# Descriptive Complexity on Non-Polish Spaces


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### Abstract

Represented spaces are the spaces on which computations can be performed. We investigate the descriptive complexity of sets in represented spaces. We prove that the standard representation of a countably-based space preserves the effective descriptive complexity of sets. We prove that some results from descriptive set theory on Polish spaces extend to arbitrary countably-based spaces. We study the larger class of coPolish spaces, showing that their representation does not always preserve the complexity of sets, and we relate this mismatch with the sequential aspects of the space. We study in particular the space of polynomials.



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## 1 Introduction

Several branches of theoretical computer science, such as semantics of programming language, domain theory or computability theory, have demonstrated the intimate relationship between computation and topology, one of the simplest manifestations of this being that computable functions are continuous. Descriptive Set Theory (DST) and its effective version provide a natural framework in which the interaction between computations and topology can be fully studied.

However, a large part of DST focuses on Polish spaces, leaving aside many topological spaces relevant to theoretical computer science. A first extension to $\omega$-continuous domains was developed by Selivanov [13]. A further extension allowing a unification with Polish spaces was achieved by de Brecht [2] who introduced the class of quasi-Polish spaces, which can be thought of as the largest class of countably-based spaces on which DST extends. Still, many topological spaces which are meaningful to theoretical computer science fall outside the class: for instance the Kleene-Kreisel spaces important in higher-order computation theory [8], the recently introduced coPolish spaces, which admit a well-behaved computational complexity theory [12], more generally the represented spaces in computable analysis.

Therefore, there is a need to extend DST to more general topological spaces. Such an extension was proposed and initiated in [10] for represented spaces. Some negative results were obtained in [5] for spaces of Kleene-Kreisel functionals. With a different approach, a study of quotients of countably-based spaces (QCB-spaces) was done in [4]. It would be interesting to explore the relationship between DST on represented spaces and equivalence relations on standard Borel spaces, a representation naturally inducing an equivalence relation on names.

In this paper, we investigate general countably-based spaces and the class of coPolish spaces, in particular the space $\mathbb{R}[X]$ of real polynomials. We investigate the validity of some classical theorems from DST on those topological spaces, identifying when they extend and when they do not.

All these spaces have natural representations, which are essential to the development of the theory. The role of representations in this work can be interpreted in two different ways:

1. Representations can simply be seen as a tool in the study of certain topological spaces. Many results can be stated without any mention of representations, and of a purely topological interest.
2. Representations can also be seen as providing a structure alternative to topology, inducing in particular different measures of descriptive complexity of sets. As we briefly mention at the end of the paper, several results from DST that fail on some topological spaces can be recovered if one uses the notions induced by the representation rather than the topology. This interpretation supports the viewpoint adopted in [10], suggesting a development of DST in the category of represented spaces rather than topological spaces (which makes no difference when restricting to Polish or quasi-Polish spaces where these two approaches are equivalent).

We think that it is too early to choose between these two viewpoints, and that they both have their merits.

We now give an overview of the results.

In a topological space, the descriptive complexity of a set $A$ measures the complexity of expressing $A$ in terms of open sets. In a represented space $(X, \delta_X)$, where $\delta_X :\subseteq \mathcal{N} \to X$ is a partial surjective function from the Baire space $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$ ($p \in \mathrm{dom}(\delta_X)$ being a code for the point $\delta_X(p) \in X$, to be given as input to a Turing machine), an alternative approach is to measure the descriptive complexity of the set of codes of points of $A$, which measures the complexity of testing membership of points in the set, when points are given by codes.

Represented spaces have a canonical topology (the final topology of the representation), so two competing measures of complexity are available on these spaces.

▶ **Problem 1.** *When do the notions of descriptive complexity induced by a representation and its corresponding topology coincide?*

We show that they coincide on countably-based spaces in a uniform computable way (effectivizing a result of de Brecht [2]) and that they can differ on other spaces, including $\mathbb{R}[X]$. In the class of coPolish spaces, we characterize the spaces on which the two notions of complexity coincide (at least in the low complexity levels) as the class of Fréchet-Urysohn spaces (the spaces in which closures and sequential closures coincide). It suggests that the mismatch between the topology and the representation is related to the difference between the topological and sequential aspects of the space, and that the complexity induced by the representation reflects the sequential rather than topological aspects of the space.

▶ **Problem 2.** *In a topological space, how to establish a lower bound on the descriptive complexity of a given set?*

On a Polish space, in order to prove that a set $A$ does not belong to a descriptive complexity class $\Gamma$, it is necessary and sufficient to prove that $A$ is $\check{\Gamma}$-hard (when $\Gamma$ is not self-dual, i.e. when $\check{\Gamma} \neq \Gamma$). We show that for complexity classes below $\boldsymbol{\Delta}^0_2$, more precisely for the classes of the difference hierarchy, this result is surprisingly valid on *arbitrary* countably-based spaces. However it becomes false on $\mathbb{R}[X]$, precisely because of Problem 1: the hardness of a set is not a measure of its topological complexity, but of the complexity of its preimage under the representation. Therefore, we need new techniques to measure the topological complexity of a set in non-countably-based spaces. We develop a criterion, which is necessary and sufficient, to prove that a set does not belong to a class below $\underset{\sim}{\mathbf{\Delta}}^0_2$, in any topological space.

Finally, we investigate the validity, outside Polish spaces, of a famous result from DST, the Hausdorff-Kuratowski Theorem. In its simplest form, it states that the class $\underset{\sim}{\mathbf{\Delta}}^0_2$ can be classified exhaustively using the difference hierarchy (the general result also considers classes $\underset{\sim}{\mathbf{\Delta}}^0_\alpha$).

▶ **Problem 3.** *On which topological spaces does the Hausdorff-Kuratowski Theorem hold?*

Using the previous results, we show that the Hausdorff-Kuratowski Theorem holds on a countably-based space if and only if that space does not contain any $\underset{\sim}{\mathbf{\Delta}}^0_2$-complete set. We show that the Hausdorff-Kuratowski Theorem does not hold on the topological space $\mathbb{R}[X]$. However, when seeing $\mathbb{R}[X]$ as a represented space and measuring the complexity of sets according to the representation rather than the topology, the Hausdorff-Kuratowski Theorem becomes true.

In Section 2 we give a minimalist summary of the background needed to state and prove our results. In Section 3 we present our results concerning Problem 2, which will be needed to give answers to Problem 1 in Section 4. In Section 5, we investigate the class of coPolish spaces, which includes spaces that are not countably-based.

## 2 Background

The Baire space is $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$ with the product topology generated by the cylinders $[\sigma]$, with $\sigma \in \mathbb{N}^*$. A *represented space* is a pair $(X, \delta_X)$ where $X$ is a set and $\delta_X :\subseteq \mathcal{N} \to X$ is onto. A *realizer* of a function $f : (X, \delta_X) \to (Y, \delta_Y)$ is any function $F : \mathrm{dom}(\delta_X) \to \mathrm{dom}(\delta_Y)$ such that $f \circ \delta_X = \delta_Y \circ F$. $f$ is *computable* if it has a computable realizer.

A represented space $(X, \delta_X)$ is *admissible* if the continuously realizable functions $f :\subseteq \mathcal{N} \to X$ are precisely the continuous functions (for the final topology of $\delta_X$).

An effective countably-based space is a countably-based topological space $X$ with a numbered basis of the topology $(B_i)_{i\in\mathbb{N}}$ such that intersection of basic open sets is computable: $B_i \cap B_j = \bigcup_{k\in W_{f(i,j)}} B_k$ for some computable $f : \mathbb{N}^2 \to \mathbb{N}$, where $(W_e)_{e\in\mathbb{N}}$ is an effective enumeration of the c.e. subsets of $\mathbb{N}$. The standard representation, which is admissible, is defined by representing $x \in X$ by any listing of the set $\{i \in \mathbb{N} : x \in B_i\}$. A particularly useful property of these spaces is that the standard representation is *effectively open*: $\delta([\sigma]) = \bigcup_{i\in W_{g(\sigma)}} B_i$, for some computable $g$. The class $\Sigma^0_1(X)$ of effective open sets consists of c.e. unions of basic open sets. More details can be found in [17, 9].

