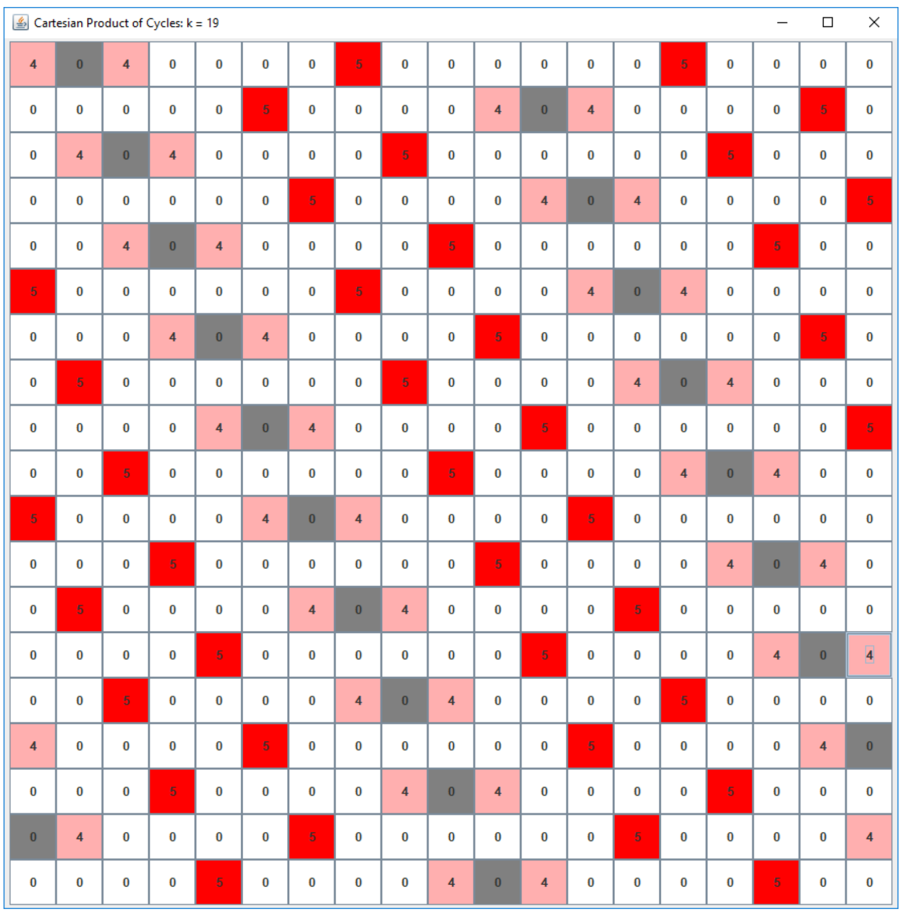


FIGURE 4.8: Returned dominating set by algorithm 4.1.1 for $k = 19$

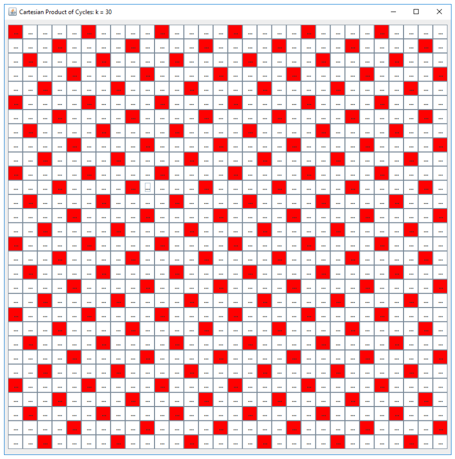
Theorem 4.1.5. *Algorithm 4.1.1 generates a dominating set of size $\frac{k^2+k}{5}$ for $C_k \square C_k$ when $k \equiv 4 \bmod 5$.*

Proof. There are k columns, and for each column $p = \frac{k+1}{5}$ vertices are added to the dominating set. Therefore, the size of the resulting dominating set of Algorithm 4.1.1 is $\frac{k^2+k}{5}$, which satisfies the conjectured value of $C_k \square C_k$ by Richard Bean [4]. $\square$

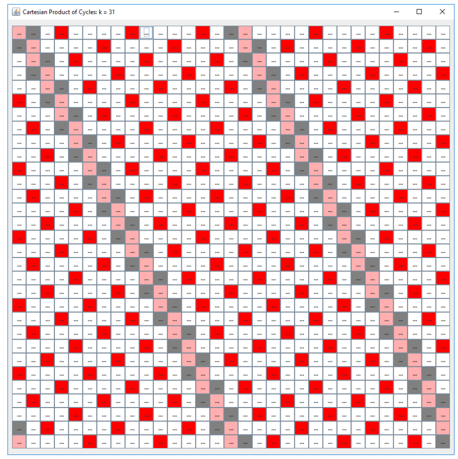
Table 4.1 summarizes the size of the dominating sets that Algorithm 4.1.1 generates, and it compares them with known results. Figure 4.9 shows the dominating sets constructed by Algorithm 4.1.1 for $30 \leq k \leq 34$.

k mod 5	Previous Results	Algorithm 4.1.1 Results
0	$\frac{k^2}{5}$(Proved)	$\frac{k^2}{5}$
1	$\frac{k^2+2k-8}{5}$(Conjectured)	$\frac{k^2+4k}{5}$
2	$\frac{k^2}{4}$ (Upper Bound)	$\frac{k^2+3k}{5}$
3	$\frac{k^2}{4}$ (Upper Bound)	$\frac{k^2+2k}{5}$
4	$\frac{k^2+k}{5}$(Conjectured)	$\frac{k^2+k}{5}$

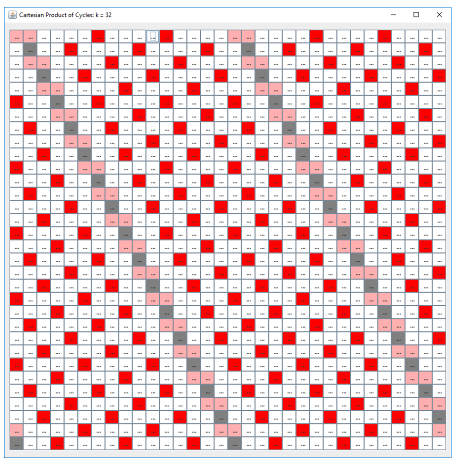
TABLE 4.1: Sizes of dominating sets constructed by Algorithm 4.1.1.



