Miroslav Fiedler; Vlastimil Pták

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ON MATRICES WITH NON-POSITIVE OFF-DIAGONAL ELEMENTS AND POSITIVE PRINCIPAL MINORS

MIROSLAV FIEDLER and VLASTIMIL PTÁK, Praha

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The authors study a class of matrices which occur frequently in applications to convergence properties of iteration processes in linear algebra and spectral theory of matrices.

1. INTRODUCTION

In many investigations concerning the convergence of iteration processes in linear algebra and spectral properties of matrices the idea of considering matrices of the type

$$\begin{pmatrix} a_{11}, & -a_{12}, & -a_{13}, & \dots \\ -a_{21}, & a_{22}, & -a_{23}, & \dots \\ -a_{31}, & -a_{32}, & a_{33}, & \dots \end{pmatrix}$$

with nonnegative a_{ik} suggests itself in quite a natural way. Matrices of this type have been extensively studied especially by A. OSTROWSKI, K. FAN, D. M. KOTELJANSKIJ and others.

It appears that the matrices investigated here play an important rôle in regularity criteria and estimates for spectra of matrices. It is our opinion that the theorems of the present paper give a deeper insight into some earlier results on regularity conditions as well as a possibility of obtaining new results; this will form the subject of a further paper.

During the authors' work in the past five years many results appeared essentially as consequences of theorems on matrices of this type. It appears useful to collect these theorems in a separate paper which is intended as a basis for further applications. In this manner the authors intend to avoid repetitions in further communicions and hope to present a unified account of the properties of this particular class of matrices. In the mean time, some of these properties have been obtained independently by other authors.

The paper is divided into seven sections, the first being introductory. The second explains the terminology and notation. In the third section, we investigate the properties of matrices whose principal minors are positive. Section four is devoted to the investigation of properties of a class of matrices called **K**.[1]) Theorem (4,2) which shows the equivalence of several properties used to define the class **K** is essentially known and is included only for the sake of completeness. In section five, we study a wider class of matrices which may be considered as the closure of **K**. In the sixth section we collect several criteria for a matrix to belong to **K**. These criteria have important applications in the study of regions in the complex plane which contain the spectrum of a given matrix. In section seven we prove some general inequalities for matrices of class **K**. These inequalities contain as special cases some known properties of minors.

2. NOTATION

In the whole paper n will be a fixed positive integer. The set of indices $1, 2, \ldots, n$ will be denoted by N. A (square) matrix is a real function on $P \times P$, where P is some index set. If A is a matrix, we shall denote by a_{ik} the value of A at the point (i, k). The transpose of a matrix A will be denoted by A^T. The determinant of a matrix A will be denoted by det A. If $M \subset N$ and if A is a matrix on $N \times N$, we denote by $A(M)$ the partial function of A on $M \times M$. We shall call it the principal submatrix of A corresponding to M. The number det $A(M)$ is called the principal minor of A corresponding to M.

A matrix A is said to be reducible if there exists a nonvoid $P \subset N$, $P \neq N$, such that $a_{ik} = 0$ for $i \in P$ and $k \in N - P$. A matrix is irreducible if it is not reducible. A matrix A is said to be nonnegative or $A \geqq 0$ if $a_{ik} \geqq 0$ for each $i, k \in N$. If $a_{ik} > 0$ for each $i, k \in N$ we say that A is positive or $A > 0$. A vector is a real function on N. We write $x \geqq 0$ if $x_i \geqq 0$ for each $i \in N$ and $x > 0$ if $x_i > 0$ for each $i \in N$. If A and B are two matrices we shall write $B \geqq A$ for $B - A \geqq 0$. We shall frequently use the following important theorem due to Perron and Frobenius.

(2,1) *Let A be a nonnegative matrix. Then there exists a proper value $p(A)$ of A, the "Perron root of A", such that $p(A) \geqq 0$ and $|\lambda| \leqq p(A)$ for every proper value λ of A. If $0 \leqq A \leqq B$ then $p(A) \leqq p(B)$. Moreover, if A is irreducible, the Perron root $p(A)$ is positive, simple and the corresponding proper vector may be chosen positive.*

Let A be a matrix. The "spectral radius" $\sigma(A)$ of A is defined as the maximum of the moduli $|\lambda|$ of all proper values λ of A. According to the Perron-Frobenius theorem, we have $\sigma(A) = p(A)$ for nonnegative matrices.

(2,2) Definition. *A matrix W is said to have dominant principal diagonal if $|w_{ii}| > \sum_{k \neq i} |w_{ik}|$ for each $i \in N$.*

(2.3) *If W is a matrix with dominant principal diagonal, then $\sigma(E - H^{-1}W) < 1$ where H is the diagonal of W.*

[1]) These matrices are sometimes called M-matrices.

Proof. Let λ be a proper value of $E - H^{-1}W$. Then there exists a vector $x \neq 0$ such that $\lambda x = x - H^{-1}Wx$. Take $i \in N$ such that $|x_i| = \max_{j \in N} |x_j| > 0$. Then

$$\lambda x_i = \sum_{j \neq i} (w_{ij}/w_{ii}) x_j \quad \text{whence} \quad |\lambda| |x_i| \leqq (\sum_{j \neq i} |w_{ij}|/|w_{ii}|) \max_{j \in N} |x_j| < |x_i| .$$

It follows that $|\lambda| < 1$ and since λ was an arbitrary proper value of $E - H^{-1}W$, we have $\sigma(E - H^{-1}W) < 1$.

In the main text, we introduce several classes of matrices: $\boldsymbol{P}$, $\boldsymbol{Z}$, $\boldsymbol{K}$ and $\boldsymbol{K}_0$. Their definitions are contained in (3,4), (4,1), (4,4) and (5,2) respectively.

3. POSITIVITY OF PRINCIPAL MINORS

This section has an auxiliary character. We prove two equivalences which show that some of the properties investigated later depend on properties of principal minors only.

(3,1) *The following two properties of a matrix A are equivalent:*

1° *the sequence of principal minors $A(M_i)$ is positive (here M_i denotes the set consisting of the indices* $1, 2, \ldots, i$);

2° *there exists a lower triangular matrix T_1 and an upper triangular matrix T_2 both with positive diagonal elements such that $A = T_1 T_2$.*

Proof. The step from 1° to 2° will proceed by induction. The case $n = 1$ being trivial, let $n > 1$ and suppose that this implication holds for all matrices of order smaller than n. Let

$$A = \begin{pmatrix} A_{n-1}, & a \\ b & , a_{nn} \end{pmatrix}$$

be a matrix of order n fulfilling condition 1°. Then $A_{n-1} = \tilde{T}_1 \tilde{T}_2$ where $\tilde{T}_1(\tilde{T}_2)$ is a lower (upper) triangular matrix with positive diagonal elements. From the relation

$$a_{nn} - bA_{n-1}^{-1} a = \det A / \det A_{n-1} > 0$$

it follows that the matrices

$$T_1 = \begin{pmatrix} \tilde{T}_1 & , 0 \\ b\tilde{T}_2^{-1}, & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} \tilde{T}_2, & \tilde{T}_1^{-1} a \\ 0 & , a_{nn} - bA_{n-1}^{-1} a \end{pmatrix}$$

fulfil condition 2°.

The other implication is obvious since the principal minors $\det A(M_i)$ are equal to the product of the first i diagonal elements of T_1 and T_2.

The following lemma (3,2) will not be used until section four.

(3,2) *Let a matrix $A = (a_{ij})$, $i, j \in N$, fulfil the conditions $a_{ij} \leqq 0$ for $i \neq j$, $i, j \in N$, and let $A = T_1 T_2$ where $T_1(T_2)$ is a lower (upper) triangular matrix with positive diagonal elements. Then the off-diagonal elements of both T_1 and T_2 are non-positive.*

Proof. Let $T_1 = (r_{ij})$, $T_2 = (s_{ij})$, so that $r_{ij} = 0$ for $i < j$, $s_{ij} = 0$ for $i > j$, $r_{ii} > 0$, $s_{ii} > 0$ $(i, j \in N)$. We shall prove the inequalities $r_{ij} \leqq 0$, $s_{ij} \leqq 0$ $(i \neq j)$ by induction with respect to $i + j$. If $i + j = 3$, the inequalities $r_{21} \leqq 0$ and $s_{12} \leqq 0$ follow from $a_{12} = r_{11}s_{12}$ and $a_{21} = r_{21}s_{11}$. Let $i + j > 3$, $i \neq j$, and suppose the inequalities $r_{kl} \leqq 0$ and $s_{kl} \leqq 0$ $(k \neq l)$ valid if $k + l < i + j$. Then, if $i < j$, in the relation

$$a_{ij} = r_{ii}s_{ij} + \sum_{k<i} r_{ik}s_{kj}$$

we have $a_{ij} \leqq 0$, $\sum_{k<i} r_{ik}s_{kj} \geqq 0$ since $r_{ik} \leqq 0$, $s_{kj} \leqq 0$ according to $i + k < i + j$, $k + j < i + j$. Thus $s_{ij} \leqq 0$. Analogously, for $i > j$ the inequality $r_{ij} \leqq 0$ can be proved.