### 2.1 Hierarchies on topological spaces

#### 2.1.1 Borel hierarchy

We should emphasize that although we work in represented spaces, we are using the topology to define the descriptive complexity classes. It contrasts with the approach developed in [10] in which the complexity of a set is defined as the descriptive complexity of the corresponding set of names, i.e. the preimage of the set under the representation. Our general goal is precisely to compare the topological complexity of sets with the complexity of its corresponding set of names.

The Borel hierarchy, usually defined on Polish spaces, can be extended immediately to any topological space $X$, with a slight modification to handle correctly the non-Hausdorff spaces, in which open sets are not always unions of closed sets [13].

- $\boldsymbol{\Sigma}^0_1(X)$ is the class of open sets,
- For $1 < \alpha < \omega_1$, $A \in \boldsymbol{\Sigma}^0_\alpha(X)$ if $A = \bigcup_{i\in\mathbb{N}} A_i \setminus B_i$ where $A_i, B_i \in \boldsymbol{\Sigma}^0_{\alpha_i}$ with $\alpha_i < \alpha$.

We also define $\boldsymbol{\Pi}^0_\alpha(X)$ as the class of complements of sets in $\boldsymbol{\Sigma}^0_\alpha(X)$, and $\boldsymbol{\Delta}^0_\alpha(X) = \boldsymbol{\Sigma}^0_\alpha(X) \cap \boldsymbol{\Pi}^0_\alpha(X)$.

### 2.1.2 Difference hierarchy

Let $X$ be a topological space. The difference hierarchy $(\mathbf{D}_\alpha(\boldsymbol{\Sigma}^0_\beta(X)))_{1\leq\alpha<\omega_1}$ based on $\boldsymbol{\Sigma}^0_\beta(X)$ is defined by transfinite induction as follows [13]:

- $\mathbf{D}_1(\boldsymbol{\Sigma}^0_\beta(X)) = \boldsymbol{\Sigma}^0_\beta(X)$,
- $A \in \mathbf{D}_{\alpha+1}(\boldsymbol{\Sigma}^0_\beta(X))$ if $A = U \setminus B$ where $U \in \boldsymbol{\Sigma}^0_\beta(X)$ and $B \in \mathbf{D}_\alpha(\boldsymbol{\Sigma}^0_\beta(X))$,
- For a limit ordinal $\lambda$, $A \in \mathbf{D}_\lambda(\boldsymbol{\Sigma}^0_\beta(X))$ if $A = \bigcup_{\alpha<\lambda,\alpha \text{ even}} B_{\alpha+1} \setminus B_\alpha$ where $(B_\alpha)_{\alpha<\lambda}$ is a growing sequence of sets in $\boldsymbol{\Sigma}^0_\beta(X)$.

We define $\check{\mathbf{D}}_\alpha(\boldsymbol{\Sigma}^0_\beta(X))$ as the class of complements of sets in the class $\mathbf{D}_\alpha(\boldsymbol{\Sigma}^0_\beta(X))$. We will mainly focus on the hierarchy based on $\boldsymbol{\Sigma}^0_1(X)$, and we denote $\mathbf{D}_\alpha(\boldsymbol{\Sigma}^0_1(X))$ by $\mathbf{D}_\alpha(X)$.

In any topological space $X$, the difference hierarchy based on $\boldsymbol{\Sigma}^0_\beta(X)$ is contained in $\boldsymbol{\Delta}^0_{\beta+1}(X)$. On Polish spaces and even quasi-Polish spaces, the Hausdorff-Kuratowski Theorem states that the hierarchy entirely exhausts $\boldsymbol{\Delta}^0_{\beta+1}(X)$ (Theorem 70 in [2]).

### 2.1.3 Representations of sets

Representations of descriptive complexity classes have been investigated in [1, 14].

As soon as one has chosen a representation of open sets, at least the finite levels of the hierarchies have an obvious representation. For instance, a set in $\mathbf{D}_2(X)$ is represented by pairing two names of open subsets of $X$. A set in $\boldsymbol{\Sigma}^0_{n+1}(X)$ is inductively represented by two sequences of names of sets in $\boldsymbol{\Sigma}^0_n(X)$.

If $(X, \delta_X)$ is a represented set, then the canonical topology on $X$ is the final topology of $\delta_X$. An open subset $U$ of $X$ can be represented by a name of any open subset of $\mathcal{N}$ whose intersection with $\mathrm{dom}(\delta_X)$ is $\delta_X^{-1}(U)$.

## 3 Measuring the topological complexity of a set

When studying the descriptive complexity of sets in topological spaces, an important task is to prove that a set does not belong to a given class. In the traditional theory on Polish spaces, it can be achieved using the notion of hardness. We investigate on which topological spaces this technique is still valid, and develop an alternative technique working on all topological spaces.

Let $\Gamma$ be a complexity class. We say that $A \subseteq X$ is $\Gamma$-***hard*** if for every $C \in \Gamma(\mathcal{N})$, there is a continuous reduction from $C$ to $A$, i.e. a continuous map $f : \mathcal{N} \to X$ such that $C = f^{-1}(A)$.

On any Polish space $X$, if a class $\Gamma$ is not self-dual (i.e., $\Gamma \neq \check{\Gamma}$) and is closed under continuous preimages, then for $A \subseteq X$,

$$A \notin \Gamma(X) \iff A \text{ is } \check{\Gamma}(X)\text{-hard.} \tag{1}$$

This is essentially Wadge's lemma in [7]. We call this equivalence the *hardness criterion*. This result can easily be extended to the countably-based spaces admitting a *total* admissible representation $\delta$ (they are called quasi-Polish spaces [2]).

As we will see in this paper, it fails in other spaces such as $\mathbb{R}[X]$, because the hardness of a set does not measure its topological complexity, but the complexity of its preimage under the representation, which may differ. Therefore, we need some techniques to prove that a set does not belong to a given class. We are going to see that for classes below $\utilde{\mathbf{\Delta}}^0_2$,

- There is a characterization that is valid in any topological space,
- The hardness criterion (1) surprisingly holds for any countably-based space.

## 3.1 Arbitrary topological spaces

In order to characterize the classes of the difference hierarchy, we adapt the proof of the Hausdorff-Kuratowski Theorem presented in [7], in which the level of a set in the difference hierachy is essentially captured by iterating an operator on the set, and identifying the level at which the set becomes empty. However, the operator used in [7], which takes a set to its boundary, is too coarse to distinguish between the classes $\utilde{\mathbf{D}}_\eta$ and $\check{\utilde{\mathbf{D}}}_\eta$. We refine it and show how it captures precisely the complexity of a set.

Intuitively, iterating the operator progressively removes the simple parts of the set, so that the level at which the set is emptied measures the complexity of the set. Let $\omega_1$ be the first uncountable ordinal.

▶ **Definition 3.1.** *Let $X$ be a topological space. For a set $A \subseteq X$, we define the sequence of closed sets $(H_\eta)_{\eta<\omega_1}$ by transfinite induction on $\eta$ as follows:*

$$\begin{aligned} H_0(A) &= X, \\ H_{\eta+1}(A) &= \overline{A \cap H_\eta(A^c)}, \\ H_\lambda(A) &= \bigcap_{\eta<\lambda} H_\eta(A) \qquad \text{for a limit ordinal } \lambda. \end{aligned}$$

Intuitively, $x \in H_2(A)$ if $x$ is arbitrarily close to points of $A$ that are arbitrarily close to points of $A^c$. At the next level, $x \in H_3(A)$ if $x$ is arbitrarily close to points of $A$ that are arbitrarily close to points of $A^c$ that are themselves arbitrarily close to points of $A$. More generally, $H_\eta(A)$ contains the points that are sufficiently deep inside the boundary of $A$.

We now study the basic properties of that sequence (the proofs are given in the appendix). First, it is decreasing,

▶ **Proposition 3.2.** *If $\alpha \leq \beta$ then $H_\alpha(A) \supseteq H_\beta(A)$.*

At the limit levels, $A$ and $A^c$ induce the same set.

▶ **Proposition 3.3.** *For each limit ordinal $\lambda$, one has $H_\lambda(A) = H_\lambda(A^c)$.*

We only consider countable ordinals because on a large class of spaces including represented spaces, the sequence reaches a fixed point at some countable ordinal. A topological space is hereditarily Lindelöf if every family of open sets contains a countable subfamily with the same union. The final topology of a representation is always hereditarily Lindelöf.