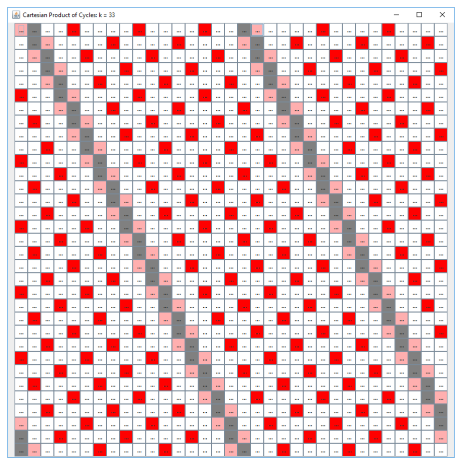
(a) $k = 30$



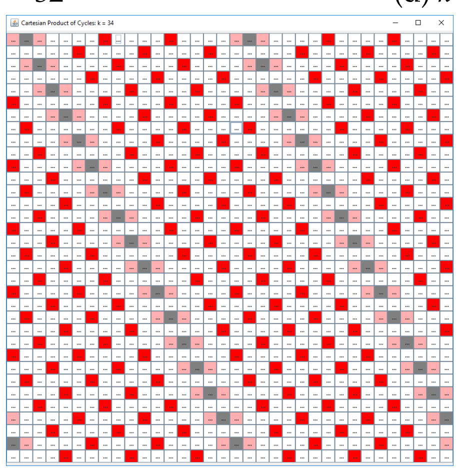
(b) $k = 31$



(c) $k = 32$



(d) $k = 33$



(e) $k = 34$

FIGURE 4.9: Dominating sets constructed by Algorithm 4.1.1 for $30 \leq k \leq 34$

4.2 A Pattern for $k \equiv 1 \bmod 5$

In this section, a construction for k congruent to one module five is described. This construction generates a dominating set of size $\frac{k^2+2k-8}{5}$ for $k \geq 41$ that matches the conjectured size by *Bean* [4]. Figure 4.10 shows the constructed pattern for $k = 51$. Figure 4.11 shows the dominating sets of size $\frac{k^2+2k-8}{5}$ for cases when k is less than 41 and k is congruent to one module five.

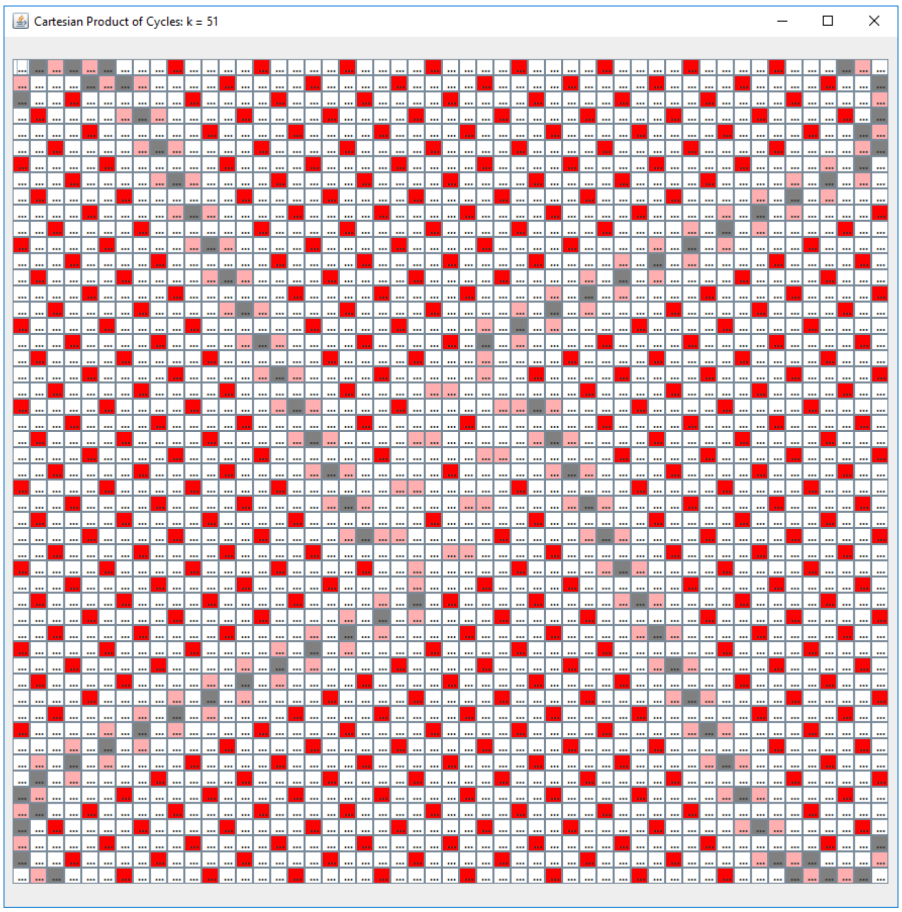
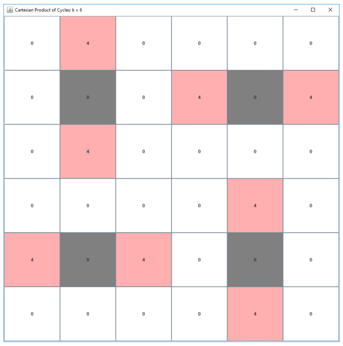


