Report from Dagstuhl Seminar 21461

Descriptive Set Theory and Computable Topology

Edited by

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Abstract

Computability and continuity are closely linked – in fact, continuity can be seen as computability relative to an arbitrary oracle. As such, concepts from topology and descriptive set theory feature heavily in the foundations of computable analysis. Conversely, techniques developed in computability theory can be fruitfully employed in topology and descriptive set theory, even if the desired results mention no computability at all. In this Dagstuhl Seminar, we brought together researchers from computable analysis, from classical computability theory, from descriptive set theory, formal topology, and other relevant areas. Our goals were to identify key open questions related to this interplay, to exploit synergies between the areas and to intensify collaboration between the relevant communities.

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1 Executive Summary

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Research area and topics

Descriptive set theory traditionally studies the complexity of subsets of and functions between Polish spaces (which are the completely metrizable separable spaces). As a mathematical area, it has well-established interactions with set theory and real analysis. Its canonical textbook is Kechris [11].

Following the developments in (classical) descriptive set theory, also the area of effective descriptive set theory flourished. In a way, this is the result of replacing *continuous* by *computable* everywhere, and by replacing arbitrary countable union by effective ones. Here, the canonical textbook is Moschovakis' [19]. While classical descriptive set theory is trivial on discrete spaces, the results from effective descriptive set theory on ℕ often generalize results from computability theory. While this is rarely emphasized (see [20] for an exception), one can recover classical descriptive set theory from effective descriptive set theory by relativization – provided that theorems are phrased in the right way.

Recent years have seen a lot of interest in the interplay between descriptive set theory and theoretical computer science going beyond the natural meeting point of effective DST. Four core developments outlined below are particularly relevant for the meeting:

DST on spaces of interest for TCS

Certain classes of topological spaces were revealed as applicable to reasoning about the semantics of programming. The most famous example is domain theory, but Escardo's synthetic topology [6] or the relationship between well-structured transition systems and Noetherian spaces revealed by Goubault-Larrecq [7] were also very influential. The spaces relevant for TCS are often not Hausdorff, and in particular not Polish. Selivanov pioneered the call for a development of descriptive set theory for these spaces [28, 29]. A break-through was achieved by de Brecht [3] with identifying the class of quasi-Polish spaces as a common generalization of Polish spaces and omega-continuous domains, and by showing that many core results of descriptive set theory can be extended to quasi-Polish spaces.

In computable analysis, we typically work with the category of admissible represented spaces (equivalently, with QCB_0-spaces, i.e. T_0-quotients of countably-based spaces) [24]. This is a Cartesian-closed category, meaning that we can form function spaces. This is a very natural requirement from a TCS-perspective, but does not preserve being countably-based. How descriptive set theory works on non-countably-based spaces is still a mystery. de Brecht, Selivanov and Schröder have undertaken initial investigations, in particular into the Kleene-Kreisel spaces in [27, 26, 5]. Hoyrup has shown that even very simple non-countably-based spaces such as $\mathcal{O}(\omega^\omega)$ exhibit very unfamiliar behaviour compared to the usual DST [8].

Synthetic DST

de Brecht and Pauly observed a connection between synthetic topology (which in turn can be seen as the theory of functional programming [6]), models of hypercomputation and descriptive set theory [22, 23, 4]. This connection opens up the opportunity to apply reasoning styles about models of computation to descriptive set theory. Work by Kihara on the Jayne-Rogers conjecture has shown significant potential of this approach for solving open questions in descriptive set theory [12]. There is also a hope that this theory can connect to other parts of TCS such as descriptive complexity.

DST and computability theory

Traditional computability theory, in particular the study of enumeration degrees, was related to the study of topological spaces via the notion of point degree spectrum introduced by Kihara and Pauly [13], building on earlier work by J. Miller [17]. This lets us reason about the degrees of individual points in a topological space, and understand properties of the space in terms of what degrees are realized there. This technique was already used to resolve a long-standing open question by Jayne ([9], also [21]) on the number of sigma-homeomorphism types of Polish spaces in [13].

This connection is bidirectional, and also allows for the application of topological arguments in computability theory. As such, it has inspired a flurry of recent developments in the area of enumeration degrees by J. Miller, M. Soskova and others [2, 18, 1, 16]. Particularly remarkable here is the existence of non-total almost-total enumeration degrees. This is a purely recursion-theoretic statement, but the various known proofs all invoke topological arguments such as Brouwer's Fixed Point theorem, Urysohn's metrization theorem or Hurewicz' and Wallmann's characterization of countably-dimensional Polish spaces.