▶ **Proposition 3.4.** *If $X$ is a hereditarily Lindelöf topological space, then for any $A \subseteq X$ the sequence $(H_\eta)_{\eta<\omega_1}$ is eventually constant.*

**Proof.** Let $U = X \setminus \bigcap_{\eta<\omega_1} H_\eta(A)$. The growing family of open sets $(X \setminus H_\eta(A))_{\eta<\omega_1}$ covers $U$, so contains a countable subfamily covering $U$. As a result, $U = X \setminus H_\eta(A)$ for some $\eta < \omega_1$, and $H_\alpha(A) = H_\eta(A)$ for all $\alpha \geq \eta$. ◀

Whether a point belongs to $H_\eta(A)$ only depends on the local behavior of $A$ around that point, which can be formulated as follows.

▶ **Lemma 3.5.** *Let $U$ be open. If $A \cap U = B \cap U$, then $H_\eta(A) \cap U = H_\eta(B) \cap U$.*

In particular, $H_\eta(A) \cap U = H_\eta(A \cap U) \cap U = H_\eta(A \cup U^c) \cap U$.

The main result of this section is that the topological complexity of a set is captured by this sequence, which gives an operative way of identifying the complexity of a set, and will be used in the sequel.

▶ **Theorem 3.6.** *Let $X$ be any topological space. For a set $A \subseteq X$, one has*

$$A \in \underset{\sim}{\mathbf{D}}_\eta(X) \iff H_{\eta+1}(A) = \emptyset.$$

The proofs are rather technical and given in the appendix. We simply mention that one direction has the following more general formulation.

▶ **Lemma 3.7.** *For any set $A \subseteq X$, one has $A \setminus H_{\eta+1}(A) \in \underset{\sim}{\mathbf{D}}_\eta(X)$.*

## 3.2 Countably-based spaces

The preceding result enables us to go further by extending the hardness criterion (1) to any countably-based space, at least for the classes of the difference hierarchy.

▶ **Theorem 3.8** (Hardness criterion)**.** *Let $X$ be countably-based. Let $\eta < \omega_1$ be a countable ordinal. For every set $A \subseteq X$, one has*

$$A \notin \underset{\sim}{\mathbf{D}}_\eta(X) \iff A \text{ is } \check{\underset{\sim}{\mathbf{D}}}_\eta\text{-hard}.$$

**Proof.** The implication $\Leftarrow$ holds on any space, we prove the other implication.

We prove by induction on $\eta$ that for any countably-based space $X$, if $C \in \underset{\sim}{\mathbf{D}}_\eta(\mathcal{N})$ and $H_{\eta+1}(A) \neq \emptyset$ then there exists a continuous reduction $\phi$ from $C$ to $A^c$ (or equivalently, from $C^c$ to $A$).

This induction hypothesis implies in particular that for any countably-based space $X$ and any open set $B \subseteq X$, if $B \cap H_{\eta+1}(A) \neq \emptyset$ then there exists a continuous reduction from $C$ to $A^c$ on any cylinder $[\sigma]$, with values in $B$. Indeed, in the subspace $B$, the set $H^B_{\eta+1}(A \cap B)$ (which is $H_{\eta+1}(A \cap B)$) defined in the space $B$) is the intersection of $H^X_{\eta+1}(A)$ with $B$.

The case $\eta = 0$ is straightforward: $C = \emptyset$ and $A \neq \emptyset$ so one can take a constant function with value in $A$.

Now assume the induction hypothesis for some $\eta$. Let $C \in \underset{\sim}{\mathbf{D}}_{\eta+1}(\mathcal{N})$ and $H_{\eta+2}(A) \neq \emptyset$. We build a continuous reduction $\phi$ from $C^c$ to $A$. We adopt the language of computability, by explaining how to compute $\phi(x)$ from $x \in \mathcal{N}$. Computing $\phi(x) \in X$ means enumerating the basic neighborhoods of $\phi(x)$. The computation is made relative to some suitable oracle encoding whatever is needed.

One has $C = U \setminus C'$ with $U$ open and $C' \in \underset{\sim}{\mathbf{D}}_\eta(\mathcal{N})$. First define $\phi : U^c \to X$ with some constant value in $A \cap H_{\eta+1}(A^c)$ which is non-empty. Given $x \in \mathcal{N}$, start computing $\phi(x)$ as if $x \in U^c$. If eventually one discovers that $x \in U$, then we obtain some $[\sigma] \subseteq U$ with $x \in [\sigma]$. So far, an open neighborhood $B$ of $\phi(x)$ has been enumerated. One has $B \cap H_{\eta+1}(A^c) \neq \emptyset$. By induction hypothesis applied to $\eta$, $[\sigma] \cap C'$ and $A^c$, one can define $\phi$ on $[\sigma]$ with values in $B$, reducing $[\sigma] \cap C'$ to $A \cap B$.

We now prove the case of limit ordinals. Let $\lambda$ be a limit ordinal, and assume the induction hypothesis for all $\eta < \lambda$.

Let $C \in \underset{\sim}{\mathbf{D}}_\lambda(\mathcal{N})$ and $H_{\lambda+1}(A) \neq \emptyset$. Let $(U_\alpha)_{\alpha<\lambda}$ be open sets defining $C$, and $U$ their union. We first define $\phi$ on $U^c$ with some constant value in $A \cap H_\lambda(A^c)$, which is non-empty. Given $x \in \mathcal{N}$, start computing $\phi(x)$ as if $x \in U^c$. If eventually one discovers that $x \in U$, then we obtain some $[\sigma] \subseteq U_\eta$ with $x \in [\sigma]$ and $\eta < \lambda$. So far, an open neighborhood $B$ of $\phi(x)$ has been enumerated. By Proposition 3.3, one has $B \cap H_\lambda(A) = B \cap H_\lambda(A^c) \neq \emptyset$ hence $B \cap H_{\eta+2}(A) \neq \emptyset$. As $[\sigma] \subseteq U_\eta$, one has $[\sigma] \cap C \in \underset{\sim}{\mathbf{D}}_{\eta+1}(\mathcal{N})$. By induction, one can define $\phi$ on $[\sigma]$ with values in $B$, reducing $[\sigma] \cap C$ to $A^c$. ◀

We leave open the question whether such the hardness criterion holds for classes above $\underset{\sim}{\mathbf{\Delta}}^0_2$ in countably-based spaces.

We will see later that in the space $\mathbb{R}[X]$ of polynomials, it already fails at the level $\underset{\sim}{\mathbf{D}}_2$.

Theorem 3.8 gives for free a characterization of the countably-based spaces on which the Hausdorff-Kuratowski Theorem holds. A set is said to be $\underset{\sim}{\mathbf{\Delta}}^0_2$-complete if it belongs to $\underset{\sim}{\mathbf{\Delta}}^0_2(X)$ and is $\underset{\sim}{\mathbf{\Delta}}^0_2$-hard.

▶ **Corollary 3.9.** *In a countably-based space, the Hausdorff-Kuratowski Theorem holds iff there is no $\underset{\sim}{\mathbf{\Delta}}^0_2$-complete set.*

**Proof.** If the HK theorem fails then there is $A \in \underset{\sim}{\mathbf{\Delta}}^0_2(X)$ such that $A \notin \underset{\sim}{\mathbf{D}}_\eta(X)$ for any $\eta < \omega_1$. By Theorem 3.8, $A$ is $\check{\underset{\sim}{\mathbf{D}}}_\eta$-hard for each $\eta$. As a result, $A$ is $\underset{\sim}{\mathbf{\Delta}}^0_2$-hard hence $\underset{\sim}{\mathbf{\Delta}}^0_2$-complete.