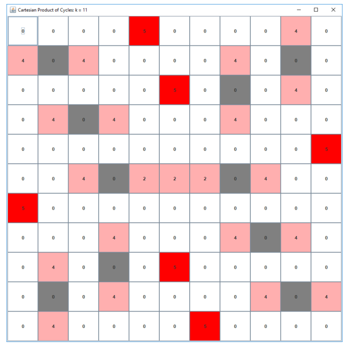
FIGURE 4.10: Constructed Dominating Set for $k = 51$

Algorithms 4.2.1 shows the pseudo-code for the construction which generates a dominating set of size $\frac{k^2+2k-8}{5}$ for $k \geq 41$.

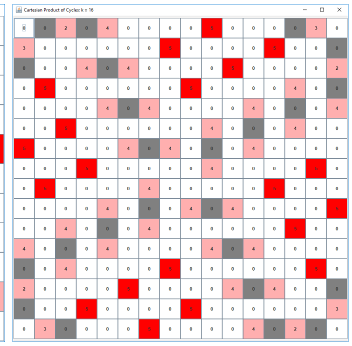
Figure 4.12 illustrates the construction steps of Algorithm 4.2.1 for $k = 46$.



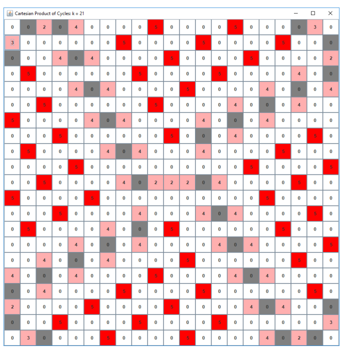
(a) Dominating set for $k = 6$



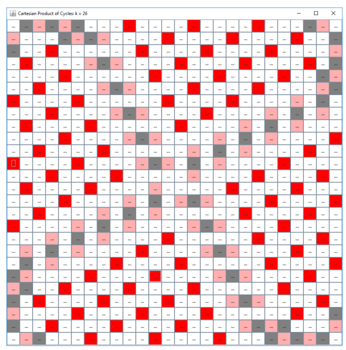
(b) Dominating set for $k = 11$



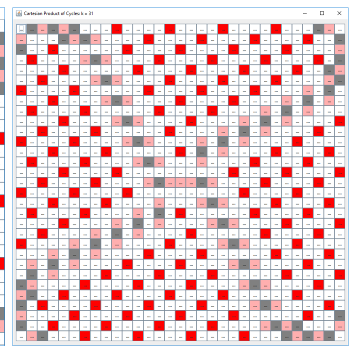
(c) Dominating set for $k = 16$



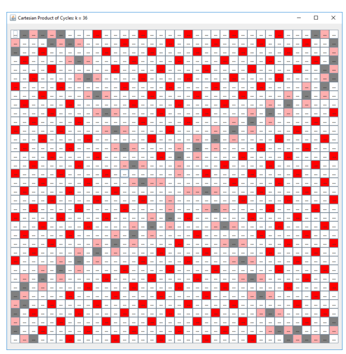
(d) Dominating set for $k = 21$



(e) Dominating set for $k = 26$



(f) Dominating set for $k = 31$



(g) Dominating set for $k = 36$

FIGURE 4.11: Dominating sets when k is congruent to one module 5 and $k < 41$

Algorithm 4.2.1 Construction Algorithm for a perfect dominating set for $C_k \square C_k$

procedure DOMINATING SET(k) ▷ Constraint: $k \equiv 1 \bmod 5, k \geq 41$

Step 1 - Double Dominations on the Boundary

    Add vertex $[1,0]$ to S
    Add vertex $[0,2]$ to S
    Add vertex $[0,4]$ to S
    Add vertex $[k-2,k-1]$ to S
    Add vertex $[k-1,k-5]$ to S
    Add vertex $[k-1,k-3]$ to S
    Add vertex $[0,k-2]$ to S
    Add vertex $[2,k-1]$ to S
    Add vertex $[4,k-1]$ to S
    Add vertex $[k-1,1]$ to S
    Add vertex $[k-5,0]$ to S
    Add vertex $[k-3,0]$ to S
    $left \leftarrow 2*((k-1)/5-3)+1$
    $right \leftarrow (k-1)/5+2$
    $middle \leftarrow (k-1)/5-7$

Step 2 - Horizontal Pairs

    for $i \leftarrow 0$ to $left-1$ **do**
        $r \leftarrow 2*i+1$
        $c \leftarrow 5+i$
        Add vertex $[r,c]$ to S
        Add vertex $[r,c+2]$ to S
        Add vertex $[k-1-r,k-1-c]$ to S
        Add vertex $[k-1-r,k-1-(c+2)]$ to S
    end for
    $r \leftarrow 2*(left-1)+1$
    $c \leftarrow 5+(left-1)+3$
    Add vertex $[r,c]$ to S
    Add vertex $[k-1-r,k-1-c]$ to S