Of a similar flavour (but the precise connections are still unclear) is the approach to fractal geometry and Hausdorff dimension via *effective dimension* of points, defined via Kolmogorov complexity [14]. This approach has already been demonstrated to provide strengthening of core results of fractal geometry, in many cases by rendering inessential restrictions to measurable sets. This includes a reproof of known answer to the two-dimensional Kakeya-conjecture [15].

coPolish spaces and computational complexity

In general, it seems that computational complexity of algorithms from computable analysis needs second-order complexity (Kawamura and Cook [10]). For certain spaces, however, runtimes of algorithms are still first-order objects [25]. Ongoing work by de Brecht and Schröder has shown that this holds for the coPolish spaces, a dual notion to the quasi-Polish spaces. As such, it seems that "spaces where descriptive set theory is well-behaved" is the dual notion to "spaces where complexity theory is well-behaved". This merits further attention by a broader community.

Seminar structure

As our seminar brought together researchers from previously rather disconnected areas, we included several tutorial talks of one hour each to introduce the various facets of our seminar topics to everyone. The talks covered *Quasi-Polish spaces* (Matthew de Brecht), *Quantitative Coding and Complexity Theory of Continuous Data* (Martin Ziegler), *CoPolish spaces and Effectively Hausdorff spaces* (Matthias Schröder), *New directions in Synthetic Descriptive Set Theory* (Takayuki Kihara), *Categorical aspects of Descriptive Set Theory* (Ruiyuan Chen), *Topology reflected in the enumeration degrees* (Joseph S. Miller), *Point-free Descriptive Set Theory* (Alex Simpson) and *Borel combinatorics fail in HYP* (Linda Westrick).

In addition, we had many short (fifteen minute) talks introducing topics or open questions. The prompt for these talks was "What theorem do you want to prove during/following this workshop?", and we are excited to learn what will come from this in the next months.

Challenges in hybrid Dagstuhl meetings

While the organizers and most participants had grown very accustomed to virtual meetings, the setting for our seminar was decidedly hybrid: About half of the participants were present in person, half were participating remotely. The same split applied to the organizing team.

The Dagstuhl team had equipped our main meeting room with multiple cameras and microphones (including microphones suspended from the ceiling throughout the room to pick up audience contributions). The equipment was controlled by several volunteers amongst the participants, and we are very grateful to Nikolay Bazhenov, Josiah Jacobsen-Grocott and Eike Neumann for having performed this crucial role. This setup made interactions in the lecture theatre between remote and in-person participants almost seamless.

A feature we felt was both crucial for a successful Dagstuhl seminar and difficult to accomplish in a hybrid setting are the informal discussions taking place in smaller groups. Our approach was to make those slightly less informal, and to use the collaboration platform Slack for arranging meetings. Slack also served for asking questions somewhat after the talks. This was somewhat successful, and several fruitful discussions involving both remote and in-person participants took place. It is difficult to ascertain though how much potential for additional discussions remained untapped.

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2 Table of Contents

3 Tutorial Talks

3.1 Categorical aspects of DST

Ruiyuan (Ronnie) Chen (McGill University – Montreal, CA)

We gave an introduction to categorical structures of interest in (classical) descriptive set theory, including axioms on limits and colimits in categories of topological and Borel spaces [3], duality with countably presented algebras, locales and point-free descriptive set theory [4], and connections with infinitary propositional and first-order logic [1, 2].

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3.2 Tutorial on Quasi-Polish Spaces

Matthew de Brecht (Kyoto University, JP)

We give a brief introduction to quasi-Polish spaces and their connections with Descriptive set theory, Domain theory, Computable topology, Geometric logic, and Duality.

3.3 New Directions in Synthetic Descriptive Set Theory

Takayuki Kihara (Nagoya University, JP)

Main reference Takayuki Kihara: "Lawvere-Tierney topologies for computability theorists", CoRR, Vol. abs/2106.03061, 2021.
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URL https://doi.org/10.1109/LICS.2015.48

Let us reconsider what an *oracle* is. At least three different perspectives of oracle can be presented. The first perspective is the most standard one, which is to think of an oracle as a *blackbox*, represented as a set, a function, an infinite string, etc. If we think of a blackbox as just a container to store an input data (whose data type is stream), as some people say, an oracle is merely an input stream. The latter idea is also quite standard nowadays.

The second perspective is based on a recent approach taken e.g. by de Brecht and Pauly to develop *synthetic descriptive set theory*, which is, according to them, the idea that descriptive set theory can be reinterpreted as the study of certain endofunctors and derived concepts, primarily in the category of *represented spaces*. We interpret this key idea of synthetic descriptive set theory as relativizing topological notions by (higher-type) oracles. In this approach, an oracle is considered to be a functor that allows us to *change the way we access spaces*.