If the HK theorem holds then no $\underset{\sim}{\mathbf{\Delta}}^0_2$-set can be complete. Indeed, if $A \in \underset{\sim}{\mathbf{\Delta}}^0_2(X)$ then $A \in \underset{\sim}{\mathbf{D}}_\eta(X)$ for some $\eta < \omega_1$ by the HK theorem. If $A$ is $\underset{\sim}{\mathbf{\Delta}}^0_2$-hard then every $C \in \underset{\sim}{\mathbf{\Delta}}^0_2(\mathcal{N})$ is continuously reducible to $A$ so $C \in \underset{\sim}{\mathbf{D}}_\eta(\mathcal{N})$, contradicting the fact that the difference hierarchy does not collapse on $\mathcal{N}$. ◀

For instance, the space $\mathcal{C} = \{f : \mathbb{N} \to \mathbb{N} : f \text{ is eventually constant}\} \subseteq \mathcal{N}$ is countably-based and contains a $\underset{\sim}{\mathbf{\Delta}}^0_2$-complete set $\mathcal{C}_0 = \{f : \mathbb{N} \to \mathbb{N} : f \text{ is eventually null}\}$. Therefore, to show that a set is $\underset{\sim}{\mathbf{\Delta}}^0_2$-hard, it is sufficient to reduce $\mathcal{C}_0$ to that set. One may ask whether it is always possible. We answer positively in the case of countably-based spaces.

▶ **Proposition 3.10.** *In a countably-based space $X$, a set $A \subseteq X$ is $\underset{\sim}{\mathbf{\Delta}}^0_2$-hard iff there exists a continuous function $\phi : \mathcal{C} \to X$ such that $\mathcal{C}_0 = \phi^{-1}(A)$.*

**Proof.** If $A$ is $\underset{\sim}{\mathbf{\Delta}}^0_2$-hard, then we show that there is a non-empty closed set $F$ such that both $F \cap A$ and $F \setminus A$ are dense in $F$. From this result, we can build a reduction as follows. Given $f \in \mathcal{C}$, one can decide with finitely many mind-changes whether $f \in \mathcal{C}_0$. We can assume that the first guess is that $f \in \mathcal{C}_0$. We start outputting a point in $F \cap A$; each time we change our mind, if our new guess is that $f \notin \mathcal{C}_0$ then we move to a point in $F \setminus A$, and if our new guess is that $f \in \mathcal{C}_0$, then we move to a point in $F \cap A$. We can do that because both sets are dense in $F$, so whatever the current neighborhood of the point we have already output, we can move. After some finite time, there is no more mind-change, so we indeed output a point, which belongs to $A$ iff $f \in \mathcal{C}_0$.

Let us now prove the existence of such a $F$. For any $\eta < \omega_1$, one has $A \notin \underset{\sim}{\mathbf{D}}_\eta(X)$ (otherwise every set in $\underset{\sim}{\mathbf{\Delta}}^0_2(\mathcal{N})$ would belong to $\underset{\sim}{\mathbf{D}}_\eta(\mathcal{N})$, so the difference hierarchy would collapse on $\mathcal{N}$). By Theorem 3.6, one has $H_{\eta+1}(A) \neq \emptyset$ for all $\eta < \omega_1$. Let $\eta$ be such that $H_\alpha(A) = H_\eta(A)$ for all $\alpha \geq \eta$. Let $F = H_\eta(A)$. It is a closed set, and both $A$ and $A^c$ are dense in it. Indeed, $H_\eta(A) = H_{\eta+2}(A) \subseteq H_{\eta+1}(A^c) \subseteq H_\eta(A)$ so they are all equal. Therefore $H_\eta(A) = H_{\eta+1}(A^c) = \overline{A^c \cap H_\eta(A)}$, so $A^c$ is dense in $H_\eta(A)$, and $H_\eta(A) = H_{\eta+2}(A) = \overline{A \cap H_\eta(A)}$ so $A$ is dense in $H_\eta(A)$. ◀

# 4 Preservation of descriptive complexity by representations

## 4.1 Countably-based spaces

It is proved in [2] that in a countably-based space $X$ with a continuous open or admissible representation $\delta :\subseteq \mathcal{N} \to X$, the descriptive complexity of $\delta^{-1}(A)$ in $\mathrm{dom}(\delta)$ coincides with the descriptive complexity of $A$ in $X$. The proof is based on [11, 16] and we point out that it is actually effective. The effective version of the $\boldsymbol{\Sigma}^0_2$ case was proved in [6]. The classes $\mathrm{D}_m$ and $\Sigma^0_n(X)$ is the effective version of $\mathbf{D}_m$ and $\boldsymbol{\Sigma}^0_n(X)$.

▶ **Theorem 4.1** (Preservation). *Let $X$ be an effective countably-based space. The inverse of the function $\delta^{-1} : \mathbf{D}_m(\boldsymbol{\Sigma}^0_n(X)) \to \mathbf{D}_m(\boldsymbol{\Sigma}^0_n(\mathrm{dom}(\delta)))$ is computable. In particular, for every $A \subseteq X$,*

$$\delta^{-1}(A) \in \mathrm{D}_m(\Sigma^0_n(\mathrm{dom}(\delta))) \iff A \in \mathrm{D}_m(\Sigma^0_n(X)).$$

This result is used and generalized in [15]. We follow the lines of the proofs given in [11, 2]. For $A \subseteq \mathcal{N}$, let $B(A) = \{x \in X : A \cap \delta^{-1}(x) \text{ is not meager in } \delta^{-1}(x)\}$.

▶ **Lemma 4.2.** $B(\bigcup_{i\in\mathbb{N}} S_i) = \bigcup_{i\in\mathbb{N}} B(S_i)$.

**Proof.** $\delta^{-1}(x)$ is Polish so in that space, $\bigcup_i S_i$ is not meager iff some $S_i$ is not meager. ◀

▶ **Lemma 4.3.** *For any Borel set $S \subseteq \mathcal{N}$, one has $B(S) = \bigcup_\sigma \delta([\sigma]) \setminus B([\sigma] \setminus S)$.*

**Proof.** One has $x \in B(S)$ iff $S \cap \delta^{-1}(x)$ is not meager in $\delta^{-1}(x)$, iff there exists $\sigma$ such that $[\sigma]$ intersects $\delta^{-1}(x)$ (which means that $x \in \delta([\sigma])$) and $[\sigma] \cap S \cap \delta^{-1}(x)$ is comeager in $[\sigma] \cap \delta^{-1}(x)$. The latter property can be reformulated as follows:

$$\begin{aligned}
&[\sigma] \cap S \cap \delta^{-1}(x) \text{ is comeager in } [\sigma] \cap \delta^{-1}(x)\\
&\iff [\sigma] \setminus S \cap \delta^{-1}(x) \text{ is meager in } [\sigma] \cap \delta^{-1}(x)\\
&\iff [\sigma] \setminus S \cap \delta^{-1}(x) \text{ is meager in } \delta^{-1}(x)\\
&\iff x \notin B([\sigma] \setminus S).
\end{aligned}$$

As a result, $x \in B(S)$ iff there exists $\sigma$ such that $x \in \delta([\sigma]) \setminus B([\sigma] \setminus S)$. ◀

▶ **Lemma 4.4.** *Let $X$ be an effective countably-based space. The function $B : \boldsymbol{\Sigma}^0_n(\mathcal{N}) \to \boldsymbol{\Sigma}^0_n(X)$ is computable.*

**Proof.** We prove it by induction on $n$. For $n = 1$, if $A \in \boldsymbol{\Sigma}^0_1(\mathcal{N})$ then $B(A) = \delta(A)$ so the result follows because $\delta$ is effectively open.

Let $A \in \boldsymbol{\Sigma}^0_{n+1}(\mathcal{N})$. $A$ is given as $\bigcup_i A_i$ with $A_i \in \boldsymbol{\Pi}^0_n(\mathcal{N})$. By Lemmas 4.2 and 4.3,

$$B(A) = \bigcup_i B(A_i) = \bigcup_{i,\sigma} \delta([\sigma]) \setminus B([\sigma] \setminus A_i).$$

We can conclude by applying the induction hypothesis to $[\sigma] \setminus A_i \in \boldsymbol{\Sigma}^0_n(\mathcal{N})$. ◀

**Proof of Theorem 4.1.** One has $B(\delta^{-1}(A)) = A$ so the inverse of $\delta^{-1}$ is exactly $B$, which is computable by Lemma 4.4. It shows the case $m = 1$. For $m > 1$, one can show as in [2] that if $\delta^{-1}(A) = S \setminus T$ then $A = B(S) \setminus B(T)$, so the result follows by induction on $m$. ◀

The possibility of converting the description of the preimage of a set into a description of the set in a uniform continuous way is actually a characterization of countably-based spaces, as shown by the next result.