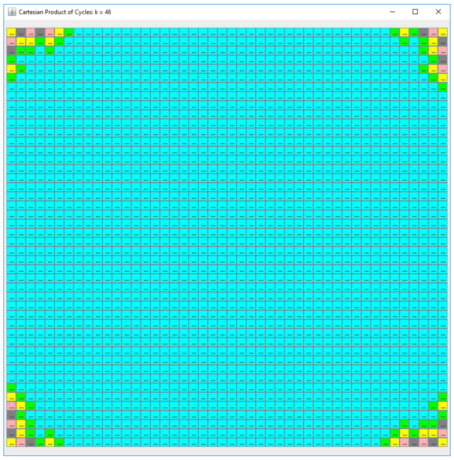
Step 3 - Vertical Pairs

    for $i \leftarrow 0$ to $right-1$ **do**
        $r \leftarrow k-1-5-i$
        $c \leftarrow 2*i+1$
        Add vertex $[r,c]$ to S
        Add vertex $[r-2,c]$ to S
        Add vertex $[k-1-r,k-1-c]$ to S
        Add vertex $[k-1-(r-2),k-1-c]$ to S
    end for
    $r \leftarrow k-1-5-right-2$
    $c \leftarrow 1+2*(right-1)$
    Add vertex $[r,c]$ to S
    Add vertex $[k-1-r,k-1-c]$ to S

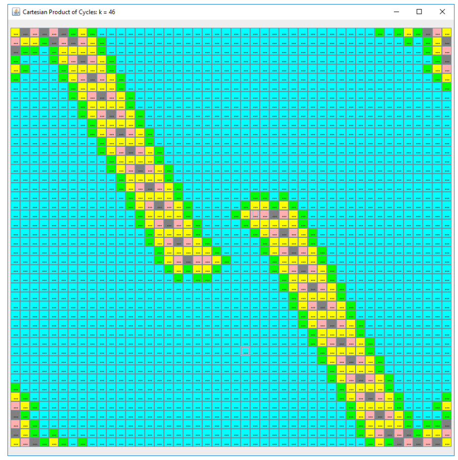
```
                Step 4 - Perfect Domination on Left and Right
r ← 2 * left + 1
c ← 5 + left
Add vertex [r, c] to S
Add vertex [k − 1 − r, k − 1 − c] to S
for i ← 0 to left do
    r_start ← 2 + 2 * i
    c_start ← 3 + i
    r ← r_start
    c ← c_start
    while c ≥ 0 do
        Add vertex [r, c] to S
        Add vertex [k − 1 − r, k − 1 − c] to S
        r ← r + 1
        c ← c − 2
    end while
end for
                Step 5 - Perfect Domination on Top and Bottom
r ← k − 1 − 5 − (right − 1) − 1
c ← 1 + 2 * right
Add vertex [r, c] to S
Add vertex [k − 1 − r, k − 1 − c] to S
for i ← 0 to left do
    r_start ← k − 4 − i
    c_start ← 2 + 2 * i
    r ← r_start
    c ← c_start
    while r < k do
        Add vertex [r, c] to S
        Add vertex [k − 1 − r, k − 1 − c] to S
        r ← r + 2
        c ← c + 1
    end while
end for
for i ← 0 to middle − 1 do
    r_start ← 2 * (left − 1) + 1 − 5 − 2 * i
    c_start ← 5 + left − 1 + 2 − i
    for j ← 0 to middle − i do
        r ← r_start − 3 * j
        c ← c_start + j
        Add vertex [r, c] to S
        Add vertex [k − 1 − r, k − 1 − c] to S
    end for
end for
```

```
                    Step 6 - Middle Part
    r ← 2 ∗ (left − 1)
    c ← 5 + (left − 1) + 5
    for i ← 0 to middle − 1 do
        Add vertex [r, c] to S
        Add vertex [r + 2, c + 1] to S
        Add vertex [r + 2, c + 2] to S
        Add vertex [r − 2, c − 1] to S
        Add vertex [r − 2, c − 2] to S
        r ← r − 3
        c ← c + 1
    end for
    return S
end procedure
```

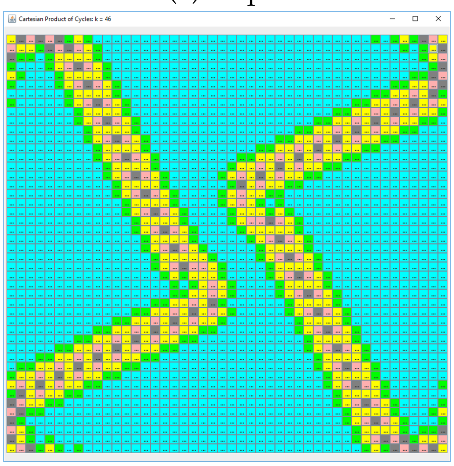
Figure 4.13 illustrates the constructed dominating set of size 1880 by Algorithm 4.2.1 for $k = 96$.