(3,3) Theorem. *The following four properties of a matrix are equivalent:*

1° *All principal minors of A are positive.*

2° *To every vector $x \neq 0$ there exists an index k such that $x_k y_k > 0$ where $y = Ax$.*

3° *To every vector $x \neq 0$ there exists a diagonal matrix D_x with positive diagonal elements such that the scalar product $(Ax, D_x x) > 0$.*

4° *To every vector $x \neq 0$ there exists a diagonal matrix $H_x \geqq 0$ such that $(Ax, H_x x) > 0$.*

5° *Every real proper value of A as well as of each principal minor of A is positive.*

Proof. To prove that 1° implies 2°, choose an arbitrary non-zero vector x and suppose that $x_i y_i \leqq 0$ for each $i \in N$ where $y = Ax$. Let M be the set of those indices i, for which $x_i \neq 0$. Obviously $M \neq 0$. If $A(M)$ is the principal submatrix with rows and columns of M, $x(M)$ the vector whose coordinates have indices of M and coincide with coordinates of x, then the coordinates z_i of the vector $z = A(M)\,x(M)$ coincide with the coordinates y_i for $i \in M$. Thus, there exists a diagonal matrix $U \geqq 0$ (over $M \times M$) such that $z = -Ux(M)$, i.e. $(A(M) + U)\,x(M) = 0$. Consequently the matrix $A(M) + U$ is singular. But all principal minors of $A(M)$ being positive, the same holds for $A(M) + U$ since U is diagonal non-negative. This contradiction proves the above implication.

To prove that 2° implies 3°, let $x \neq 0$ be a vector, $y = Ax$ and k the index for which $x_k y_k > 0$. There exists a number $\varepsilon > 0$ such that

$$x_k y_k + \varepsilon \sum_{j \neq k} x_j y_j > 0\,.$$

It is sufficient to take as D_x the diagonal matrix with $d_{kk} = 1$ and $d_{jj} = \varepsilon$ for $j \neq k$.

Since 4° follows from 3° immediately, we shall prove that 4° implies 5°. Thus, let $0 \neq M \subset N$ and let λ be a real proper value of $A(M)$ with the proper vector $x(M)$. Denote by x the vector the coordinates x_i of which coincide with those of $x(M)$ for

$i \in M$ and are zero for $i \in M$. According to 4° there exists a diagonal matrix $H_x \geqq 0$ such that $(Ax, H_x x) > 0$. But obviously

$$(Ax, H_x x) = (A(M)\,x(M),\ H_x(M)\,x(M)) = \\ = \lambda(x(M),\ H_x(M)\,x(M)) = \lambda(x, H_x x)\,.$$

Since $(x, H_x x) \geqq 0$, we have $\lambda > 0$.

The proof of the theorem will be complete if we prove that 5° implies 1°. But this follows easily from the fact that the determinant of a matrix A is equal the product of all proper values of A and that the product of the non-real proper values of a real matrix is positive.

(3,4) Definition. We shall denote by $\boldsymbol{P}$ the class of all matrices fulfilling one of the conditions of the preceding theorem.

(3,5) *Let $A \in \boldsymbol{P}$. If D is a diagonal matrix with positive diagonal elements, then both DA nad AD belong to $\boldsymbol{P}$ as well.*

Proof. This assertion is an easy consequence of condition 1°.

4. MATRICES OF CLASS $\boldsymbol{K}$

This section contains the definition as well as the main theorems on the class $\boldsymbol{K}$.

The main theorem (4,3) is essentially known and has been included for the sake of completeness.

(4,1) Definition. We shall denote by $\boldsymbol{Z}$ the class of all real square matrices whose off-diagonal elements are all non-positive.

(4,2) *Let each real proper value of a matrix $A \in \boldsymbol{Z}$ be positive. Let $B \in \boldsymbol{Z}$ fulfil the inequality $A \leqq B$. Then*

1° *both A^{-1} and B^{-1} exist and $A^{-1} \geqq B^{-1} \geqq 0$;*
2° *each real proper value of the matrix B is positive;*
3° *$\det B \geqq \det A > 0$.*

Proof. There exists a positive number σ such that $U = E - \sigma B \geqq 0$. Then $V = E - \sigma A \geqq E - \sigma B = U \geqq 0$ and if $p(V)$ is the Perron-root of V, we have

$$\det\left[(1 - p(V))\,E - \sigma A\right] = \det\left[V - p(V)\,E\right] = 0\,.$$

According to our assumption, $1 - p(V) > 0$, i.e. $0 \leqq p(V) < 1$. Thus, the series $E + V + V^2 + \ldots$ converges to the matrix $(E - V)^{-1} = (\sigma A)^{-1}$, the last matrix being obviously non-negative. Since $0 \leqq U^k \leqq V^k$ for $k = 1, 2, \ldots$, $E + U + U^2 + \ldots$ converges to $(E - U)^{-1} = (\sigma B)^{-1}$ as well; the inequalities $(\sigma A)^{-1} \geqq (\sigma B)^{-1} \geqq 0$, and consequently, $A^{-1} \geqq B^{-1} \geqq 0$ are fulfilled. Let now $\alpha \leqq 0$. Then $B - \alpha E \geqq A$. According to 1°, $B - \alpha E$ is regular and all real proper values of B are thus positive. It remains to prove 3°. We shall do that by induction. If the order n of A is 1, everything is obvious. Let $n > 1$ and suppose that 3° is fulfilled for all pairs of k by k matrices satisfying our hypotheses with $1 \leqq k < n$. The principal submatrix $A_1 = A(M)$ as well as $B_1 = B(M)$, $M = \{1, 2, \ldots, n-1\}$, belong to $\boldsymbol{Z}$ and, obviously $A_1 \leqq B_1$. Since the matrix

$$\tilde{A} = \begin{pmatrix} A_1 , & 0 \\ 0 , & a_{nn} \end{pmatrix}$$

fulfills the inequality $A \leqq \tilde{A}$ and $\tilde{A} \in \boldsymbol{Z}$, each real proper value of $\tilde{A}$, and consequently of A_1, is positive. Thus, according to the induction hypothesis, $\det B_1 \geqq \det A_1 > 0$. From $A^{-1} \geqq B^{-1} \geqq 0$ follows (if elements with indices (n, n) are considered)

$$\frac{\det A_1}{\det A} \geqq \frac{\det B_1}{\det B} \geqq 0 .$$

Hence $\det A > 0$, $\det B > 0$ and

$$\det B \geqq \frac{\det B_1}{\det A_1} \det A \geqq \det A > 0 .$$

The proof is complete.