The third perspective of oracle is the one that we promote in this talk. In this third perspective, we consider an oracle to be an *operation on truth-values* that may cause a transformation of one world into another. One might say that this is based on the idea that there is a correspondence between *computations using oracles* and *proofs using transcendental axioms*". Such an idea is used as a very standard technique in, for example, classical reverse mathematics. Our approach is similar, but with a newer perspective that deals more directly with operations on truth-values. More explicitly, it is formulated using topos-theoretic notions such as Lawvere-Tierney topology, which is a kind of generalization of Grothendieck topology to an arbitrary topos.

In this talk, we clarify the connection between these three perspectives of oracle. In this way, we attempt to bridge the gap between computability theory, synthetic description set theory, and effective topos theory.

3.4 Topology reflected in the enumeration degrees

Joseph S. Miller (University of Wisconsin – Madison, US)

This was an expository talk on connections between pure topology and the enumerations degrees.

The continuous degrees were introduced by the speaker (2004) to measure the computability-theoretic content of elements of computable metric spaces. They properly extend the Turing degrees. All known constructions of nontotal (i.e., non-Turing) continuous degrees involve a nontrivial topological component. Indeed, the fact that there is a nontotal continuous degrees in every upper cone is equivalent to the fact that the Hilbert cube is not a countable union of (subspaces homeomorphic to) subspaces of Cantor space.

The continuous degrees naturally embed in the enumeration degrees, where there are more connections to topology. Many of these were described by Kihara and Pauly, who assigned enumeration degrees to the points of any second countable T_0 topological space. This work was continued by Kihara, Ng, and Pauly. Among their many results, they characterized the cototal degrees as the degrees of points in $(\omega_{\mathrm{cof}})^\omega$, where ω_{cof} is ω with the cofinite topology.

It is not know if every continuous degree is graph cototal, but the work above allows us to translate this into a topological question. In particular, the following are equivalent:

- There is a continuous degree that is not graph cototal in every upper cone.
- The Hilbert cube is not a countable union of (subspaces homeomorphic to) subspaces of $(\omega_{\mathrm{cof}})^\omega$.

3.5 CoPolish Spaces and Effectively Hausdorff Spaces

Matthias Schröder (TU Darmstadt, DE)

This talk presented two classes of topological spaces which play a big role in Computable Analysis, namely CoPolish spaces and effectively Hausdorff spaces. CoPolish spaces are a generalisation of locally compact spaces in the realm of QCB-spaces. They form exactly the class of topological spaces which admit Simple Complexity, i.e. the measurement of Time Complexity in terms of a discrete parameter on the input and the desired output precision. Moreover, we show that there exists a universal CoPolish space, which is a CoPolish space into which every other CoPolish space embeds as a closed subspace. Effectively Hausdorff spaces generalise computable metric spaces and yield a better effectivisation of Hausdorffness than the current notion of a computable Hausdorff space. Unlike computable Hausdorff spaces they admit computability of a certain form of overt compact choice. Moreover, we characterise computability of multivalued functions from computable metric spaces to effectively Hausdorff spaces.

3.6 Tutorial: Quantitative Coding and Complexity Theory of Compact Metric Spaces

Martin Ziegler (KAIST – Daejeon, KR)

Joint work of Donghyun Lim, Martin Ziegler
Main reference Donghyun Lim, Martin Ziegler: "Quantitative Coding and Complexity Theory of Compact Metric Spaces", in Proc. of the Beyond the Horizon of Computability – 16th Conference on Computability in Europe, CiE 2020, Fisciano, Italy, June 29 – July 3, 2020, Proceedings, Lecture Notes in Computer Science, Vol. 12098, pp. 205–214, Springer, 2020.
URL https://doi.org/10.1007/978-3-030-51466-2_18
URL http://youtu.be/QGTkZfUzhrI

Specifying a computational problem includes fixing encodings for input and output: encoding graphs as adjacency matrices, characters as integers, integers as bit strings, and vice versa. For such discrete data, the actual encoding is usually straightforward and/or complexity-theoretically inessential (up to linear or polynomial time, say). Concerning continuous data, already real numbers naturally suggest various encodings (formalized as historically so-called *representations*) with very different algorithmic properties, ranging from the computably "unreasonable" binary expansion [doi:10.1112/plms/s2-43.6.544] via qualitatively to polynomially and even linearly complexity-theoretically "reasonable" signed-digit expansion. But how to distinguish between un/suitable encodings of other spaces common in Calculus and Numerics, such as Sobolev?