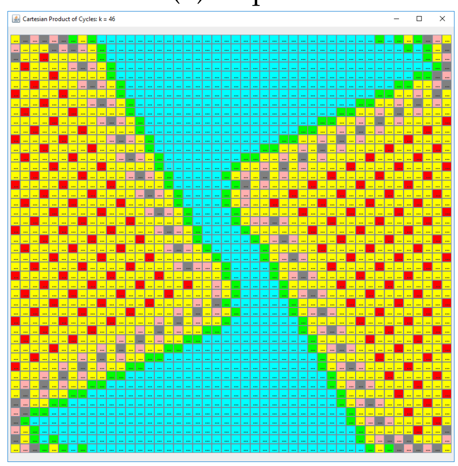
(a) Step 1



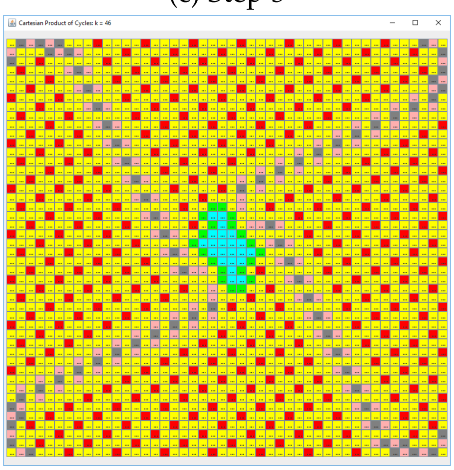
(b) Step 2



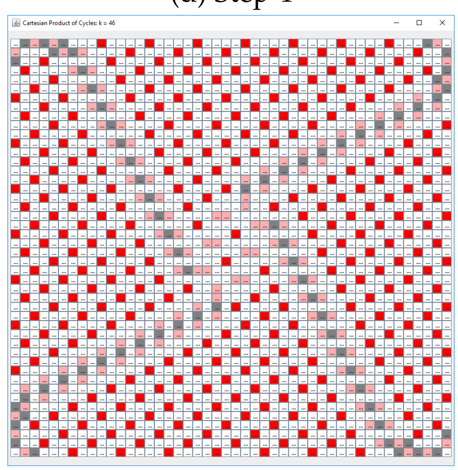
(c) Step 3



(d) Step 4



(e) Step 5



(f) Step 6

FIGURE 4.12: Dominating set construction for $k = 46$

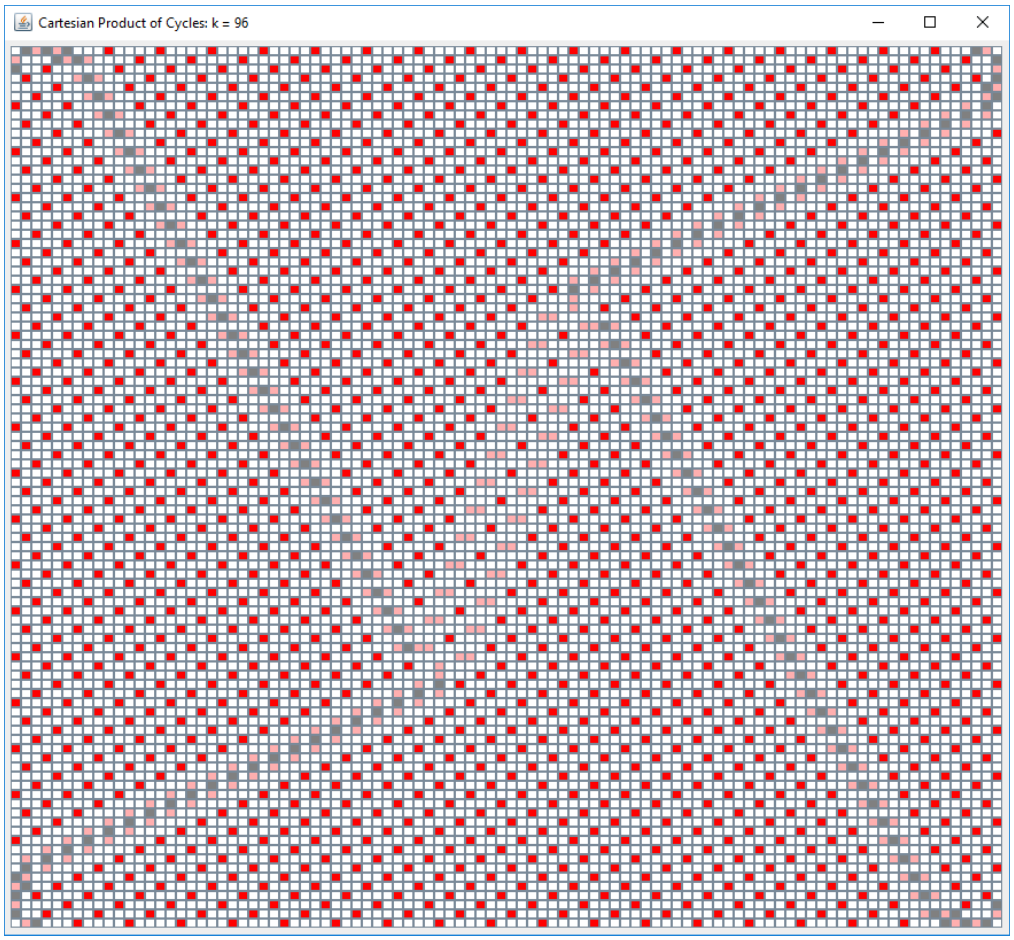
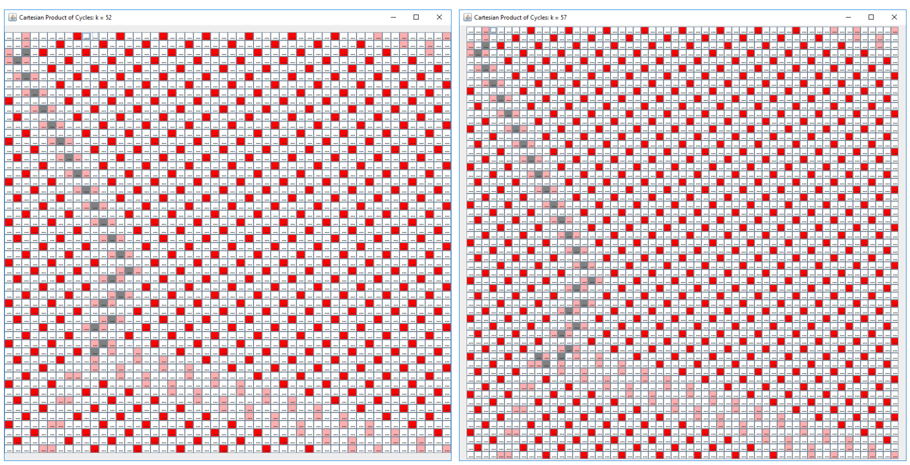