(4,3) Theorem. *Let $A \in \boldsymbol{Z}$. Then the following conditions are equivalent to each other*:

1° *There exists a vector $x \geqq 0$ such that $Ax > 0$;*

2° *there exists a vector $x > 0$ such that $Ax > 0$;*

3° *there exists a diagonal matrix D with positive diagonal elements such that $ADe > 0$ (here e is the vector whose all coordinates are 1);*

4° *there exists a diagonal matrix D with positive diagonal elements such that the matrix $W = AD$ is a matrix with dominant positive principal diagonal;*

5° *for each diagonal matrix R such that $R \geqq A$ the inverse R^{-1} exists and $\sigma(R^{-1}(P - A)) < 1$, where P is the diagonal of A;*

6° *if $B \in \boldsymbol{Z}$ and $B \geqq A$, then B^{-1} exists;*

7° *each real proper value of A is positive;*

8° *all principal minors of A are positive;*

9° *there exists a strictly increasing sequence $0 \neq M_1 \subset M_2 \subset \ldots \subset M_n = N$ such that the principal minors $\det A(M_i)$ are positive;*

10° *there exists a permutation matrix P such that PAP^{-1} may be written in the form RS where R is a lower triangular matrix with positive diagonal elements such that $R \in \boldsymbol{Z}$ and S is an upper triangular matrix with positive diagonal elements such that $S \in \boldsymbol{Z}$;*

11° *the inverse A^{-1} exists and $A^{-1} \geqq 0$;*

12° *the real part of each proper value of A is positive;*

13° *for each vector $x \neq 0$ there exists an index k such that $x_k y_k > 0$ for $y = Ax$.*

Proof. Suppose that $x \geqq 0$ and $Ax > 0$. Let us denote by e the vector with coordinates $1, 1, \ldots, 1$. Let ε be a positive number. Then $x + \varepsilon e > 0$ and $A(x + \varepsilon e) = Ax + \varepsilon Ae$ will be positive if ε is small enough.

If x is a positive vector such that $Ax > 0$, let us denote by D the diagonal matrix with diagonal elements $x_1, \ldots, x_n$. It follows that D has positive diagonal elements and $ADe = Ax > 0$.

Suppose that D is a diagonal matrix with positive diagonal elements such that $We > 0$ for $W = AD$. It follows that $w_{ii} > -\sum_{j \neq i} w_{ij}$ for each i. Since $A \in \mathbf{Z}$ and $d_{ii} > 0$, we have $w_{ij} \leqq 0$ for $i \neq j$. The inequality above may thus be written in the form $w_{ii} > \sum_{j \neq i} |w_{ij}|$ so that W is a matrix with dominant positive principal diagonal.

Suppose that D is a diagonal matrix with positive diagonal elements such that the matrix $W = AD$ fulfills $w_{ii} > \sum_{j \neq i} |w_{ij}|$ for each i. Note first that w_{ii} and, consequently, a_{ii} are positive. It follows from lemma (2,3) that $\sigma(E - H^{-1}W) < 1$ where H is the diagonal of W. If P is the diagonal of A, we have $H = PD$ and

$$\sigma(E - P^{-1}A) = \sigma(D^{-1}(E - P^{-1}A)\, D) =$$
$$= \sigma(E - D^{-1}P^{-1}AD) = \sigma(E - H^{-1}W) < 1 .$$

If R is a diagonal matrix such that $R - A \geqq 0$, we have $r_{ii} \geqq a_{ii} > 0$ and, accordingly, the matrix R^{-1} exists and $R^{-1} \leqq P^{-1}$. According to the theorem of Perron-Frobenius (2,1) we have $\sigma(M) = p(M)$ for each nonnegative matrix M. The matrix $P - A$ being nonnegative, we have

$$\sigma(R^{-1}(P - A)) = p(R^{-1}(P - A)) \leqq$$
$$\leqq p(P^{-1}(P - A)) = p(E - P^{-1}A) = \sigma(E - P^{-1}A) < 1 .$$

Let $B \in \mathbf{Z}$ and $B \geqq A$; suppose that A fulfills condition 5°. Let R and P be the diagonals of B and A respectively. It follows from 5° that R^{-1} exists, has positive diagonal elements and $\sigma(R^{-1}(P - A)) < 1$. Since B belongs to $\mathbf{Z}$ and $B \geqq A$, we have $0 \leqq R - B \leqq P - A$. It follows that $0 \leqq R^{-1}(R - B) \leqq R^{-1}(P - A)$. Hence

$$\sigma(R^{-1}(R - B)) = p(R^{-1}(R - B)) \leqq p(R^{-1}(P - A)) = \sigma(R^{-1}(P - A)) < 1$$

so that the series $E + (E - R^{-1}B) + (E - R^{-1}B)^2 + \ldots$ converges to the sum $(R^{-1}B)^{-1}$; it follows that B^{-1} exists.

Suppose that A fulfills 6° and let $\omega \leqq 0$. Consider the matrix $B = A - \omega E$. We have $B \in \mathbf{Z}$, $B \geqq A$ so that B^{-1} exists according to 6°. It follows that ω is not a proper value of A.

Suppose that all real proper values of A are positive. Let $M \subset N$ and let us show that $\det A(M)$ is positive. Define a matrix B in the following manner: $b_{ij} = a_{ij}$ if both $i, j \in M$, $b_{ii} = a_{ii}$ for i non $\in M$ and $b_{ij} = 0$ for the remaining pairs of indices. Clearly $B \geqq A$ and $B \in \mathbf{Z}$. According to theorem (4,2) we have $\det B > 0$ and every proper value of B is positive. Especially all a_{ii} are positive for $i \notin M$. Since $\det B$ equals the product of $\det A(M)$ and the a_{ii} with $i \notin M$, the minor $\det A(M)$ is positive as well.

The implication 8° to 9° is obvious. The step from 9° to 10° is an immediate consequence of (3,1) and (3,2).

Suppose that A may be written in the form RS where R, S are (lower and upper) triangular matrices with positive diagonals and $R, S \in \mathbf{Z}$. It is easy to see that both R^{-1} and S^{-1} exist and are non negative. Hence A^{-1} exists and $A^{-1} = S^{-1}R^{-1} \geqq 0$.

Suppose that A^{-1} exists and $A^{-1} \geqq 0$. Put $x = A^{-1}e$ so that $x \geqq 0$ and $Ax = e > 0$. It follows that 1° is fulfilled.

This completes a cycle of implications connecting the first eleven properties. Properties 8° and 13° are equivalent according to (3,3). Condition 12° clearly implies 7°. The proof will be complete if we show that 12° follows from 7°. To see that, suppose that all real proper values of A are positive. Take a positive ϱ large enough that $\varrho E - A \geqq 0$. Then $|\varrho - \xi| \leqq p(\varrho E - A)$ for each proper value ξ of A. Further, there exists a real proper value ξ_0 of A such that $\varrho - \xi_0 = p(\varrho E - A)$. According to our assumption, we have $\xi_0 > 0$ whence $|\varrho - \xi| \leqq \varrho - \xi_0 < \varrho$ for each proper value ξ of A. This completes the proof.

(4,4) Definition. We shall denote by $\mathbf{K}$ the class of all matrices fulfilling one of the conditions of the preceding theorem.

(4,5) *Let $A \in \mathbf{K}$. Then there exists a positive proper value $q(A)$ of A such that the real part of any proper value of A is at least $q(A)$.*

Proof. There exists a positive σ such that $\sigma E - A \geqq 0$. Put $q(A) = \sigma - p(\sigma E - A)$. Since $A \in \mathbf{K}$, we have $q(A) > 0$. Let λ be a proper value of A. Then $\sigma - \lambda$ is a proper value of $\sigma E - A$ so that

$$|\sigma - \lambda| \leqq p(\sigma E - A) = \sigma - q(A)$$

which proves the theorem.

(4,6) *Let $A \in \mathbf{K}$, $B \in \mathbf{Z}$ and $B \geqq A$. Then $B \in \mathbf{K}$ as well and B possesses the following properties:*

1° $0 \leqq B^{-1} \leqq A^{-1}$;

2° $\det B \geqq \det A > 0$;

3° $A^{-1}B \geqq E$ and $BA^{-1} \geqq E$;

4° the matrix $B^{-1}A$ as well as AB^{-1} belongs to $\mathbf{K}$ and $B^{-1}A \leqq E$, $AB^{-1} \leqq E$;

5° $\sigma(E - B^{-1}A) < 1$, $\sigma(E - AB^{-1}) < 1$;

6° $q(B) \geqq q(A)$.