▶ **Theorem 4.5.** *Let $X$ be an admissibly represented space. The function $\delta^{-1} : \underset{\sim}{\mathbf{D}}_2(X) \to \underset{\sim}{\mathbf{D}}_2(\mathrm{dom}(\delta))$ is computably invertible relative to an oracle iff $X$ is countably-based.*

To prove the theorem, we need the next Lemma, whose proof is given in the appendix. In a topological space, the specialization preorder is $x \leq y$ if every neighborhood of $x$ is a neighborhood of $y$.

▶ **Lemma 4.6.** *For every admissibly represented space, there is an admissible representation $\delta$ such that the sets $\delta([\sigma])$ are upward closed for the specialization preorder.*

**Proof.** We use the following characterization of admissibly represented spaces [9]: the canonical injection $X \to \mathcal{O}(\mathcal{O}(X))$ sending $x$ to the set of its open neighborhoods has a continuously realizable inverse.

For any represented set $Y$, the canonical representation of $\mathcal{O}(Y)$ has the upward closedness property: the specialization preorder on $\mathcal{O}(Y)$ is inclusion and every prefix of a name of an open set $U$ can be extended to the name of any open set $V \supseteq U$: the prefix encodes a finite list of cylinders whose intersection with $\mathrm{dom}(\delta_Y)$ is contained in $\delta_Y^{-1}(U)$, hence in $\delta_Y^{-1}(V)$.

Now by taking $Y = \mathcal{O}(X)$ we get that the representation of $\mathcal{O}(\mathcal{O}(X))$ has this property. Therefore any subspace also has this property. In particular, as $X$ is admissibly represented, $X$ is a represented subspace of $\mathcal{O}(\mathcal{O}(X))$. ◀

**Proof of Theorem 4.5.** We can assume w.l.o.g. that $\delta$ has the upward closedness property as in Lemma 4.6. Indeed, $\delta$ is equivalent to such a representation $\delta_*$, and if $\delta^{-1}$ is continuously invertible as in the statement then so is $\delta_*^{-1}$. Indeed, one has $\delta = \delta_* \circ F$ for some continuous $F : \mathrm{dom}(\delta) \to \mathrm{dom}(\delta^*)$, the function $F^{-1} : \underset{\sim}{\mathbf{D}}_2(\mathrm{dom}(\delta_*)) \to \underset{\sim}{\mathbf{D}}_2(\mathrm{dom}(\delta))$ is continuous, so given $\delta_*^{-1}(A) \in \underset{\sim}{\mathbf{D}}_2(\mathrm{dom}(\delta_*))$ one can continuously obtain $\delta^{-1}(A) = F^{-1}(\delta_*^{-1}(A)) \in \underset{\sim}{\mathbf{D}}_2(\mathrm{dom}(\delta))$, from which one can continuously obtain $A \in \underset{\sim}{\mathbf{D}}_2(X)$.

Assume that $X$ is not countably-based. For each finite union of cylinders $C_i \subseteq \mathcal{N}$, the interior $B_i$ of $\delta(C_i)$ is an open subset of $X$. As $X$ is not countably-based, the sequence $(B_i)_{i\in\mathbb{N}}$ is not a basis, therefore there exists an open set $U \subseteq X$ which is not a union of $B_i$'s. It means that there exists $x \in U$ such that $x \notin B_i$ for each $B_i \subseteq U$.

Assume that the inverse $\phi$ of $\delta^{-1} : \underset{\sim}{\mathbf{D}}_2(X) \to \underset{\sim}{\mathbf{D}}_2(\mathrm{dom}(\delta))$ is continuously realizable, let $\Phi :\subseteq \mathcal{N} \to \mathcal{N}$ be a continuous realizer of $\phi$.

We build a set $A \in \underset{\sim}{\mathbf{D}}_2(X)$ by producing a name of $\delta^{-1}(A) \in \underset{\sim}{\mathbf{D}}_2(\mathrm{dom}(\delta))$, feeding it to $\Phi$ and observing its output, which must be a name of $A \in \underset{\sim}{\mathbf{D}}_2(X)$. In other words, we feed $\Phi$ with a pair of names of open sets $E_0, E_1 \subseteq \mathcal{N}$ such that $\delta^{-1}(A) = E_1 \setminus E_0 \cap \mathrm{dom}(\delta)$ and we observe names of open sets $F_0, F_1 \subseteq \mathcal{N}$ such that $A = A_1 \setminus A_0$ with $\delta^{-1}(A_i) = F_i \cap \mathrm{dom}(\delta)$. Our goal is to make $\Phi$ fail.

We pick a particular name $p$ of $x$. We start with $E_0 = \emptyset$ and $E_1 = \delta^{-1}(U)$, and start feeding names of them to $\Phi$. We wait for $p$ to appear in $F_1$ (which must happen, otherwise $A = U$ but $x \notin A_1 \setminus A_0$). When $p$ appears in $F_1$, we stop our enumeration of $E_1$, which is currently some $C_i$, and let $E_0 = C_i$. We start extending the names of $E_0$ and $E_1$ so that $E_0 = E_1 = C_i$, and wait for $p$ to appear in $F_0$ (it must happen, otherwise $A = \emptyset$ but $x \in A_1 \setminus A_0$). When $p$ appears in $F_0$, we do the following.

As $\delta(C_i) \subseteq U$, $x$ does not belong to the interior of $\delta(C_i)$. As $x$ belongs to the open set $A_0$, this open set cannot be contained in $\delta(C_i)$. As a result, there exists $y \in A_0 \setminus \delta(C_i)$, and we wait until we find a name of such a $y$ in $F_0$. The representation $\delta$ has the property that the image of any cylinder is upward closed, in particular $\delta(C_i)$ is upward closed. As a result, the closure of $\{y\}$, denoted by $\overline{\{y\}}$, is disjoint from $\delta(C_i)$ (indeed, $z \in \overline{\{y\}}$ iff $z \leq y$, so $z \notin \delta(C_i)$).

Now, we switch to $A = \overline{\{y\}}$, with $E_1 = \mathcal{N}$ and $E_0$ an open set such that $\mathrm{dom}(\delta) \cap E_0 = \mathrm{dom}(\delta) \setminus \delta^{-1}(\overline{\{y\}})$. It is indeed possible to find such an open set containing $C_i$, because $\delta(C_i)$ is disjoint from $\overline{\{y\}}$, and $C_i$ is the part of $E_0$ that has already been enumerated. The output of $\Phi$ is mistaken, because $y \in A$ but $y \notin A_1 \setminus A_0$ (we chose $y \in A_0$).

Therefore we get a contradiction, which implies that $\Phi$ cannot exist. ◀

## 5 CoPolish spaces

So far, we have essentially obtained positive results, in particular that the standard representation of countably-based spaces preserve descriptive complexity of sets.

We now investigate what happens on non-countably based spaces. It is however a very vast class, so we focus on one of the simplest classes of spaces, the coPolish spaces.

CoPolish spaces are a nice class of spaces going beyond countably-based spaces. They were studied in [12] where they arise as a natural class of spaces on which complexity theory can be developed. We start by showing that in coPolish spaces, the representation does not always preserve descriptive complexity. Moreover, we give a simple characterization of the coPolish spaces whose representation preserves the descriptive complexity of sets (at least for low level complexity classes). We take the next definition from [3].

▶ **Definition 5.1.** *A* **coPolish space** *is a direct limit of an increasing sequence of compact metrizable subspaces $X_n$.*

In other words $X = \bigcup_n X_n$ and a set $U \subseteq X$ is open if for each $n$, $U \cap X_n$ is open on $X_n$. In this topology, a converging sequence is entirely contained in some $X_n$ [12]. An admissible representation $\delta_X$ is obtained as follows: a point $x \in X$ is represented by a pair $(n, p) \in \mathbb{N} \times \mathcal{N} \cong \mathcal{N}$ where $n \in \mathbb{N}$ is such that $x \in X_n$, and $p \in \mathcal{N}$ is a name of $x$ in $X_n$.

▶ Remark 5.2. For a descriptive complexity class $\Gamma$ in the Borel and difference hierarchies, one has

$$\delta_X^{-1}(A) \in \Gamma(\mathrm{dom}(\delta_X)) \iff \forall n, A \cap X_n \in \Gamma(X_n).$$

Indeed, $\delta_X^{-1}(A)$ is the disjoint union over $n \in \mathbb{N}$ of $[n] \cap \delta_X^{-1}(A)$, so $\delta_X^{-1}(A)$ belongs to a class iff each member of the disjoint union belongs to that class. Now observe that on $[n]$, $\delta_X$ is simply $\delta_{X_n}$. We conclude by observing that as $X_n$ is countably-based, the complexity of $A \cap X_n$ is the same as the complexity of its preimage under $\delta_{X_n}$.