FIGURE 4.13: Constructed Dominating Set by Algorithm 4.2.1 for $k = 96$

4.3 Improving the Upper-bound for $\gamma(C_k \,\square\, C_k)$ when $k \equiv 2 \bmod 5$ and $k \geq 22$

This section gives a construction for creating a dominating set of size $\frac{k^2+2k-8}{5}$ for cases where k is congruent to 2 modulo 5 and $k \geq 22$. This improves the previous upper bound of $\frac{k^2+3k}{5}$ provided by our construction in Section 4.1. Moreover, we conjecture that the exact value of $\gamma(C_k \,\square\, C_k)$ when k is congruent to two modulo five is $\frac{k^2+2k-8}{5}$.

Algorithm 4.3.1 shows the pseudo-code for the construction which generates a dominating set of size $\frac{k^2+2k-8}{5}$ for $k \geq 22$ when k is congruent to 2 modulo 5. Figure 4.14 illustrates the constructed pattern for $k = 52$ and $k = 57$.



(a) k = 52 (b) k = 57

FIGURE 4.14: The patterns for $k = 42$ and $k = 47$

Figure 4.15 illustrates the construction steps of Algorithm 4.3.1 for $k = 42$.

Algorithm 4.3.1 Construction Algorithm for a perfect dominating set for $C_k \square C_k$

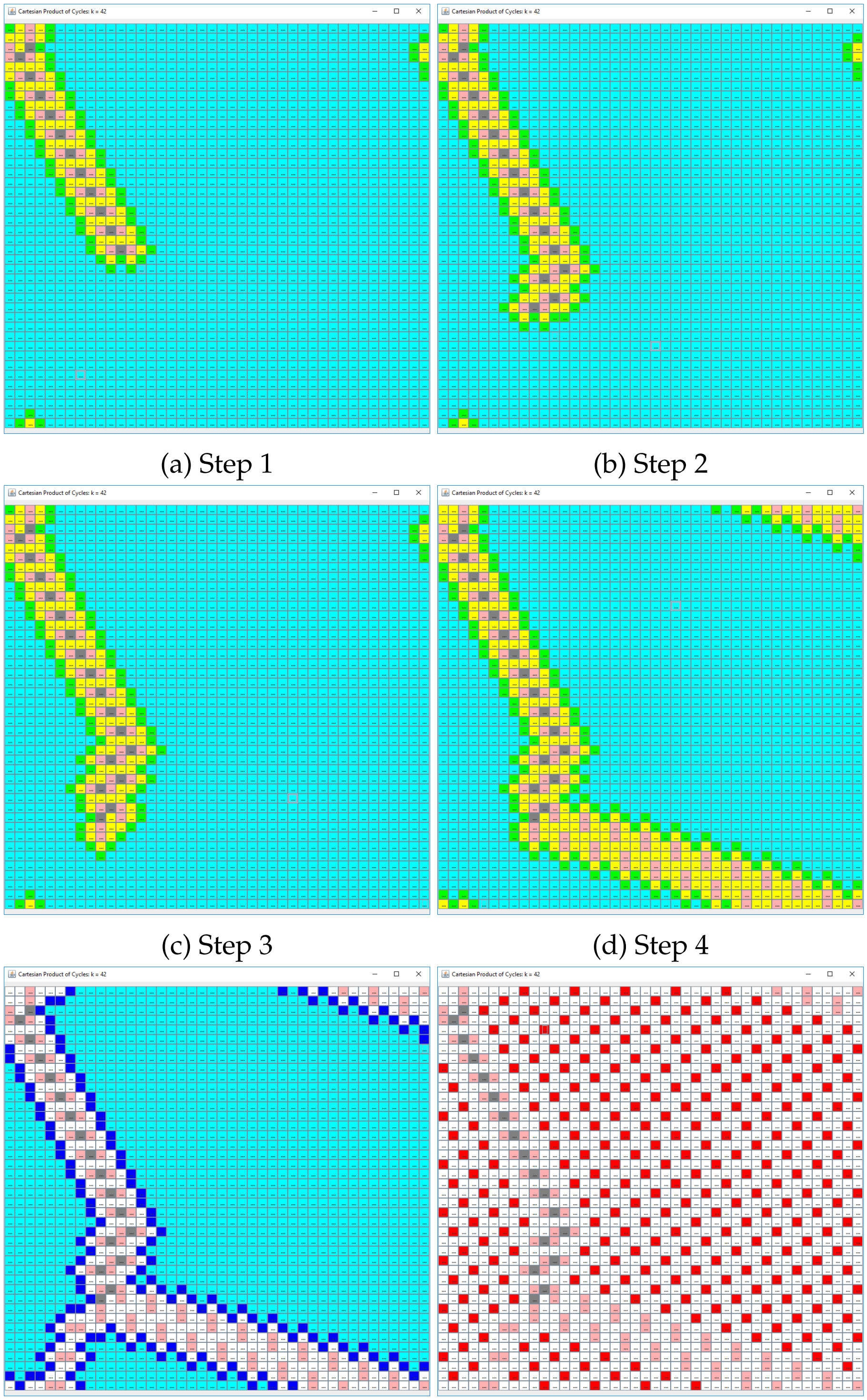
procedure DOMINATING SET(k) $\triangleright$ Constraint: $k \equiv 2 \bmod 5, k \geq 22$
  if ($k\%10 == 2$) **then**
    $\bar{k} \leftarrow (k-2)/5+3$
    $k_{jig} \leftarrow (k-2)/10-2$
    $k_{pink_v} \leftarrow 3*(k-2)/10-2$
    $k_{pink_h} \leftarrow (k-2)/10-1$
  else
    $\bar{k} \leftarrow (k-7)/5+5$
    $k_{jig} \leftarrow (k-7)/10-2$
    $k_{pink_v} \leftarrow 3*(k-7)/10-1$
    $k_{pink_h} \leftarrow (k-7)/10$
  end if
Step 1 - The J-shape on Top and the staircase-shape
  Add vertex $[0,2]$ to S
  Add vertex $[1,2]$ to S
  Add vertex $[2,0]$ to S
  $r \leftarrow 3$
  for $c \leftarrow 0$ to $\bar{k}$ **do**
    Add vertex $[r,c]$ to S
    Add vertex $[r,c+2]$ to S
  end for
Step 2 - Jig-shapes in the middle
  for $i \leftarrow 0$ to k_{jig} **do**
    Add vertex $[r,c]$ to S
    Add vertex $[r,c+2]$ to S
    Add vertex $[r+1,c]$ to S
    Add vertex $[r+1,c-2]$ to S
    $r \leftarrow r+3$
    $c \leftarrow c-1$
  end for
Step 3 - Upside down J-shape in the middle
  Add vertex $[r,c]$ to S
  Add vertex $[r,c+2]$ to S
  Add vertex $[r+1,c+2]$ to S
  Add vertex $[r+2,c]$ to S
  Add vertex $[r+3,c]$ to S
  if ($k\%10 == 7$) **then**
    Add vertex $[r+1,c-2]$ to S
  end if