Proof. Since $A \in \mathbf{K}$, we have $A \in \mathbf{Z}$ and each real proper value of A is positive by 7° of (4,3). It follows from theorem (4,2) that $\det B \geqq \det A$ and, further, that B^{-1} exists and $A^{-1} \geqq B^{-1} \geqq 0$. By 11° of (4,3) the matrix B belongs to $\mathbf{K}$. Since $A^{-1} \geqq 0$ and $B - A \geqq 0$, we have $A^{-1}(B - A) \geqq 0$ so that $A^{-1}B \geqq E$. Similarly, $BA^{-1} \geqq E$. Since $B^{-1} \geqq 0$ and $B - A \geqq 0$, we have $B^{-1}(B - A) \geqq 0$ so that $B^{-1}A \leqq E$. Especially $B^{-1}A \in \mathbf{Z}$. The inverse of $B^{-1}A$ exists and is nonnegative since $A^{-1}B \geqq 0$. According to 11° of (4,3) the matrix $B^{-1}A$ belongs to $\mathbf{K}$. The matrix AB^{-1} belongs to $\mathbf{Z}$ since $AB^{-1} \leqq E$. The inverse of AB^{-1} is $BA^{-1} \geqq E$, hence nonnegative, so that $AB^{-1} \in \mathbf{K}$ by 11° of (4,3). To prove 5°, note first that $E - B^{-1}A \geqq 0$ so that

$\sigma(E - B^{-1}A) = p(E - B^{-1}A) = 1 - \mu$ for some proper value μ of $B^{-1}A$. Now μ is real and $B^{-1}A \in \mathbf{K}$. It follows from 7° of (4,3) that $\mu > 0$ so that $1 - \mu < 1$. Analogously, $\sigma(E - AB^{-1}) < 1$. To prove 6°, it is sufficient to show that $\lambda E - B$ is regular if $\lambda < q(A)$. To see that, take a $\lambda < q(A)$. The matrix $A - \lambda E$ belongs to $\mathbf{Z}$ and we have $\alpha - \lambda \geqq q(A) - \lambda > 0$ of reach real proper value α of A so that $A - \lambda E \in \mathbf{K}$ by 7° of (4,3). Now $B - \lambda E \geqq A - \lambda E$ and $B - \lambda E \in \mathbf{Z}$. It follows from the first part of the present proof that $B - \lambda E \in \mathbf{K}$ and, consequently, $B - \lambda E$ is regular.

(4,7) *Let $A \in \mathbf{K}$. Then $q(A) \leqq a_{ii}$ for each $i \in N$.*

Proof. Define a matrix B in the following manner: $b_{ij} = a_{ij}$ if $i = j$ and $b_{ij} = 0$ for $i \neq j$. It follows that $B \in \mathbf{Z}$ and $B \geqq A$. According to the preceding theorem $B \in \mathbf{K}$ and $q(B) \geqq q(A)$. Clearly $q(B) = \min a_{ii}$ and the proof is complete.

(4,8) *If $A \in \mathbf{K}$, let us denote by $\gamma(A)$ the circular region with centre $a = \max a_{ii}$ and radius $p(aE - A)$. The set $\gamma(A)$ contains the whole spectrum of A. Especially, the point $q(A)$ lies on the boundary circle. If $M \subset N$ then $A(M)$ belongs to $\mathbf{K}$ as well, $q(A(M)) \geqq q(A)$ and $\gamma(A(M)) \subset \gamma(A)$.*

Proof. We have $aE - A \geqq 0$ so that $|a - \lambda| \leqq p(aE - A)$ for each proper value of A. Now let $M \subset N$. Clearly $A(M) \in \mathbf{Z}$; all principal minors of $A(M)$ are positive whence $A(M) \in \mathbf{K}$. Define a matrix B in the following manner: $b_{ij} = a_{ij}$ for $i, j \in M$ or $i = j$ and $b_{ij} = 0$ for all other pairs of indices. We have clearly $B \in \mathbf{Z}$ and $B \geqq A$ so that $B \in K$ and $q(B) \geqq q(A)$ according to the preceding theorem. Now $q(B)$ is equal to the minimum of $q(A(M))$ and some a_{ii}. It follows that $q(A(M)) \geqq$ $\geqq q(B) \geqq q(A)$. Further, the centre of $\gamma(A(M))$, being the maximum of a_{ii} with $i \in M$, lies to the left of a. The set $\gamma(A(M))$ is therefore contained in $\gamma(A)$ and the proof is complete.

(4,9) *Let $A \in \mathbf{K}$ and let D be a diagonal matrix with positive diagonal elements. Then both DA and AD belong to $\mathbf{K}$ as well.*

Proof. Note that $\mathbf{K} = \mathbf{Z} \cap \mathbf{P}$ and apply (3,5).

(4,10) *Let $A = (a_{ij}) \in \mathbf{K}$, let $C = (c_{ij})$ be a diagonal non-negative matrix such that $c_{ii}c_{jj} \geqq a_{ii}a_{jj}$ whenever $i \neq j$. Then the matrix B with diagonal elements of C and off-diagonal elements of A belongs to $\mathbf{K}$.*

Proof. The numbers a_{ii} as well as c_{ii} being positive, there exist diagonal matrices D_1 with diagonal elements $a_{ii}^{-\frac{1}{2}}$ and D_2 with diagonal elements $c_{ii}^{-\frac{1}{2}}$. According to (4,9) we have $D_1AD_1 = E - U \in \mathbf{K}$, where $U \geqq 0$. Similarly, $D_2BD_2 = E - V$ where $V \geqq 0$. According to our assumption $V \leqq U$, since

$$v_{ij} = \frac{|a_{ij}|}{(c_{ii}c_{jj})^{\frac{1}{2}}} \leqq \frac{|a_{ij}|}{(a_{ii}a_{jj})^{\frac{1}{2}}} = u_{ij}$$

for $i \neq j$. Thus $E - V \in \mathbf{Z}$, $E - V \geqq E - U$. From (4,6) it follows that $E - V \in \mathbf{K}$ and according to (4,9) $B \in \mathbf{K}$ as well. The proof is complete.

5. MATRICES OF CLASS $\boldsymbol{K}_0$

In this section we introduce another class ot matrices which may be considered as the closure of class $\boldsymbol{K}$.

(5,1) *The following properties of a matrix $A \in \boldsymbol{Z}$ are equivalent:*

1° *all real proper values of A as well as of all principal submatrices are nonnegative;*

2° *all principal minors of A are nonnegative;*

3° $A + \varepsilon E \in \boldsymbol{K}$ *whenever* $\varepsilon > 0$;

4° *all real proper values of A are nonnegative.*

Proof. The step from 1° to 2° is obvious, each principal minor of A being equal to the product of all its proper values.

If all principal minors of A are nonnegative, it is to see that all principal minors of $A + \varepsilon E$ are positive whenever $\varepsilon > 0$. It follows from (4,3) that $A + \varepsilon E \in \boldsymbol{K}$.

Suppose now that $A + \varepsilon E \in \boldsymbol{K}$ for each $\varepsilon > 0$. Let λ be a real proper value of A or some principal submatrix of A. Then $\lambda + \varepsilon$ is a real proper value of $A + \varepsilon E$ or some principal submatrix of $A + \varepsilon E$. Since $A + \varepsilon E \in \boldsymbol{K}$, we have $\lambda + \varepsilon > 0$. This is true for every $\varepsilon > 0$ whence $\lambda \geqq 0$. The proof will be complete if we show that 4° implies 3°. But this is an easy consequence of 7° in (4,3).

(5,2) Definition. We denote by $\boldsymbol{K}_0$ the set of all matrices which belong to $\boldsymbol{Z}$ and fulfil one of the conditions of the preceding theorem.

(5,3) *The class $\boldsymbol{K}$ is contained in $\boldsymbol{K}_0$.*

Proof. Obvious.

(5,4) Theorem. *Let $A \in \boldsymbol{Z}$ and suppose that there exists a vector $x > 0$ such that $Ax \geqq 0$. Then $A \in \boldsymbol{K}_0$.*

Proof. If $\varepsilon > 0$, then $(A + \varepsilon E)x = Ax + \varepsilon x > 0$. Thus, $A + \varepsilon E \in \boldsymbol{K}$ according to 2° of theorem (4,3). From 3° of theorem (5,1) it follows that $A \in \boldsymbol{K}$.

(5,5) Theorem. *If $A \in \boldsymbol{K}_0$ is regular, then $A \in \boldsymbol{K}$.*

Proof. Let λ be a proper value of A. According to (5,1) we have $\lambda \geqq 0$. Since $\det A \neq 0$, no proper value of A can be zero. Thus all real proper values of A are positive and $A \in \boldsymbol{K}$ by 7° of theorem (4,3).