We recall that a topological space is Fréchet-Urysohn if the closure of each set is the set of limits of sequences of points in the set.

▶ **Theorem 5.3.** *For a coPolish space $X$, the following statements are equivalent:*
- *$X$ is Fréchet-Urysohn,*
- *For every $A \subseteq X$, $\delta_X^{-1}(A) \in \mathbf{D}_2(\mathrm{dom}(\delta_X))$ implies $A \in \mathbf{D}_2(X)$,*
- *For every $A \subseteq X$ and every $n < \omega$, $\delta_X^{-1}(A) \in \mathbf{D}_n(\mathrm{dom}(\delta_X))$ implies $A \in \mathbf{D}_n(X)$.*

We need the following results.

▶ **Lemma 5.4.** *Let $X$ be a Fréchet-Urysohn coPolish space. If a sequence $x_i$ converges to some $x$, with $x_i \neq x$ for all $i$, then there exists $p$ such that $x_i \in \mathrm{int}(X_p)$ for almost all $i$.*

**Proof.** Assume that for each $p$, there exist infinitely many $i$ such that $x_i \notin \mathrm{int}(X_p)$. We can extract a subsequence $x_{i_p} \notin \mathrm{int}(X_p)$ with $i_p < i_{p+1}$. Let $U_p$ be a neighborhood of $x_{i_p}$ such that $x \notin \overline{U_p}$ (it exists as $x_{i_p} \neq x$ and $X$ is Hausdorff). Let $U = \bigcup_p U_p \setminus X_p$. One has $x_{i_p} \in \overline{U}$

for all $p$, so $x \in \overline{U}$. As the space if Fréchet-Urysohn, there exists $p_0$ such that $x \in \overline{U \cap X_{p_0}}$ (indeed, there exists a sequence in $U$ converging to $x$, and that sequence must belong to some $X_{p_0}$). However, $U \cap X_{p_0} \subseteq \bigcup_{p<p_0} U_p \setminus X_p$ so its closure does not contain $x$, giving a contradiction. ◀

Let $H_n^{X_p}(A \cap X_p)$ be the set $H_n(A \cap X_p)$, but defined in the space $X_p$.

▶ **Lemma 5.5.** *If $X$ is a Fréchet-Urysohn coPolish space, then $H_n(A) = \bigcup_p H_n^{X_p}(A \cap X_p)$.*

**Proof.** We prove it by induction on $n$. For $n = 0$ the result is clear. Assume the result for $n \in \mathbb{N}$.

Let $x \in H_{n+1}(A) = \overline{A \cap H_n(A^c)}$. There exists a sequence $x_i \in A \cap H_n(A^c)$ converging to $x$. If $x = x_i$ for some $i$, then $x \in A \cap H_n(A^c)$ so $x \in A \cap H_n^{X_p}(A^c \cap X_p)$ for some $p$ by induction hypothesis, hence $x \in H_{n+1}^{X_p}(A \cap X_p)$. If $x \neq x_i$ for all $i$, then by Lemma 5.4 there exists $p$ such that $x_i \in \text{int}(X_p)$ for almost all $i$. As a result, using Lemma 3.5 we see that $x_i \in H_n^{X_p}(A^c \cap X_p)$ for almost all $i$. We can take $p$ large enough so that $x \in X_p$, and obtain that $x \in H_{n+1}^{X_p}(A \cap X_p)$. ◀

**Proof of Theorem 5.3.** Let $X$ be Fréchet-Urysohn. If $\delta^{-1}(A) \in \underset{\sim}{\mathbf{D}}_n(\text{dom}(\delta_X))$ then for each $p \in \mathbb{N}$, $A \cap X_p \in \underset{\sim}{\mathbf{D}}_n(X_p)$ by Remark 5.2 so $H_{n+1}^{X_p}(A \cap X_p) = \emptyset$. As a result, Lemma 5.5 implies that $H_{n+1}(A) = \emptyset$ which implies that $A \in \underset{\sim}{\mathbf{D}}_n(X)$.

If $X$ is not Fréchet-Urysohn, then there exists a double-sequence $x_{n,p} \in X$ such that for each $n$, $x_{n,p}$ converges to some $x_n$ as $p \to \infty$, which in turn converges to some $x$ as $n \to \infty$, with no sequence in $\{x_{n,p} : n, p \in \mathbb{N}\}$ converging to $x$, and such that $C = \{x\} \cup \{x_n : n \in \mathbb{N}\} \cup \{x_{n,p} : n, p \in \mathbb{N}\}$ is closed (it was proved in [3], Proposition 66). Let $A := \{x\} \cup \{x_{n,p} : n, p \in \mathbb{N}\}$. One easily checks that $H_3(A) = \{x\} \neq \emptyset$ so $A \notin \underset{\sim}{\mathbf{D}}_2(X)$. However, $\delta_X^{-1}(A) \in \underset{\sim}{\mathbf{D}}_2(\text{dom}(\delta_X))$. For each $i$, the set $\{n : \exists p, x_{n,p} \in A \cap X_i\}$ is finite. Therefore, one can easily see that $A \cap X_i \in \underset{\sim}{\mathbf{D}}_2(X_i)$. It implies that $\delta_X^{-1}(A) \in \underset{\sim}{\mathbf{D}}_2(\text{dom}(\delta_X))$. ◀

The result does not extend to higher complexity levels. The space $\mathbb{R}/\mathbb{Z}$ is coPolish and Fréchet-Urysohn, but the representation does not preserve the level $\omega$ of the difference hierarchy.

▶ **Proposition 5.6.** *There exists $A \subseteq \mathbb{R}/\mathbb{Z}$ such that $\delta^{-1}(A) \in \underset{\sim}{\mathbf{D}}_\omega(\text{dom}(\delta))$ but $A \notin \underset{\sim}{\mathbf{D}}_\omega(\mathbb{R}/\mathbb{Z})$.*

**Proof.** The space $X = \mathbb{R}/\mathbb{Z}$ is the direct limit of $X_n = [-n, n]/\mathbb{Z}$. For each $n$, let $A_n \subseteq [n, n + 1/2]$ be such that $A_n \in \underset{\sim}{\mathbf{D}}_{n+1} \setminus \underset{\sim}{\mathbf{D}}_n$, with $n \in H_{n+1}(A_n)$. Let $A$ be the quotient of $\bigcup_n A_n$. For each $n$, one has $A \cap X_n \in \underset{\sim}{\mathbf{D}}_{n+1}(X_n) \subseteq \underset{\sim}{\mathbf{D}}_\omega(X_n)$, so $\delta^{-1}(A) \in \underset{\sim}{\mathbf{D}}_\omega(\text{dom}(\delta))$.

However, $A \notin \underset{\sim}{\mathbf{D}}_\omega(X)$ because $0 \in H_{\omega+1}(A) \neq \emptyset$. Indeed, $0 \in H_{n+1}(A) \subseteq H_n(A^c)$ for all $n$, so $0 \in A \cap H_\omega(A^c) \subseteq H_{\omega+1}(A)$. ◀

On coPolish spaces, the representation may not preserve low complexity classes. However, it always preserves classes $\underset{\sim}{\mathbf{\Delta}}_2^0$ and above.