Step 4 - Vertical Pairs
$r \leftarrow r + 1$
$c \leftarrow c + 5$
for $i \leftarrow 0$ to k_{pink_v} **do**
  Add vertex $[r, c]$ to S
  Add vertex $[(r + 1)\%k, c]$ to S
  Add vertex $[(r + 2)\%k, c - 2]$ to S
  Add vertex $[(r + 3)\%k, c - 2]$ to S
  $r \leftarrow (r + 1)\%k$
  $c \leftarrow c + 3$
end for
Step 5 - Horizontal Pairs
$r \leftarrow k - 1$
$c \leftarrow 4$
for $i \leftarrow 0$ to k_{pink_h} **do**
  Add vertex $[r, c]$ to S
  Add vertex $[r, c + 1]$ to S
  $r \leftarrow r - 3$
  $c \leftarrow c + 1$
end for
Step 6 - Perfect Dominating Vertices
Finish-Perfect(S, k) ▷ Picks the perfect dominating vertices
return S
end procedure

Algorithm 4.3.1 constructs a dominating set for $C_k \square C_k$ for all the values of k when k is congruent to two modulo five. So we achieved the upper bound of size $\frac{k^2+2k-8}{5}$ for $\gamma(C_k \square C_k)$ in this case. This upper bound matches the domination number for computed cases. This leads us to the following conjecture.

Conjecture 4.3.1. *For k congruent to two modulo five and $k \geq 22$,*

$$\gamma(C_k \square C_k) = \frac{k^2 + 2k - 8}{5}.$$



(a) Step 1 (b) Step 2

(c) Step 3 (d) Step 4

(e) Step 5 (f) Step 6

FIGURE 4.15: Dominating set construction for $k = 42$

Chapter 5

Conclusions and Future Research

A summary of the work that is done in this project is provided in Section 5.1 . In Section 5.2, various open problems that arose from the study are summarized for further work.

5.1 Thesis Summary

This thesis provides constructions that give dominating sets of the following sizes:

k mod 5	Best Formula	Bounds
0	$\frac{k^2}{5}$(Proved)	$k \geq 5$
1	$\frac{k^2+2k-8}{5}$(Conjectured)	$k \geq 41$
2	$\frac{k^2+2k-8}{5}$ (Conjectured)	$k \geq 22$
3	$\frac{k^2+2k}{5}$ (Upper Bound)	$k \geq 3$
4	$\frac{k^2+k}{5}$(Conjectured)	$k \geq 4$

TABLE 5.1: Sizes of dominating sets constructed by our algorithms.

It is known that for zero modulo five the construction is optimal. According to Bean's conjectures [4], these constructions are optimal for cases when k is one or four modulo five. We have conjectured that the construction gives an optimal result for two modulo five. The construction for three modulo five is not optimal, because it does not match the best known results for small cases as indicated in Table 5.2, but it improves the previous upper bound. The main contributions made on this project were the *DomGrid App* in Chapter 3, and improving the upper bounds of $\gamma(C_k \square C_k)$ when k is congruent to two or three modulo five in Chapter 4.

k	Best Known Results	Our Construction in Chapter 4
13	38	39
18	71	72
23	114	115
28	166	168
33	229	231
38	302	304
43	385	387

TABLE 5.2: Comparing the best known results on $\gamma(C_k \square C_k)$ for $k \equiv 3 \bmod 5$ with our upper bound.

Finding a dominating set is inherently a harder problem than finding a dominating number. In Chapter 4, we developed algorithms for construction of dominating sets of desired sizes. The running time of these algorithms is proportional to the size of the dominating set. Table 5.1 summarizes the known results on domination number of Cartesian product of two k-cycles.