(5,6) *Let $A \in \boldsymbol{K}_0$ be singular of order n and irreducible. Then A has rank $n - 1$ and there exists a vector $y > 0$ such that $Ay = 0$.*

Proof. For a suitable $c > 0$ the matrix $cE - A$ is nonnegative. According to the theorem of Perron-Frobenius the Perron root $p(cE - A)$ is simple and the proper vector y corresponding to $p(cE - A)$ may be chosen positive. Since $c - p(cE - A)$ belongs to the spectrum of A, we have $c - p(cE - A) \geqq 0$ by 3° of (5,1) whence $p(cE - A) \leqq c$. The matrix A being singular, c is a proper value of $cE - A$ so that

$c \leqq p(cE - A)$. It follows that $p(cE - A) = c$ so that $Ay = 0$. Since 0 is a simple proper value of A, A is of rank $n - 1$.

(5,7) Theorem. *Let $A \in \mathbf{K}_0$ be irreducible. Then all proper principal minors of A are positive.*

Proof. If A is regular, we have $A \in \mathbf{K}$ by (5,5) and the conclusion follows from 8° of (4,3). Thus let A be singular of order n. It is sufficient to consider the case $n > 1$. According to (5,6), there exists a vector $y > 0$ such that $Ay = 0$. Analogously, there exists a row-vector $z' > 0$ such that $z'A = 0$. The adjoint matrix B of the singular matrix A, whose elements are the cofactors A_{ik} of the elements a_{ik} of A, is known to be of the form $\varrho z'y$. Here $\varrho \neq 0$, since the rank of A is $n - 1 \geqq 1$ by (5,6). If $\varrho < 0$, there would exist an $\varepsilon > 0$ such that the adjoint matrix B_1 of $A + \varepsilon E$ satisfies $B_1 < 0$, which is a contradiction with $A + \varepsilon E \in \mathbf{K}$ (its principal minors of order $n - 1$ are positive). Thus, $\varrho > 0$ and $B > 0$.

It follows that all principal minors of order $n - 1$ of the matrix A are positive. According to (5,5), all its principal minors of order $\leqq n - 1$ are positive. The proof is complete.

(5,8) *Let $A \in \mathbf{K}_0$ be irreducible. Then there exists a vector $x > 0$ such that $Ax \geqq 0$.*

Proof. If A is regular, it follows from 2° of (4,3). If A is singular, it follows from (5,6).

(5,9) *If $A \in \mathbf{K}_0$, there exists a non-zero vector $x \geqq 0$ such that $Ax \geqq 0$.*

Proof. If $A = 0$, the theorem is obvious. If $A \neq 0$ there exists a set M, $0 \neq M \subset N$ such that $A(M)$ is irreducible and $a_{pq} = 0$ for $p \in M$ and $q \in N - M$. From (5,8), follows that there exists a vector $x(M) > 0$ fulfilling the inequality $A(M)\, x(M) \geqq 0$. Thus, the vector x whose coordinates x_i are equal to those of $x(M)$ for $i \in M$, and $x_j = 0$ for $j \notin M$, fulfills the condition $Ax \geqq 0$.

(5,10) *Let $A \in \mathbf{K}$. Then $A - q(A)\, E$ belongs to $\mathbf{K}_0$ and every singular matrix of class $\mathbf{K}_0$ may be obtained in this manner.*

Proof. If $\varepsilon > 0$, all real proper values of $A - q(A)\, E + \varepsilon E$ are at least ε so that $A - q(A)\, E + \varepsilon E \in \mathbf{K}$. It follows that $A - q(A)\, E \in \mathbf{K}_0$. If A is singular of class $\mathbf{K}_0$, choose a positive α and put $B = A + \alpha E$. We have $B \in \mathbf{K}$ and $q(B) = \alpha$ so that $A = B - q(B)\, E$.

(5,11) *Let $A \in \mathbf{K}_0$, let $B \geqq A$ and $B \in \mathbf{Z}$. Then $B \in \mathbf{K}_0$.*

Proof. Follows from (4,6) and 3° of (5,1).

6. SOME CRITERIA FOR MATRICES OF CLASS $\mathbf{K}$

The results of this section have important applications to regularity conditions for matrices and estimates for spectra of matrices.

(6,1) *Let* $A = (a_{ij}) \in \mathbf{Z}$. *Let* g *and* g' *be a pair of adjoint norms in* E_n. *Put* $g_i = g(|a_{i,1}|, \ldots, |a_{i,i-1}|, 0, |a_{i,i+1}|, \ldots, |a_{in}|)$, *and suppose that* $g_i > 0$ *and* $a_{ii} > 0$ *for each* i. *Suppose further that there exist numbers* v_i *such that*

$$g'(v_1, \ldots, v_n) \leqq 1$$

and $v_i \geqq g_i/a_{ii}$ *for each* i. *Then* $A \in \mathbf{K}_0$. *If* $g'(v_1, \ldots, v_n) < 1$ *or if* $v_i > g_i/a_{ii}$ *for each* i, *the matrix* A *belongs to* $\mathbf{K}$.

Proof. The vector v is positive since $g_i > 0$ and $a_{ii} > 0$. We have, for each i, $\sum_{s \neq i} |a_{is}| v_s \leqq g_i g'(v) \leqq g_i \leqq a_{ii} v_i$ so that $Av \geqq 0$. According to (5,4) this is sufficient for A to be of class $\mathbf{K}_0$. If $g'(v) < 1$ we have, since $g_i > 0$, the estimate $g_i g'(v) < g_i \leqq a_{ii} v_i$ so that $Av > 0$. If $v_i > g_i/a_{ii}$ for each i, we obtain $g_i g'(v) \leqq g_i < a_{ii} v_i$ whence $Av > 0$. According to 2° of (4,3) the matrix A belongs to $\mathbf{K}$.

(6,2) *Let* $A = (a_{ij}) \in \mathbf{Z}$, $a_{ii} \geqq 0$. *If* $q \geqq 1$, *let us denote by* $g_i(q)$ *the sum* $(\sum_{k \neq i} |a_{ik}|)^q)^{1/q}$. *Further, let* $g_i(\infty) = \max_{k \neq i} |a_{ik}|$. *Let* $p_1, \ldots, p_n$ *be real numbers such that* $p = \min p_i \geqq 1$. *For each* p_i, *let* q_i *be the conjugate exponent defined by the relation* $1/p_i + 1/q_i = 1$; *if* $p_i = 1$, *we put* $g_i(q_i) = g_i(\infty)$. *Let* $g_i(q_i) > 0$ *for each* i *and suppose that*

$$\sigma = \sum_{i \in N} \frac{1}{\left(1 + \left(\dfrac{a_{ii}}{g_i(q_i)}\right)^{p_i}\right)^{p/p_i}} < 1 \,.$$

Then A *belongs to* $\mathbf{K}$.

Proof. The numbers

$$x_i = \frac{1}{\left(1 + \left(\dfrac{a_{ii}}{g_i(q_i)}\right)^{p_i}\right)^{1/p_i}}$$

being positive, consider the vector $x > 0$ with coordinates x_i. Let us show that $Ax > 0$. This is sufficient for the validity of the inclusion $A \in \mathbf{K}$ by 2° of (4,3). We have

$$\sum_{k \neq i} |a_{ik}| x_k \leqq g_i(q_i) (\sum_{k \neq i} x_k^{p_i})^{1/p_i} \leqq g_i(q_i) (\sum_{k \neq i} x_k^{p})^{1/p} =$$

$$= g_i(q_i) \left(\sigma - \frac{1}{\left(1 + \left(\dfrac{a_{ii}}{g_i(q_i)}\right)^{p_i}\right)^{p/p_i}} \right)^{1/p} < g_i(q_i) \left(1 - \frac{1}{\left(1 + \left(\dfrac{a_{ii}}{g_i(q_i)}\right)^{p_i}\right)^{p/p_i}} \right)^{1/p},$$

the number $g_i(q_i)$ being positive. Now $1 + b \geqq (1 + b^r)^{1/r}$ for every $b > 0$ and $r \geqq 1$. This inequality for $b = (a_{ii}/g_i(q_i))^p$ and $r = p_i/p$ gives

$$1 + \left(\frac{a_{ii}}{g_i(q_i)}\right)^p \geqq \left(1 + \left(\frac{a_{ii}}{g_i(q_i)}\right)^{p_i}\right)^{p/p_i}$$

whence

$$1 - \frac{1}{\left(1 + \left(\dfrac{a_{ii}}{g_i(q_i)}\right)^{p_i}\right)^{p/p_i}} \leqq 1 - \frac{1}{1 + \left(\dfrac{a_{ii}}{g_i(q_i)}\right)^p} .$$

Together with the inequality above, this gives

$$-\sum_{k \neq i} a_{ik} x_k < g_i(q_i) \left(1 - \frac{1}{1 + \left(\dfrac{a_{ii}}{g_i(q_i)}\right)^p}\right)^{1/p} =$$

$$= g_i(q_i) \frac{a_{ii}}{g_i(q_i)} \frac{1}{\left(1 + \left(\dfrac{a_{ii}}{g_i(q_i)}\right)^p\right)^{1/p}} \leqq a_{ii} x_i$$

and the proof is complete.