With respect to qualitative computability over topological spaces, *admissibility* had been identified [doi:10.1016/0304-3975(85)90208-7] as a crucial criterion for a representation over the Cantor space of infinite binary sequences to be 'reasonable': It requires the representation to be (sequentially) continuous, and to be maximal with respect to (sequentially) continuous reduction [doi:10.1007/11780342_48]. Such representations are guaranteed to exist for a large class of spaces. And for (precisely) these does the sometimes so-called *Main Theorem* hold: which characterizes continuity of functions by the continuity of mappings translating codes, so-called *realizers*.

qualitative	computability	topology	(uniform) continuity	compactness	equilogical
quantitative	complexity	metric	modulus of continuity	entropy	ultrametric

Following this "dictionary", we refine qualitative computability over topological spaces to quantitative complexity over metric spaces, by developing the theory of *polynomially* and of *linearly admissible* representations. Informally speaking, these are 'optimally' continuous, namely linearly/polynomially relative to the space's entropy; and maximal with respect to relative linearly/polynomially continuous reductions defined below. A large class of spaces is shown to admit a quantitatively admissible representation, including a generalization of the signed-digit encoding; and these exhibit a quantitative strengthening of the qualitative *Main Theorem*, namely now characterizing quantitative continuity of functions by quantitative continuity of realizers. Our quantitative admissibility thus provides the desired criterion for complexity-theoretically 'reasonable' encodings.

3.7 Borel combinatorics fail in HYP

Linda Westrick (Pennsylvania State University – University Park, US)

Joint work of Linda Westrick, Henry Towsner, Rose Weisshaar
Main reference H. Towsner, R. Weisshaar, L. Westrick: "Borel combinatorics fail in HYP", To appear in the Journal of Mathematical Logic
URL http://arxiv.org/abs/2106.13330

Of the principles just slightly weaker than ATR, the most well-known are the theories of hyperarithmetic analysis (THA). By definition, such principles hold in HYP. Motivated by the question of whether the Borel Dual Ramsey Theorem is a THA, we consider several theorems involving Borel sets and ask whether they hold in HYP. To make sense of Borel sets without ATR, we formalize the theorems using completely determined Borel sets. We characterize the completely determined Borel subsets of HYP as precisely the sets of reals which are Δ^1_1 in $L_{\omega_1^{ck}}$. Using this, we show that in HYP, Borel sets behave quite differently than in reality. For example, in HYP, the Borel dual Ramsey theorem fails, every n-regular Borel acyclic graph has a Borel 2-coloring, and the prisoners have a Borel winning strategy in the infinite prisoner hat game. Thus the negations of these statements are not THA.

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4 Short Talks

4.1 Continuity and Computability

Vasco Brattka (Bundeswehr University Munich, DE)

We discuss relations between continuity and computability. From the folklore fact that LPO is the weakest discontinuous function with respect to the topological version of Weihrauch reducibility, we deduce a characterization of discontinuity as the class of those functions whose parallelization realizes every Turing jump on some cone. We also show that the parallelization of a function being computably reducible to the identity is a condition that sits in between computability and computability with respect to the halting problem and we raise the question whether this condition can be separated from computability.

4.2 When does Wadge meet Tang and Pequignot?

Riccardo Camerlo (University of Genova, IT)

Wadge hierarchy on topological spaces has been introduced by W.W. Wadge to compare subsets according to their complexity. A variation of this hierarchy has been introduced by A. Tang for the Scott domain, and more recently generalized by Y. Pequignot to every T_0 second countable spaces. I discuss the question of when these two hierarchies coincide, presenting what is known and which problems are still open.

4.3 Algorithmic Learning of Structures

Ekaterina Fokina (TU Wien, AT)

Joint work of Nikolay Bazhenov, Ekaterina Fokina, Dino Rossegger, Luca San Mauro, Alexandra Soskova, Mariya Soskova, Stefan Vatev
Main reference Nikolay Bazhenov, Ekaterina B. Fokina, Luca San Mauro: "Learning families of algebraic structures from informant", Inf. Comput., Vol. 275, p. 104590, 2020.
URL https://doi.org/10.1016/j.ic.2020.104590

In this talk we summarize some of the recent results and mention several open questions on algorithmic learning of structures. We combine the ideas of computable structure theory and algorithmic learning theory (inductive inference) to study the question of what classes of structures are learnable under various learning criteria and restrictions. A class of structures is said to be learnable if there is a learner (a function) that correctly learns each structure from the class. This means, that the learner observes larger and larger finite pieces of the structure and makes guesses about which structure it is observing. After finitely many steps the learner must converge to a correct hypothesis. In general, we do not care about the complexity of the learner, but sometimes we do.