5.2 Future Work

In Chapter 3 we developed the *DomGrid App* for visualization of dominating sets for the Cartesian product of two k-cycles. The *DomGrid App* can be generalized to support more families of graphs which can be represented on a grid, such as Cartesian product of two arbitrary cycles, grid graphs, hypercube graphs and triangular grid graphs. This tool has proved extremely helpful for providing human guidance and insight for the graphs that are a product of two k-cycles.

They will likely also prove a very valuable research tool for these other dominating set problems. The smallest open case sizes and their bounds for hypercube graphs, triangular grid graphs and queen graphs are summarized in Table 5.3.

Graph Family	Smallest Open Case	Bounds for γ
Hypercube Graphs (Q_k)	10	$107 - 120$
Triangular Grid Graphs (T_k)	32	87 (Conjectured)
Queen Graphs ($Queen_k$)	26	$13 - 14$

TABLE 5.3: Smallest open cases for the dominating number of three families of graphs.

Another interesting problem that arises from Chapter 4 and Table 5.1 is to either prove the conjectured values for dominating number of Cartesian product of two k-cycles when k is congruent to one, two or four modulo five are optimal, or to find smaller dominating sets for the cases where cycle size k is not zero modulo five. Improving the upper bound when k is congruent to three modulo five is another interesting research problem.

Bibliography

[1] Robert B. Allan and Renu Laskar. "On domination and independent domination numbers of a graph". In: *Discrete Math.* 23.2 (1978), pp. 73–76.

[2] Jernej Azarija, Michael A. Henning, and Sandi Klavžar. "(Total) domination in prisms". In: *Electron. J. Combin.* 24.1 (2017), Paper 1.19, 11.

[3] W. W. Rouse Ball. *A short account of the history of mathematics*. Dover Publications, Inc., New York, 1960, pp. xxiv+522.

[4] Richard Bean. *The Online Encyclopedia of Integer Sequences*. Entry A094087 (domination number of cartesian product of n-cycles). URL: https://oeis.org/A094087.

[5] Claude Berge. *The theory of graphs and its applications*. Translated by Alison Doig. Methuen & Co. Ltd., London; John Wiley & Sons Inc., New York, 1962.

[6] Riccardo Bertolo, Patric R. J. Östergå rd, and William D. Weakley. "An updated table of binary/ternary mixed covering codes". In: *J. Combin. Des.* 12.3 (2004), pp. 157–176.

[7] William Herbert Bird. "Computational Methods for Domination Problems". University of Victoria. PhD thesis. 2017, p. 187.

[8] Tony Yu Chang. "Domination numbers of grid graphs". University of South Florida. PhD thesis. 1992, p. 120.

[9] Dmitry Finozhenok and William D. Weakley. “An improved lower bound for domination numbers of the queen’s graph”. In: *Australas. J. Combin.* 37 (2007), pp. 295–300.

[10] Daniel Gonçalves, Alexandre Pinlou, Michaël Rao, and Stéphan Thomassé. “The domination number of grids”. In: *SIAM J. Discrete Math.* 25.3 (2011), pp. 1443–1453.

[11] Teresa W. Haynes, Stephen T. Hedetniemi, and Peter J. Slater. *Fundamentals of domination in graphs*. Vol. 208. Monographs and Textbooks in Pure and Applied Mathematics. Marcel Dekker, Inc., New York, 1998, pp. xii+446.

[12] Richard M. Karp. “Reducibility among combinatorial problems”. In: *Complexity of computer computations (Proc. Sympos., IBMThomas J. Watson Res. Center, Yorktown Heights,N.Y., 1972)* (1972), pp. 85–103.

[13] Matthew D. Kearse and Peter B. Gibbons. “Computational methods and new results for chessboard problems”. In: *Australas. J. Combin.* 23 (2001), pp. 253–284.

[14] Sandi Klavžar and Norbert Seifter. “Dominating Cartesian products of cycles”. In: *Discrete Appl. Math.* 59.2 (1995), pp. 129–136.

[15] “Literatur-Berichte: Mathematische Unterhaltungen und Spiele”. In: *Monatsh. Math. Phys.* 12.1 (1901). Von Dr. W. Ahrens in Magdeburg. Mit 1 Tafel und vielen Figuren im Text. XII + 428 S. gr. 8 º. B. G. Teubner, Leipzig, 1901. Ladenpreis geb. 10 M, A41–A42.

[16] Patric R. J. Östergård and William D. Weakley. “Values of domination numbers of the queen’s graph”. In: *Electron. J. Combin.* 8.1 (2001), Research Paper 29, 19.

[17] Hans Siegfried. *Schachclub Ansbach*. Retrieved March 2019. URL: `http://www.schachclub-ansbach.de/chronik_bezzel.htm`.

[18] V. G. Vizing. "Some unsolved problems in graph theory". In: *Uspehi Mat. Nauk* 23.6 (144) (1968), pp. 117–134.

[19] V. G. Vizing. "The Cartesian product of graphs". In: *Vyčisl. Sistemy No.* 9 (1963), pp. 30–43.

[20] Stan Wagon. "Graph theory problems from hexagonal and traditional chess". In: *College Math. J.* 45.4 (2014), pp. 278–287.

[21] Eric W. Weisstein. *The Online Encyclopedia of Integer Sequences*. Entry A251419 (domination number of triangle grid graph). URL: `https://oeis.org/A251419`.

[22] Douglas B. West. *Introduction to graph theory*. Prentice Hall, Inc., Upper Saddle River, NJ, 1996, pp. xvi+512.