(6,3) *Let $A = (a_{ij}) \in \mathbf{Z}$, $a_{ii} \geqq 0$. Suppose that, in each row, at least one off-diagonal element is different from zero. Let $p > 1$, $q = p/(p - 1)$ and let R be the set of those i for which $a_{ii} \leqq g_i(1)$. If R is empty, then $A \in \mathbf{K}$. Let $R \neq 0$ and let μ be the maximum of the numbers*

$$\frac{1}{\left(1 + \left(\dfrac{a_{ii}}{g_i(p)}\right)^q\right)^{1/q}} \quad \textit{for} \quad i \in R .$$

Suppose that

$$\sum_{i \in R} \frac{1}{1 + \left(\dfrac{a_{ii}}{g_i(p)}\right)^q} + \mu^q \sum_{i \in N - R} \left(\frac{g_i(1)}{a_{ii}}\right)^q < 1 .$$

Then $A \in \mathbf{K}$.

Proof. If $R = 0$, the theorem follows from 2° of (4,2). Let $R \neq 0$. There exists a positive number ε such that the inequalities

$$\sigma = \sum_{i \in R} \frac{1}{1 + \left(\dfrac{a_{ii}}{g_i(p)}\right)^q} + (\mu + \varepsilon)^q \sum_{i \in N - R} \left(\frac{g_i(1)}{a_{ii}}\right)^q < 1$$

as well as $\mu + \varepsilon \leqq \mu(a_{ii}/g_i(1))$ for each $i \in N - R$ are valid. Put

$$\xi_i = \frac{1}{\left(1 + \left(\dfrac{a_{ii}}{g_i(p)}\right)^q\right)^{1/q}} \quad \text{if} \quad i \in R$$

and $\xi_i = (\mu + \varepsilon)(g_i(1)/a_{ii})$ if $i \in N - R$.

Note that $g_i(1) > 0$ according to our assumption. It follows that the vector x with coordinates ξ_i is positive. We intend to show that $Ax > 0$.

If $i \in R$, we have

$$-\sum_{j \neq i} a_{ij}\xi_j \leqq g_i(p)\left(\sum_{j \neq i} \xi_j^q\right)^{1/q} = g_i(p)\left(\sigma - \xi_i^q\right)^{1/q} < g_i(p)\left(1 - \frac{1}{1 + \left(\frac{a_{ii}}{g_i(p)}\right)^q}\right)^{1/q} =$$

$$= g_i(p)\frac{a_{ii}}{g_i(p)}\xi_i = a_{ii}\xi_i\,.$$

If $i \in N - R$, we have

$$-\sum_{j \neq i} a_{ij}\xi_j \leqq g_i(1)\,.\,\max \xi_j\,.$$

Consider now ξ_j for $j \in N - R$; since $\mu + \varepsilon \leqq \mu(a_{ii}/g_i(1))$, we have

$$\xi_j = (\mu + \varepsilon)\frac{g_j(1)}{a_{jj}} \leqq \mu$$

so that $\max \xi_r = \mu$. Hence

$$-\sum_{j \neq i} a_{ij}\xi_j \leqq g_i(1)\,\mu = g_i(1)\frac{(\mu + \varepsilon)}{a_{ii}}\,.\,\frac{a_{ii}\mu}{\mu + \varepsilon} = \frac{\mu}{\mu + \varepsilon}\,a_{ii}\xi_i < a_{ii}\xi_i\,.$$

The proof is complete.

(6,4) *Let $A = (a_{ij}) \in \mathbf{Z}$. Suppose that, in each row, at least one off-diagonal element is different from zero. Suppose that $p > 1$ and that k_i are positive numbers such that $a_{ii} > k_i g_i(p)$ for each $i \in N$. Let W be the set of those j for which $g_j(1)/g_j(p) > k_j$. If $W = 0$ then $A \in \mathbf{K}$. Let $W \neq 0$ and put $m = \max 1/[(1 + k_j^q)^{1/q}]$ for $j \in W$. Let M_0 be the set of those j for which*

$$\frac{1}{1 + k_j^q} < \frac{m^q}{k_j^q}\sigma_j \quad where \quad \sigma_i = \left(\frac{g_i(1)}{g_i(p)}\right)^q \quad and \quad q = \frac{p}{p - 1}\,.$$

Let

$$\sigma = \sum_{i \in M_0} \frac{1}{1 + k_i^q} + m^q \sum_{i \in N - M_0} \frac{\sigma_i}{k_i^q} \leqq 1\,.$$

Then $A \in \mathbf{K}$.

Proof. If $W = 0$, then $a_{ii} > g_i(1)$ for each $i \in N$ and $A \in \mathbf{K}$ according to 2° of (4,3). Now let $W \neq 0$ and let $i \in W$. We have then

$$\frac{1}{1 + k_i^q} \leqq m^q < m^q\left(\frac{1}{k_i}\frac{g_i(1)}{g_i(p)}\right)^q = \frac{m^q}{k_i^q}\sigma_i\,.$$

It follows that $W \subset M_0$. Put $\xi_i = 1/(1 + k_i^q)^{1/q}$ for $i \in M_0$ and $\xi_i = m(\sigma_i^{1/q}/k_i)$ for $i \in N - M_0$. The vector x with coordinates ξ_i is positive. Let us show that $Ax > 0$.

If $i \in M_0$, we have

$$\sum_{j \neq i} |a_{ij}|\,\xi_j \leqq g_i(p)\left(\sum_{j \neq i} \xi_j^q\right)^{1/q} \leqq g_i(p)\left(\sigma - \frac{1}{1 + k_i^q}\right)^{1/q} \leqq g_i(p)\left(1 - \frac{1}{1 + k_i^q}\right)^{1/q} =$$

$$= g_i(p)\frac{k_i}{(1 + k_i^q)^{1/q}} = g_i(p)\,k_i\xi_i < a_{ii}\xi_i\,.$$

If $i \in N - M_0$, we have $i \in N - W$ so that $g_i(1)/g_i(p) \leqq k_i$. Further,

$$\sum_{j \neq i} a_{ij}\xi_j \leqq g_i(1) \,.\, \max \xi_r \,.$$

Let us show now that $\max \xi_r = m$. Since $W \subset M_0$, we have $m = \max \xi_r$ for $r \in W$.

If $r \in N - M_0$, we have $r \in N - W$ so that $g_r(1)/g_r(p) \leqq k_r$ and

$$\xi_r = \frac{m}{k_r} \frac{g_r(1)}{g_r(p)} \leqq m \,.$$

If $r \in M_0 - W$, we have $g_r(1)/g_r(p) \leqq k_r$ and

$$\xi_r = \frac{1}{(1 + k_r^q)^{1/q}} < \frac{m}{k_r} \sigma_r^{1/q} \leqq m \,.$$

This proves the relation $m = \max \xi_r$. We see thus that, for $i \in N - M_0$,

$$\sum_{j \neq i} a_{ij}\xi_j \leqq g_i(1)\, m = \frac{g_i(1)}{g_i(p)\, k_i} \,.\, k_i g_i(p) \,.\, m = \xi_i k_i g_i(p) < a_{ii}\xi_i \,.$$

The proof is complete.

The following theorem has a close connection with a result obtained by K. Fan and A. J. Hoffman ([3], Th. 1.5).

(6,5) *Suppose that, in the notation of the preceding theorem, there exists a positive α such that $a_{ii} > \alpha g_i(p)$ for each i and $\sum_{i \in N} \sigma_i \leqq \alpha^q(1 + \alpha^q)$. Then $A \in$* **K**.