In the talk we explain the main result of [1] which gives a syntactic characterization of explanatory learnability of classes of structures from informant and also gives an upper bound on the complexity of the learner. We then mention a similar result for the notion of learning of structures from text (work in progress [3]). Furthermore, we mention results from [2] that reveal an interesting relation between explanatory learning of structures from informant and descriptive set theory. We wonder what other learning criteria can be characterized syntactically and/or in terms of equivalence relations.

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4.4 Refuting Selman's theorem in the hyperenumeration degrees

Jun Le Goh (University of Wisconsin – Madison, US)

We report on discussions by the participants in the #e-degrees Slack channel, specifically on hyperenumeration reducibility $\leq_{he}$ (see M. Soskova's abstract in the present report).

We came up with a possible strategy for refuting the analog of Selman's theorem for $\leq_{he}$, i.e., for constructing sets $A \not\leq_{he} B$ such that whenever $B \leq_{he} C \oplus C^c$, we have $A \leq_{he} C \oplus C^c$. The idea is to construct a Δ^1_1-pointed tree $T \subseteq \omega^{<\omega}$ with no dead ends such that $T^c \not\leq_{he} T$. It then suffices to consider $A = T^c$ and $B = T$: If $T \leq_{he} C \oplus C^c$, then T is $\Pi^1_1(C)$, so T has a path P which is $\Pi^1_1(C)$. Since T is Δ^1_1-pointed, it is $\Delta^1_1(P)$, hence $\Delta^1_1(C)$. We conclude that T^c is $\Pi^1_1(C)$, i.e., $T^c \leq_{he} C \oplus C^c$ as desired.

Josiah Jacobsen-Grocott has made progress on implementing the above strategy.

4.5 A characterization of Π^0_3-completeness

Vassilios Gregoriades (National Technical University of Athens, GR)

Main reference Vassilios Gregoriades: "Intersections of ℓ^q spaces in the Borel hierarchy", Journal of Mathematical Analysis and Applications, Vol. 498(1), p. 124922, 2021.
URL https://doi.org/10.1016/j.jmaa.2021.124922

Given $0 < a < q$, the intersection of all spaces ℓ^p for $p > a$ is a Π^0_3-complete subset of ℓ^q. This answers a question by Nestoridis [1]. The proof motivates a characterization of Π^0_3-completeness of sets in Polish spaces.

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4.6 There is no Good Notion of Quasi-Polish Convergence Spaces

Reinhold Heckmann (AbsInt – Saarbrücken, DE)

We looked for a full subcategory QP-CONV of the category CONV of convergence spaces that is closed under countable product, equalizers, and exponentials and whose topological spaces are exactly the quasi-Polish spaces. A natural candidate is QPE, the least full subcategory of CONV that contains the Sierpinski space and is closed under isomorphism, countable products, equalizers, and exponentials. Yet QPE contains the subspace Q of R, which is not quasi-Polish, and this implies that there is no category QP-CONV with the desired properties. Nevertheless, we think that QPE is an interesting category for further study.

4.7 Descriptive complexity on represented spaces

Mathieu Hoyrup (Loria, Inria – Nancy, FR)

Our goal is to better understand the relationship between two notions of descriptive complexity for subsets of a represented space, one using the topology, the other one using the representation.

4.8 Regularity properties, determinacy, and Solovay models

Daisuke Ikegami (Shibaura Institute of Technology – Tokyo, JP)

Main reference Daisuke Ikegami: "I-regularity, determinacy, and ∞-Borel sets of reals", CoRR, Vol. 2108.06632, 2021.
URL https://arxiv.org/abs/2108.06632

Regularity properties for sets of reals have been extensively studied since the early 20th century. A set of reals with a regularity property can be approximated by simple sets (such as Borel sets) modulo some small sets. Typical examples of regularity properties are Lebesgue measurability, the Baire property, the perfect set property, and Ramseyness.

For each σ-ideal I on the Baire space, Khomskii introduced a regularity property called I-regularity, and developed a general theory of I-regularity. Khomskii asked if strong axioms of determinacy (such as the Axiom of Determinacy) imply every set of reals is I-regular for any I such that the associated preorder P_I is proper.

In this talk, we discuss some results and questions concerning I-regularity, determinacy of infinite games, and Solovay models.

4.9 Resource-bounded effective dimension and the point-to-set principle

Elvira Mayordomo (University of Zaragoza, ES)

Joint work of Jack H. Lutz, Neil Lutz, Elvira Mayordomo
Main reference Jack H. Lutz, Neil Lutz, Elvira Mayordomo: "Dimension and the Structure of Complexity Classes", CoRR, Vol. abs/2109.05956, 2021.
URL https://arxiv.org/abs/2109.05956

In this short talk I review the recent results on the point to set principle for resource-bounded dimensions [1] stating that if Δ is a resource bound more general than Γ then Δ-dimension can be characterized in terms of Γ-dimension relativized to oracles dependent on Δ. I also include a few questions on the optimality and complexity of the corresponding oracles for different resource-bounds and gauge functions.