▶ **Proposition 5.7.** *Let $X$ be coPolish. For each $\alpha \geq 1$ and $A \subseteq X$, one has*

$$\delta_X^{-1}(A) \in \underset{\sim}{\mathbf{\Sigma}}_\alpha^0(\text{dom}(\delta_X)) \iff A \in \underset{\sim}{\mathbf{\Sigma}}_\alpha^0(X).$$

**Proof.** For $\alpha = 1$, it follows from the admissibility of $\delta_X$. Let $\alpha \geq 2$. As observed earlier, one has $\delta_X^{-1}(A) \in \underset{\sim}{\mathbf{\Sigma}}_\alpha^0(\text{dom}(\delta_X)) \iff \forall n, A \cap X_n \in \underset{\sim}{\mathbf{\Sigma}}_\alpha^0(X_n)$. As $X_n \in \underset{\sim}{\mathbf{\Pi}}_1^0(X) \subseteq \underset{\sim}{\mathbf{\Sigma}}_\alpha^0(X)$, it implies that $A \cap X_n \in \underset{\sim}{\mathbf{\Sigma}}_\alpha^0(X)$. Therefore, $A = \bigcup_n A \cup X_n \in \underset{\sim}{\mathbf{\Sigma}}_\alpha^0(X)$. ◀

In particular, if $\delta_X^{-1}(A) \in \mathbf{D}_2(\mathrm{dom}(\delta_X))$ then $A \in \mathbf{\Delta}_2^0(X)$. We show that this gap cannot be reduced in the space of polynomials $\mathbb{R}[X]$. This space is obtained as the direct limit of the spaces $\mathbb{R}^n$, consisting of the polynomials of degree $\leq n$, identified with their coefficients. A polynomial is then represented by giving any upper bound on its degree, and the finite list of its coefficients.

▶ **Theorem 5.8.** *There exists a set $A \subseteq \mathbb{R}[X]$ such that $\delta^{-1}(A) \in \mathrm{D}_2(\mathrm{dom}(\delta))$ but $A \notin \mathbf{D}_\alpha(\mathbb{R}[X])$ for any $\alpha < \omega_1$.*

**Proof.** Let

$$A = \left\{ \frac{1}{k_1} + \frac{1}{k_2}X^{k_1} + \ldots + \frac{1}{k_{n+1}}X^{k_n} : k_1 < k_2 < \ldots < k_{n+1}, k_n \text{ even} \right\}.$$

First, the closure of $A$ can be easily obtained by taking the definition of $A$ without the evenness condition on $n$. Inside $\overline{A}$, the complement of $A$ is dense. No set of the difference hierarchy can be dense and with a dense complement.

However, we now show that $A \cap X_d \in \mathrm{D}_2(\mathbb{R}^d)$. Let $P = \sum_{i \leq d} p_i X^i \in \overline{A}$ be given by a name (in particular, we know $d$). One can compute the degree of $P$ with one mind-change. Indeed, all the non-null coefficients except the dominant one must be at least $\frac{1}{d}$. So for each $i \leq d$, we test in parallel whether $p_i < \frac{1}{d}$ and whether $p_i > 0$, and wait that one of them stops (which must happen). Let $i \leq d$ be maximal such that the test $p_i > 0$ stops first. That $i$ is our current guess for the degree of $P$. We then start testing $p_j > 0$ for all $i < j \leq d$. If such a $j$ is eventually found, then it is the degree of $P$.

We now show how to decide whether $P \in A$ with at most two mind changes, starting with rejection. We start rejecting $P$. If $i$ is even then we change our mind and accept $P$. If we eventually realize that the degree of $P$ is some $j > i$, then if $j$ is even, we accept $P$, otherwise we reject $P$.

Now, given $P \in \mathbb{R}^d$, we run the previous algorithm and in parallel test whether $P \notin \overline{A}$. If the latter condition is eventually found true, then we stop the algorithm and definitely reject $P$. ◀

This example has several consequences, showing that many results working on countably-based spaces fail on $\mathbb{R}[X]$. First, the hardness criterion (1), which can be extended to countably-based spaces (Theorem 3.8) does not hold on the space of polynomials.

▶ **Corollary 5.9.** *In $\mathbb{R}[X]$, there exists a set $A \notin \mathbf{D}_2(X)$ that is not $\check{\mathbf{D}}_2$-hard.*

**Proof.** We take the set $A \notin \mathbf{D}_2(\mathbb{R}[X])$ with $\delta^{-1}(A) \in \mathbf{D}_2(\mathrm{dom}(\delta))$. Take some $C \in \check{\mathbf{D}}_2(\mathcal{N}) \setminus \mathbf{D}_2(\mathcal{N})$. A continuous reduction $\phi : \mathcal{N} \to X$ from $C$ to $A$ is continuously realizable because the representation is admissible. Any continuous realizer is a continuous reduction from $C$ to $\delta^{-1}(A)$, which implies that $C \in \mathbf{D}_2(\mathcal{N})$, contradicting the choice of $C$. Hence $C$ is not continuously reducible to $A$. ◀

According to Corollary 3.9, the Hausdorff-Kuratowski Theorem holds on a countably-based space iff it contains no $\mathbf{\Delta}_2^0$-complete set. We show that this characterization fails on $\mathbb{R}[X]$.

▶ **Proposition 5.10.** *The Hausdorff-Kuratowski Theorem fails in the topological space $\mathbb{R}[X]$.*

**Proof.** Theorem 5.8 provides a set $A$ such that $\delta^{-1}(A) \in \mathbf{D}_2(\mathrm{dom}(\delta))$, hence $A \in \mathbf{\Delta}_2^0(\mathbb{R}[X])$, but $A \notin \mathbf{D}_\eta(\mathbb{R}[X])$ for any $\eta < \omega_1$. ◀

▶ **Proposition 5.11.** *A space with a total admissible representation has no $\mathbf{\Delta}_2^0$-complete set.*

**Proof.** If $A \in \underset{\sim}{\mathbf{\Delta}}^0_2(X)$ then $\delta^{-1}(A) \in \underset{\sim}{\mathbf{\Delta}}^0_2(\mathcal{N})$. If $A$ is $\underset{\sim}{\mathbf{\Delta}}^0_2$-hard, then $\delta^{-1}(A)$ is $\underset{\sim}{\mathbf{\Delta}}^0_2$-hard by admissibility. As there is no $\underset{\sim}{\mathbf{\Delta}}^0_2$-complete set in $\mathcal{N}$, there is no $\underset{\sim}{\mathbf{\Delta}}^0_2$-complete set in $X$. ◀

Note that every coPolish space, in particular $\mathbb{R}[X]$, has a total admissible representation.

## 5.1 Complexity via representation

As we have just seen, several results from descriptive set theory fail in $\mathbb{R}[X]$. However, when the complexity of a set is measured using the representation, these results can be recovered. The next results apply to any space with a total admissible representation (investigated in [14]), in particular to any coPolish space. First, the hardness criterion (1) can be recovered by measuring the complexity of a set via the representation.

▶ **Proposition 5.12.** *Let $(X, \delta_X)$ be an admissibly represented space with $\delta_X$ total. Let $\Gamma$ be a class of the Borel or difference hierarchies that is non-self-dual in $\mathcal{N}$. For every $A \subseteq X$,*

$$\delta_X^{-1}(A) \notin \Gamma(\mathcal{N}) \iff A \text{ is } \check{\Gamma}\text{-hard.}$$

**Proof.** As $\delta_X$ is admissible, any continuous reduction from some $C \in \check{\Gamma}(\mathcal{N})$ to $A$ has a continuous realizer, which is a continuous reduction from $C$ to $\delta_X^{-1}(A)$. As a result, $A$ is $\check{\Gamma}$-hard iff $\delta_X^{-1}(A)$ is $\check{\Gamma}$-hard iff $\delta_X^{-1}(A) \notin \Gamma(\mathcal{N})$. ◀

When the admissible representation is not total, we still have the equivalence for low complexity classes.

▶ **Proposition 5.13.** *Let $(X, \delta_X)$ be an admissibly represented space. Let $\Gamma = \underset{\sim}{\mathbf{D}}_\eta(X)$ for some $\eta < \omega_1$. For every $A \subseteq X$, one has*

$$\delta_X^{-1}(A) \notin \Gamma(\mathrm{dom}(\delta_X)) \iff A \text{ is } \check{\Gamma}\text{-hard.}$$

**Proof.** Again, admissibility of $\delta_X$ implies that $A$ is $\check{\Gamma}$-hard iff $\delta_X^{-1}(A)$, as a subset of the space $\mathrm{dom}(\delta_X)$, is $\check{\Gamma}$-hard. As $\mathrm{dom}(\delta_X)$ is a countably-based space (it is a subspace of $\mathcal{N}$), $\delta_X^{-1}(A)$ is $\check{\Gamma}$-hard there iff $\delta_X^{-1}(A) \notin \Gamma(\mathrm{dom}(\delta_X))$ by Theorem 3.8. ◀

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## A Proofs in Section 3.1

**Proof of Proposition 3.2.** If $\beta$ is a limit ordinal then by definition of $H_\beta(A)$, if $\alpha < \beta$ then $H_\beta(A) \subseteq H_\alpha(A)$.