Proof. We shall use the preceding result; put $k_i = \alpha$. The set W is the set of those j for which $\sigma_j > \alpha^q$. If W is empty, the proof is complete. If $W \neq 0$, we have $m = 1/[(1 + \alpha^q)^{1/q}]$ and M_0 is the set of those i for which

$$\frac{1}{1 + \alpha^q} < \frac{m^q}{\alpha^q} \sigma_i = \frac{1}{1 + \alpha^q} \frac{\sigma_i}{\alpha^q}$$

and is thus equal to W. Now

$$\sum_{i \in M_0} \frac{1}{1 + k_i^q} + m^q \sum_{i \in N - M_0} \frac{\sigma_i}{k_i^q} =$$

$$= \sum_{i \in W} \frac{1}{1 + \alpha^q} + \frac{1}{1 + \alpha^q} \sum_{i \in N - W} \frac{\sigma_i}{\alpha^q} < \sum_{i \in N} \frac{1}{1 + \alpha^q} \frac{\sigma_i}{\alpha^q} \leqq 1 \,.$$

Thus the theorem (6,4) is applicable and $A \in$ **K**.

(6,6) Notation. Let $A = (a_{ij})$ be a matrix. We shall denote by $P(A)$ the matrix

$$\begin{pmatrix} |a_{11}|, & -|a_{12}|, & -|a_{13}|, & \dots \\ -|a_{21}|, & |a_{22}|, & -|a_{23}|, & \dots \\ -|a_{31}|, & -|a_{32}|, & |a_{33}|, & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

so that $P(A) \in$ **Z**.

(6,7) *Let* $A = (a_{ik}) \in \mathbf{K}$, $B = (b_{ik}) \in \mathbf{K}$ *and let* $0 < \alpha < 1$. *If* C *is the matrix with* $c_{ik} = |a_{ik}|^\alpha |b_{ik}|^{1-\alpha}$, *then* $P(C) \in \mathbf{K}$. *If* A *and* B *belong to* $\mathbf{K}_0$, *then* $P(C) \in \mathbf{K}_0$.

Proof. Consider first the case $A, B \in \mathbf{K}$. According to 2° of (4,3), there exist positive vectors x and y such that $Ax > 0$ and $By > 0$. We are going to show that $P(C)\, z > 0$ where z is the vector with coordinates $z_i = x_i^\alpha y_i^{1-\alpha}$. Indeed, we have

$$\sum_{k \neq i} |a_{ik}|^\alpha |b_{ik}|^{1-\alpha} z_k = \sum_{k \neq i} (|a_{ik}|\, x_k)^\alpha (|b_{ik}|\, y_k)^{1-\alpha} \leqq$$
$$\leqq \Big(\sum_{k \neq i} |a_{ik}|\, x_k\Big)^\alpha \Big(\sum_{k \neq i} |b_{ik}|\, y_k\Big)^{1-\alpha} < (a_{ii}x_i)^\alpha (b_{ii}y_i)^{1-\alpha} = a_{ii}^\alpha b_{ii}^{1-\alpha} z_i .$$

This completes the proof of the first assertion. Suppose now that A and B belong to $\mathbf{K}_0$. We are going to show that $P(C) + \varepsilon E$ belongs to $\mathbf{K}$ for each positive ε. Clearly there exist positive numbers s_i and t_i such that

$$c_{ii} + \varepsilon = (a_{ii} + s_i)^\alpha (b_{ii} + t_i)^{1-\alpha} .$$

If S and T are diagonal matrices with s_i and t_i as diagonal elements, we have $A + S \in \mathbf{K}$ and $B + T \in \mathbf{K}$ by (5,11) and 3° of (5,1). Hence $P(C) + \varepsilon E \in \mathbf{K}$ by the first assertion of the present theorem. It follows that $P(C) \in \mathbf{K}_0$.

(6,8) *Let* $A = (a_{ij}) \in \mathbf{Z}$. *Let* $0 \leqq \alpha \leqq 1$. *Suppose that* $a_{ii} \geqq P_i^\alpha Q_i^{1-\alpha}$ *where* $P_i = \sum_{j \neq i} |a_{ij}|$ *and* $Q_i = \sum_{j \neq i} |a_{ji}|$. *Then* $A \in \mathbf{K}_0$. *If* $a_{ii} > P_i^\alpha Q_i^{1-\alpha}$, *then* $A \in \mathbf{K}$.

Proof. It is sufficient to consider the case $0 < \alpha < 1$. Let us denote by P the matrix with the same off-diagonal elements as A and with P_i instead of a_{ii}. Clearly $P \in \mathbf{K}_0$ by 3° of (5,1) and 2° of (4,3). Let Q be the analogous matrix with Q_i instead of a_{ii} so that $Q \in \mathbf{K}_0$. According to the preceding theorem the matrix W with off-diagonal elements a_{ik} and diagonal elements $P_i^\alpha Q_i^{1-\alpha}$ belongs to $\mathbf{K}_0$. Since $a_{ii} \geqq w_{ii}$, we have $A \in \mathbf{K}_0$ by (5,11). If $a_{ii} > w_{ii}$, then $A \in \mathbf{K}$ by 3° of (5,1) and (4,6).

(6,9) *Let* $A = (a_{ij}) \in \mathbf{Z}$, *let* $P_i = \sum_{j \neq i} |a_{ij}|$. *Suppose that* $a_{ii}a_{jj} > P_iP_j$ *for each pair* i, j, $i \neq j$. *Then* $A \in \mathbf{K}$.

Proof. There exists an $\varepsilon > 0$ such that $a_{ii}a_{jj} > (P_i + \varepsilon)(P_j + \varepsilon)$ for each pair i, j, $i \neq j$. Since the matrix P with the same off-diagonal elements as A and with P_i instead of a_{ii} belongs to $\mathbf{K}_0$, $P + \varepsilon E \in \mathbf{K}$ and $A \in \mathbf{K}$ according to (4,10).

(6,10) *Let* $A = (a_{ij}) \in \mathbf{Z}$, $P_i = \sum_{j \neq i} |a_{ij}|$, $Q_i = \sum_{j \neq i} |a_{ji}|$. *Let* $0 \leqq \alpha \leqq 1$. *Suppose that*

$$a_{ii}a_{jj} > P_i^\alpha Q_i^{1-\alpha} P_j^\alpha Q_j^{1-\alpha}$$

for each pair i, j, $i \neq j$. *Then* $A \in \mathbf{K}$.

Proof. There exists an $\varepsilon > 0$ such that

$$a_{ii}a_{jj} > (P_i^\alpha Q_i^{1-\alpha} + \varepsilon)(P_j^\alpha Q_j^{1-\alpha} + \varepsilon)$$

for each pair i, j, $i \neq j$. Since the matrix R with the same off-diagonal elements as A and with $P_i^\alpha Q_i^{1-\alpha}$ instead of a_{ii} belongs to $\mathbf{K}_0$ according to (6,8), we have $R + \varepsilon E \in \mathbf{K}$ and $A \in \mathbf{K}$ by (4,10).

(6,11) *Let* $A = (a_{ik}) \in \mathbf{K}_0$ *let* $0 < \alpha < 1$. *Then the matrix* $B = (b_{ik})$ *with* $b_{ii} = a_{ii}$, $b_{ik} = -|a_{ik}|^{\alpha} |a_{ki}|^{1-\alpha}$ *for* $i \neq k$, *belongs to* $\mathbf{K}_0$ *and*

$$\det B(M) \geqq \det A(M) \quad \text{for each} \quad M \subset N = \{1, \ldots, n\} .$$

Proof. Since $A \in \mathbf{K}_0$, the transpose $A^T \in \mathbf{K}_0$ as well. From (6,7) it follows that $B \in \mathbf{K}_0$. The remaining inequality is an easy consequence of the following general theorem.

(6,12) *Let* φ *be an operation in* $\mathbf{K}_0$ *with the following properties*:

1° *if* $A \in \mathbf{K}_0$, *then* $\varphi(A)$ *is a matrix of the same order and* $\varphi(A) \in \mathbf{K}_0$;

2° *if* $A \in \mathbf{K}_0$ *and* $M \subset N$, *then* $\varphi(A(M))$ *is less than or equal to the corresponding submatrix of* $\varphi(A)$;

3° *if* $A \in \mathbf{K}_0$ *and* D *is a diagonal matrix with positive diagonal elements, then* $\varphi(A) + D \leqq \varphi(A + D)$.