References

1 Jack H. Lutz, Neil Lutz, and Elvira Mayordomo. *Dimension and the Structure of Complexity Classes*. Arxiv arXiv:2109.05956, 2021

4.10 Computable presentations in topology

Alexander Melnikov

Joint work of Harrison-Trainor, Melnikov, and Ng
Main reference Matthew Harrison-Trainor, Alexander G. Melnikov, Keng Meng Ng: "Computability of Polish Spaces up to Homeomorphism", J. Symb. Log., Vol. 85(4), pp. 1664–1686, 2020.
URL https://doi.org/10.1017/jsl.2020.67

Computable presentations in effective algebra have been studied extensively for over 60 years. Classical results of Turing, Novikov, Boone, Feiner, and Khisamiev (in chronological order) illustrate that the standard notions of computable presentability for discrete algebraic structures differ in the standard classes such as semigroups, finitely presented groups, Boolean algebras, and abelian groups, respectively. Similar results are well-known for other common classes of structures such as, e.g., linear orders.

Similarly, investigations into the algorithmic content of abstract topological structures can be traced back to Maltcev in the 1960s. There are many definitions in the literature of what it means for a Polishable space to be computably presentable. These include computable complete metrization, computable topological presentation, and an effectively compact (completely metrized) presentation. These three notions seem to be the most commonly used notions throughout the literature. Nonetheless, in contrast with effective algebra, until very recently it was not known whether these notions of computable presentability differed (up to homeomorphism). We discuss several very recent works in which, using classical and advanced modern techniques, these notions have been separated in several common classed of compact spaces.

4.11 Topological spaces of countable structures

Russell G. Miller (CUNY Queens College – Flushing, US)

Main reference Russell Miller: "Isomorphism and classification for countable structures", Comput., Vol. 8(2), pp. 99–117, 2019.
URL https://doi.org/10.3233/COM-180095

We describe a natural way to view a collection of (isomorphism types of) countable structures as a topological space. The space is T_0 provided that the structures all have distinct existential theories: sometimes it is useful to adjoin definable predicates to the signature to achieve this. The notion of a (boldface) *Turing-computable embedding*, developed by Knight et al., is simply a continuous injective map from one such space to another.

We consider the specific example of algebraic field extensions of the rational numbers. Here the topology turns out to be that of a spectral space, meaning that (by a theorem of Hochster) there is some commutative ring R whose spectrum of prime ideals, under the Zariski topology, is homeomorphic to this space and thus can serve as a classification of these fields.

The main point of this talk is to raise questions. First, what is this ring R whose spectrum classifies the algebraic fields? (Well-known polynomial rings and other obvious guesses at R have all so far turned out to be wrong.) Second, the procedure above gives rise to many more computable topological spaces, some of which are spectral and others not. In what ways do the separate, well-developed disciplines of computable topology and computable structure theory interact here, and how can we use the interaction to develop these disciplines further and to link them together?

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3 R. Miller, Isomorphism and Classification for Countable Structures, *Computability* 8 (2019) 2, 99–117.

4.12 Computable Endofunctors, Markov-computability and Relativization

Arno Pauly (Swansea University, GB)

Joint work of Arno Pauly, Matthew de Brecht

The notion of a computable endofunctor was introduced by Pauly and de Brecht [4] in order to give a somewhat unified and principled approach to develop descriptive set theory for arbitrary represented spaces. The technology was used in [3] to obtain a computable version of the Jayne Rogers theorem, and in [1] to effectivize the property of being a Noetherian topological space (in a way that revealed it to be a higher-order analogue of both compactness and overtness).

(As pointed out by Neumann in [2], the terminology “locally computable endofunctor” would be more appropriate.)

If the endofunctors generating the usual notions of interest for descriptive set theory had left adjoints, we could use abstract category theory to draw conclusions in a way that generalizes retopologization arguments. Alas, adjoints seem to be rare over the category of represented spaces and continuous functions. If instead, we take Markov-computable maps as morphisms, adjoints become abundant. A challenging question now is whether we can incorporate relativization arguments into the category-theoretic framework in a way that links the Markov-computable setting with the usual one.