Let $\lambda$ be a limit ordinal. We prove by induction on $n \in \mathbb{N}$ that

$$\text{for all } A, H_{\lambda+n+1}(A) \subseteq H_{\lambda+n}(A). \tag{2}$$

We first prove it for $n = 0$. One has $H_{\lambda+1}(A) = \overline{A \cap H_\lambda(A^c)} \subseteq H_\lambda(A^c)$. If $\eta < \lambda$ then $H_\lambda(A^c) \subseteq H_{\eta+1}(A^c) \subseteq H_\eta(A)$ by definition, so $H_{\lambda+1}(A) \subseteq \bigcap_{\eta<\lambda} H_\eta(A) = H_\lambda(A)$.

Assuming (2) for $n \in \mathbb{N}$, we apply it to $A^c$ and obtain $H_{\lambda+n+2}(A) = \overline{A \cap H_{\lambda+n+1}(A^c)} \subseteq \overline{A \cap H_{\lambda+n}(A^c)} = H_{\lambda+n+1}(A)$. ◀

**Proof of Proposition 3.3.** One has

$$\begin{aligned} H_\lambda(A) = \bigcap_{\eta<\lambda} H_\eta(A) &= \bigcap_{\eta<\lambda} H_{\eta+1}(A) \\ &\subseteq \bigcap_{\eta<\lambda} H_\eta(A^c) = H_\lambda(A^c), \end{aligned}$$

and the other inclusion is obtained by exchanging $A$ and $A^c$. ◀

**Proof of Lemma 3.5.** First observe that for an open set $U$ and any set $S$,

$$\overline{S \cap U} \cap U = \overline{S} \cap U. \tag{3}$$

We show by induction on $\eta$ that

$$\forall A, B, A \cap U = B \cap U \text{ implies } H_\eta(A) \cap U = H_\eta(B) \cap U. \tag{4}$$

For $\eta = 0$, one has $H_\eta(A) = H_\eta(B) = X$.

Assume (4) for $\eta$. Let $A, B$ satisfy $A \cap U = B \cap U$. One has $A^c \cap U = B^c \cap U$ so $H_\eta(A^c) \cap U = H_\eta(B^c) \cap U$ by induction hypothesis. Therefore, $A \cap H_\eta(A^c) \cap U = B \cap H_\eta(B^c) \cap U$ so

$$\begin{aligned} H_{\eta+1}(A) \cap U &= \overline{A \cap H_\eta(A^c)} \cap U, \\ &= \overline{A \cap H_\eta(A^c) \cap U} \cap U && \text{by (3)} \\ &= \overline{B \cap H_\eta(B^c) \cap U} \cap U \\ &= H_{\eta+1}(B) \cap U. \end{aligned}$$

If $\lambda$ is a limit ordinal, assuming (4) for all $\eta < \lambda$, one has $H_\lambda(A) \cap U = \bigcap_{\eta<\lambda} H_\eta(A) \cap U = \bigcap_{\eta<\lambda} H_\eta(B) \cap U = H_\lambda(B) \cap U$. ◀

## B Proof of Theorem 3.6

We first prove Lemma 3.7, which states that for any set $A \subseteq X$, one has $A \setminus H_{\eta+1}(A) \in \underset{\sim}{\mathbf{D}}_\eta(X)$.

**Proof of Lemma 3.7.** First observe that because $H_\eta(A^c)$ is closed, one has

$$A \cap H_{\eta+1}(A) = A \cap H_\eta(A^c), \tag{5}$$
$$A \setminus H_{\eta+1}(A) = A \setminus H_\eta(A^c). \tag{6}$$

We prove the statement by induction on $\eta$. Assume the result for some $\eta$ and every $A$. Let $A \subseteq X$. One has

$$\begin{aligned} A \setminus H_{\eta+2}(A) &= A \setminus H_{\eta+1}(A^c) \\ &= H_{\eta+1}(A^c)^c \setminus A^c \\ &= H_{\eta+1}(A^c)^c \setminus (A^c \setminus H_{\eta+1}(A^c)) \\ &= U \setminus C, \end{aligned}$$

where $U = H_{\eta+1}(A^c)^c$ is open and $C = A^c \setminus H_{\eta+1}(A^c) \in \underset{\sim}{\mathbf{D}}_\eta(X)$ by induction hypothesis. As $C \subseteq U$, one has $A = U \setminus C \in \underset{\sim}{\mathbf{D}}_{\eta+1}(X)$, which is what we wanted to prove.

The case of limit ordinals $\lambda$ is proved without the induction hypothesis.

▷ Claim B.1. One has

$$A \setminus H_{\lambda+1}(A) = \bigcup_{\eta<\lambda,\text{even}} H_\eta(A) \setminus H_{\eta+1}(A^c).$$

The claim implies that $A \setminus H_{\lambda+1}(A) \in \underset{\sim}{\mathbf{D}}_\lambda(X)$: let $B_\eta = H_\eta(A)^c$ if $\eta$ is even, $B_\eta = H_\eta(A^c)$ is $\eta$ is odd, so the right-hand side in the claim equality can be rewritten as $\bigcup_{\eta<\lambda,\text{even}} B_{\eta+1} \setminus B_\eta$, which fits the definition of $\underset{\sim}{\mathbf{D}}_\lambda(X)$. We now prove the claim.

$$\begin{aligned} A \setminus H_{\lambda+1}(A) &= A \setminus H_\lambda(A^c) \\ &= A \setminus H_\lambda(A) \\ &= \bigcup_{\eta<\lambda,\text{even}} A \setminus H_\eta(A) \\ &= \bigcup_{\eta<\lambda,\text{even}} A \setminus H_{\eta+2}(A) \setminus (A \setminus H_\eta(A)) \\ &= \bigcup_{\eta<\lambda,\text{even}} A \cap H_\eta(A) \setminus H_{\eta+2}(A). \end{aligned}$$

Now,

$$\begin{aligned} A\cap H_\eta(A)\setminus H_{\eta+2}(A) &= A\cap H_\eta(A)\setminus H_{\eta+1}(A^c)\\ &= A\cap H_\eta(A)\setminus \overline{A^c\cap H_\eta(A)}\\ &= H_\eta(A)\setminus H_{\eta+1}(A^c). \end{aligned}$$

◀

We now prove the other direction in Theorem 3.6.

▶ **Lemma B.2.** *If $A\in \underset{\sim}{\mathbf{D}}_\eta(X)$ then $H_{\eta+1}(A)=\emptyset$.*

**Proof.** Assume the result for $\eta$. Let $B\in \underset{\sim}{\mathbf{D}}_{\eta+1}(X)$. One has $B=U\setminus A$ for some open set $U$ and some $A\in \underset{\sim}{\mathbf{D}}_\eta(X)$ with $A\subseteq U$. One has

$$\begin{aligned} H_{\eta+2}(B) &= \overline{U\setminus A\cap H_{\eta+1}(A\cup U^c)}\\ &= \overline{U\setminus A\cap H_{\eta+1}(A)} \quad \text{by Lemma 3.5}\\ &= \emptyset \quad \text{as } H_{\eta+1}(A)=\emptyset \text{ by induction.} \end{aligned}$$

If $A\in \underset{\sim}{\mathbf{D}}_\lambda(X)$ then let $(A_\eta)_{\eta<\lambda}$ be an increasing sequence of open sets such that $A=\bigcup_{\eta<\lambda,\text{even}} A_{\eta+1}\setminus A_\eta$.

It is not hard to see that for each $\eta<\lambda$, one has $A^c\cap A_\eta\in \underset{\sim}{\mathbf{D}}_{\eta+1}(X)$. By induction hypothesis, $H_{\eta+2}(A^c\cap A_\eta)=\emptyset$, so $H_{\eta+2}(A^c)\cap A_\eta=\emptyset$ by Lemma 3.5. As a result, $H_\lambda(A^c)\cap A_\eta=\emptyset$ for each $\eta<\lambda$. As $A\subseteq\bigcup_{\eta<\lambda}A_\eta$, one has $H_\lambda(A^c)\cap A=\emptyset$, so $H_{\lambda+1}(A)=\overline{A\cap H_\lambda(A^c)}=\emptyset$. ◀