Under these conditions, the operation φ *transforms* $\mathbf{K}$ *into* $\mathbf{K}$ *and has the further property*

4° $\det \varphi(A) \geqq \det A$ *for each* $A \in \mathbf{K}_0$.

Proof. Let $A \in \mathbf{K}$ so that $A - q(A)\,E \in \mathbf{K}_0$. It follows from our assumption that

$$\varphi(A) - q(A)\,E \geqq \varphi(A - q(A)\,E) \in \mathbf{K}_0$$

so that $\varphi(A) - q(A)\,E \in \mathbf{K}_0$ by (5,11). It follows from (5,1) that

$$\varphi(A) = (\varphi(A) - q(A)\,E) + q(A)\,E \in \mathbf{K} .$$

The inequality for determinants will be proved by induction with respect to the order n of the matrix. For $n = 1$, the assertion is obvious since for $a \geqq 0$ and any $\varepsilon > 0$, we have

$$\varphi(a) = \varphi(\varepsilon a + (1 - \varepsilon)\,a) \geqq \varphi(\varepsilon a) + (1 - \varepsilon)\,a \geqq (1 - \varepsilon)\,a .$$

Let $n > 1$ and suppose the theorem proved for all matrices of order smaller than n. Let $A \in \mathbf{K}_0$ be given and put $B = \varphi(A)$. If A is singular, we have $\det B \geqq 0 = \det A$ and the proof is complete. We may limit ourselves accordingly to the case $A \in \mathbf{K}$. Put $\lambda = q(A)$ so that $\lambda > 0$. We have

$$A - \lambda E \in \mathbf{K}_0, \ \det B = \det(\lambda E + (B - \lambda E)) = \\ = \lambda^n + b_1 \lambda^{n-1} + \ldots + b_{n-1}\lambda + \det(B - \lambda E) .$$

According to our assumption, we have $\varphi(A - \lambda E) + \lambda E \leqq \varphi(A) = B$ whence $B - \lambda E \geqq \varphi(A - \lambda E) \in \mathbf{K}_0$.

It follows that $B - \lambda E \in \mathbf{K}_0$ so that $\det(B - \lambda E) \geqq 0$. Similarly, $\det A = \lambda^n + a_1 \lambda^{n-1} + \ldots + a_{n-1}\lambda$. Here b_i and a_i are sums of all principal minors of order i of the matrices $B - \lambda E$ and $A - \lambda E$ respectively. Let M be a proper nonvoid subset of N. Then, according to conditions 2° and 3°, we have

$$B(M) - \lambda E(M) \geqq \varphi(A(M)) - \lambda E(M) \geqq \varphi(A(M) - \lambda E(M)) \in \mathbf{K}_0$$

since $q(A(M)) \geqq q(A) = \lambda$ by (4,8).

From 2° of (4,6) and by induction hypothesis it follows that

$$\det (B(M) - \lambda E(M)) \geqq \det \varphi(A(M) - \lambda E(M)) \geqq \det (A(M) - \lambda E(M)) .$$

Thus $b_i \geqq a_i$ for each i and the proof is complete.

7. INEQUALITIES FOR MATRICES AND THEIR DETERMINANTS

The inequalities presented below contain as special cases some known properties of minors.

(7,1) *Suppose that the matrices U, V, W fulfil the inequalities $U \leqq V \leqq W$ and let $V^{-1} \geqq 0$. Then $UV^{-1}W \leqq U - V + W$; if $U \in \mathbf{K}$ and $W \in \mathbf{Z}$ then both matrices $UV^{-1}W$ and $U - V + W$ belong to $\mathbf{K}$.*

Proof. Obviously $(V - U)\, V^{-1}(W - V) \geqq 0$, all three matrices in the product being non-negative. A simple computation gives the above inequality. Further, if $U \in \mathbf{K}$ and $W \in \mathbf{Z}$ then $U - V + W = (U - V) + W$ is a sum of a matrix $W \in \mathbf{Z}$ and a non-positive matrix $U - V$. Thus, $U - V + W \in \mathbf{Z}$ and consequently $UV^{-1}W \in \mathbf{Z}$. But $(UV^{-1}W)^{-1} = (W^{-1}V)\, U^{-1}$; since $U^{-1} \geqq 0$ and $W^{-1}V \geqq 0$ by (4,6), we have $UV^{-1}W \in \mathbf{K}$ according to 11° of (4,3). From (4,6) it follows that $U - V + W \in \mathbf{K}$. The proof is complete.

(7,2) Notation. If $A = (a_{ij})$, $B = (b_{ij})$ are matrices, let us denote by $A \vee B$ the matrix $(\max (a_{ij}, b_{ij}))$, by $A \wedge B$ the matrix $(\min (a_{ij}, b_{ij}))$.

(7,3) *If A, B, C are matrices such that $A \leqq B$ and $B^{-1} \geqq 0$, then $AB^{-1}(B \vee C) \leqq A \vee C$.*

Proof. Since $B \leqq B \vee C$, it follows from (7,1) that

$$AB^{-1}(B \vee C) \leqq A - B + (B \vee C) .$$

But $A - B + (B \vee C) \leqq A \vee C$ which completes the proof.

(7,4) *Let A, B, C be matrices such that $A \leqq B$, $A^{-1} \geqq 0$, $B \vee C \in \mathbf{Z}$. Then both matrices AB^{-1} and $(A \vee C)(B \vee C)^{-1}$ belong to $\mathbf{K}$ and $AB^{-1} \leqq (A \vee C) . (B \vee C)^{-1}$. Finally,*

$$\frac{\det A}{\det B} \leqq \frac{\det (A \vee C)}{\det (B \vee C)} .$$

Proof. From (4,6) it follows that $B^{-1} \geqq 0$ (since $A, B, C \in \mathbf{Z}$), $AB^{-1} \in \mathbf{K}$ as well as $(A \vee C)(B \vee C)^{-1} \in \mathbf{K}$. The inequality in the preceding theorem multiplied by $(B \vee C)^{-1} \geqq 0$ gives $AB^{-1} \leqq (A \vee C)(B \vee C)^{-1}$. From 2° of (4,6) follows the remaining inequality for the determinants.

(7,5) *Let $A \in \mathbf{K}$, $B \in \mathbf{Z}$ be matrices $A \leqq B$. Let $0 \neq M_1 \subset M_2 \subset N$. Then*

$$\frac{\det A(M_2)}{\det B(M_2)} \leqq \frac{\det A(M_1)}{\det B(M_1)} .$$

Proof. We may suppose that $M_2 = N$. Then the inequality considered is a consequence of the preceding one if we put $C = (c_{ij})$, $c_{ij} = a_{ij}$ for $i, j \in M_1$ or $i = j$ and $c_{kl} = 0$ for the remaining pairs of indices k, l.

This section will be conclued by a modification of (7,4) and a consequence of (7,5).

(7,6) *If matrices $A, B \in \mathbf{Z}$ fulfil the condition $A \wedge B \in \mathbf{K}$, then*

$$\det A \det B \geqq \det (A \vee B) \det (A \wedge B) .$$

Proof. This follows directly from (7,4).

(7,7) *Let A, B be matrices of order n, $M_i = \{1, 2, \ldots, i\}$. Let $A, B \in K$, $A \leqq B$. Then*

$$\frac{\det A}{\det B} \leqq \frac{\det A(M_{n-1})}{\det B(M_{n-1})} \leqq \ldots \leqq \frac{\det A(M_1)}{\det B(M_1)} \leqq 1 .$$

Proof. This is an easy consequence of (7,5).

It is easy to see that the inequalities in (7,1) may be generalized for the case of $2n + 1$ matrices.

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Резюме

О МАТРИЦАХ С НЕПОЛОЖИТЕЛЬНЫМИ НЕДИАГОНАЛЬНЫМИ ЭЛЕМЕНТАМИ И ПОЛОЖИТЕЛЬНЫМИ ГЛАВНЫМИ МИНОРАМИ

МИРОСЛАВ ФИДЛЕР и ВЛАСТИМИЛ ПТАК (Miroslav Fiedler a Vlastimil Pták), Прага

Работа посвящена систематическому описанию свойств тех матриц, все недиагональные элементы которых неположительны, между тем как все главные миноры положительны. Работа служит основанием для изучений некоторых спектральных свойств матриц; результаты этих изучений будут опубликованы в дальнейших сообщениях.