References

1 Matthew de Brecht and Arno Pauly. Noetherian Quasi-Polish spaces. In Valentin Goranko and Mads Dam, editors, *26th EACSL Annual Conference on Computer Science Logic (CSL 2017)*, volume 82 of *LIPIcs*, pages 16:1–16:17. Schloss Dagstuhl, 2017.
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3 Arno Pauly and Matthew de Brecht. Non-deterministic computation and the Jayne Rogers theorem. *Electronic Proceedings in Theoretical Computer Science*, 143, 2014. DCM 2012.
4 Arno Pauly and Matthew de Brecht. Descriptive set theory in the category of represented spaces. In *30th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, pages 438–449, 2015.

4.13 Effective overtness of generalised Cantor spaces

Philipp Schlicht (University of Bristol, GB)

Joint work of Arno Pauly, Philipp Schlicht

The generalised Cantor space 2^κ for an uncountable regular cardinal κ is the space of binary sequences of length κ. One can translate the notion of representable space to this context, since 2^κ comes with a natural notion of computability with time bound κ. While 2^κ need not have a κ-computable dense subset of size κ, we discuss the weaker notion of effective overtness for these spaces.

4.14 Effective embedding and interpretations

Alexandra A. Soskova (University of Sofia, BG)

Joint work of Rachael Alvir, Wesley Calvert, G. Goodman, Valentina S. Harizanov, Julia F. Knight, R. Miller, Andrei S. Morozov, Alexandra A. Soskova, Stefan Vatev, Rose Weisshaar
Main reference Julia F. Knight, Alexandra A. Soskova, Stefan V. Vatev: "Coding in graphs and linear Orderings", J. Symb. Log., Vol. 85(2), pp. 673–690, 2020.
URL https://doi.org/10.1017/jsl.2019.91

Friedman and Stanley [2] introduced Borel embeddings as a way of comparing classification problems for different classes of structures. A Borel embedding for a class K in a class K' represents a uniform procedure for coding structures from K in structures from K'. Many Borel embeddings are actually Turing computable. When a structure $\mathcal{A}$ is coded in a structure $\mathcal{B}$, effective decoding is represented by a Medvedev reduction of $\mathcal{A}$ to $\mathcal{B}$. Harrison-Trainor, Melnikov, Miller, and Montalbán [3] defined a notion of effective interpretation of $\mathcal{A}$ in $\mathcal{B}$ and proved that this is equivalent with the existing of computable functor.

The class of undirected graphs and the class of linear orderings both lie "on top" under Turing computable embeddings. The standard Turing computable embeddings of structures in undirected graphs come with uniform effective interpretations. We [4] give examples of graphs that are not Medvedev reducible to any linear ordering, or to the jump of any linear ordering. Any graph can be interpreted in a linear ordering using computable Σ_3 formulas. Friedman and Stanley gave a Turing computable embedding L of directed graphs in linear orderings. We show that there does not exist a Borel interpretation, i.e. there are no $L_{\omega_1\omega}$ formulas that, for all graphs G, interpret G in $L(G)$. Our conjecture is: *For any Turing computable embedding Θ of graphs in orderings, there do not exist $L_{\omega_1\omega}$ formulas that, for all graphs G, define an interpretation of G in $\Theta(G)$.*

We [1] succeed to find an effective interpretation of a field in its Heisenberg group without parameters, generalising an old result of Maltsev, who gave a definition of a field in its Heisenberg group with a pair of parameters. We could define an algebraically closed field C in the group $SL_2(C)$ using finitary existential formulas with a pair of parameters. The question is: *Are there formulas that, for all algebraically closed fields C of characteristic 0, define an effective interpretation of C in $SL_2(C)$? Are there existential formulas that serve?*

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4.15 The hyper enumeration degrees

Mariya I. Soskova (University of Wisconsin – Madison, US)

In this talk I outlined some main aspects of the enumeration degrees and their relationship to the Turing degrees, so that I can draw a parallel between the enumeration degrees and the hyperenumeration degrees. We say that A is hyper enumeration reducible to B if there is a c.e. set W such that $x \in A$ if and only if for every $f \in \omega^{<\omega}$ there is some n and some finite set D such that $(f \upharpoonright n, x, D) \in W$ and $D \subseteq B$. This notion was introduced and studied by Sanchis [1], who showed that in many ways hyper enumeration reducibility relates to hyperarithmetic reducibility in the same way that enumeration reducibility relates to Turing reducibility.

I focused on two open questions:

1. Do we have an analog of Selman's theorem for hyper-enumeration reducibility: Is it true that $A \leq_{he} B$ if and only if for every X if B is $\Pi^1_1(X)$ then A is $\Pi^1_1(X)$?
2. Is there a way to stratify hyper-enumeration reducibility: We know that $A \leq_h B$ if and only if $A \leq_T B^{(\alpha)}$ for some B-computable ordinal α. Do we have some analogous result for hyper enumeration reducibility, perhaps using the skip instead of the jump?

References

1 Luis Sanchis. *Hyperenumeration reducibility*. Notre Dame Journal of Formal Logic, Volume XIX, Number 3, July 1978.

5 Working groups

5.1 Computable categoricity of Polish spaces

Nikolay Bazhenov (Sobolev Institute of Mathematics – Novosibirsk, RU), Ivan Georgiev (Sofia University "St. Kliment Ohridski", BG), Jun Le Goh (University of Wisconsin – Madison, US), Vassilios Gregoriades (National Technical University of Athens, GR), Mathieu Hoyrup (LORIA & INRIA Nancy, FR), Iskander Shagitovich Kalimullin (Kazan Federal University, RU), Steffen Lempp (University of Wisconsin – Madison, US), Alexander Melnikov (Victoria University – Wellington, NZ), Russell G. Miller (CUNY Queens College – Flushing, US), Eike Neumann (MPI für Informatik – Saarbrücken, DE), Keng Meng Ng (Nanyang TU – Singapore, SG), Arno Pauly (Swansea University, GB), Alexandra A. Soskova (University of Sofia, BG), and Daniel Turetsky (Victoria University – Wellington, NZ)

This working group followed talks by Ng and Melnikov.

Galicki, Melnikov and Ng have studied categoricity of Polish spaces. A Polish space X is computably categorical if all computable presentations of X are computably homeomorphic. More generally, a set A is the degree of categoricity of a space X if A is the minimal oracle such that all computable copies of X are A-computably homeomorphic.

They proved the following results, among others:

- The space of natural numbers N has degree 0',
- The Cantor space has degree 0',
- The Baire space is not 0'-computably categorical,
- The unit interval [0,1] has degree 0".

They also have a sketch proof that no compact Polish space is computably categorical.

In this group we have discussed the case of compact Polish spaces, trying to complete the proof, and obtained that X a compact space is not computably categorical in the following cases:

- If X has a computable copy containing a nowhere dense non-empty Pi01-set,
- If X has a computable copy such that N computably embeds in the isolated points of X.

The arguments also extend to sigma-compact spaces.

It remains open whether there is computably categorical compact Polish space, more generally if there is a computably categorical Polish space.

5.2 AE-theory of enumeration degree structures

Steffen Lempp (University of Wisconsin – Madison, US), Jun Le Goh (University of Wisconsin – Madison, US), Keng Meng Ng (Nanyang TU – Singapore, SG), and Mariya I. Soskova (University of Wisconsin – Madison, US)

This is to follow up on the short talk I gave on progress toward deciding the AE-fragments of the first-order theories of two degree structures, the global enumeration degrees and the local Σ^0_2-enumeration degrees. For the global structure, significant progress was already reported on from the paper [1]. Plans are in place to extend our results toward a full solution. For the local structure, significant progress has been made during and since the workshop by the four of us: We now have a working conjecture for 1-point extensions of antichains, which we hope to check and write up carefully over the next few months, whereas at the time of the workshop, we only had an analysis of the very special case where the antichain has size 3!

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1 Lempp, Steffen; Soskova, Mariya I.; and Slaman, Theodore A., *Fragments of the theory of the enumeration degrees*, Advances in Mathematics, Vol. 383, 2021, paper 107686, 39 pages.

6 Open problems

6.1 Questions on left-c.e. reals

Iskander Shagitovich Kalimullin (Kazan Federal University, RU)

Joint work of Iskander Shagitovich Kalimullin, Marat Kh. Faizrahmanov
Main reference Marat Kh. Faizrahmanov, Iskander Sh. Kalimullin: "Limitwise monotonic sets of reals", Math. Log. Q., Vol. 61(3), pp. 224–229, 2015.
URL https://doi.org/10.1002/malq.201400015

The talk is devoted to possible applications and problems in computable topology related to the paper [1]. In this paper the authors found a countable subset of the reals which is not left-c.e. but is non-uniformly left-c.e. relative to any non-computable oracle. This has an applications in computable structure theory, but it is interesting also to know what effects we have studying uncountable subsets of the reals.

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6.2 Which Compact Metric spaces do/don't admit polynomially admissible representations?

Martin Ziegler (KAIST – Daejeon, KR)

Donghyun Lim and Martin Ziegler [arXiv:2002.04005v5] have quantitatively refined the qualitative notion of "admissible representation" [Kreitz&Weihrauch'85]; see the tutorial in this very seminar.

Many spaces admit polynomially admissible representations, the reals even a linearly admissible (namely the signed-digit) representations.

We wonder about compact metric spaces that provably do NOT admit a polynomially admissible representatios; and perhaps even a characterization of those that do.

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