Minimum $(12, 6, 3)$ Covers

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Abstract

A $(12, 6, 3)$ cover is a family of 6-element subsets, called blocks, chosen from a 12-element universe, such that each 3-element subset is contained in at least one block. This paper constructs a $(12, 6, 3)$ cover with 15 blocks, and it shows that any $(12, 6, 3)$ cover has at least 15 blocks; thus the covering number $C(12, 6, 3) = 15$. It also shows that the 68 nonisomorphic $(12, 6, 3)$ covers with 15 blocks fall into just two classes using a very natural classification scheme.

1 Introduction

A general (v, k, t) *cover* is a family of k-sets, called *blocks*, chosen from a v-element universe, such that each of the $\binom{v}{t}$ possible t-sets is contained in at least one of the blocks. The *size* of a cover is its number of blocks. A *minimum* (v, k, t) cover is one with minimum size; that size is called the *covering number* and is denoted by $C(v, k, t)$. W. H. Mills and R. C. Mullin [2] survey the known covering numbers for small v, k, and t.

The present paper has two main results. The first is that the covering number $C(12, 6, 3) = 15$. The second, loosely, is that the size-15 covers form two natural classes; a more precise statement requires a few definitions. A (v, k, t) *covering family* is the same as a (v, k, t) cover except that some of its sets, still called blocks, might have fewer than k elements (but be nonempty). An *element-minimal* (v, k, t) covering family is one for which the removal of any element from any block leaves some t-set uncovered. Element-minimal covering families, then, are covers with the fat trimmed away, and hence are in a sense more natural than covers. A *completion* of a $(k-d)$-set, for $0 \le d < k$, is the k-set that results from adding to it d other elements of the universe. A completion of a family of sets is the family of k-sets that results from a completion of each of its sets. So a more precise statement of the second main result is: Every $(12, 6, 3)$ cover of size 15 is isomorphic to a completion of exactly one of two specific element-minimal covering families (shown in Figures 1 and 6).

Here's the layout of the rest of the paper. Section 2 constructs a $(12, 6, 3)$ cover of size 15, providing the upper bound. (The construction generalizes, yielding, for example, $C(18, 9, 4) \leq 43$, which currently is the best bound for that case.) Section 3 lists some properties of minimum $(11, 5, 2)$ covers; R. G. Stanton [5] showed, in essence, that every such cover is a completion of a single specific element-minimal covering family. Those $(11, 5, 2)$ properties are useful, because the blocks of a (v, k, t) cover that contain a specific element e, after e has been deleted, form a $(v-1, k-1, t-1)$ cover, called the *e-induced cover*, or sometimes just *e-cover*. So in general the properties of $(v-1, k-1, t-1)$ covers restrain (v, k, t) covers. Accordingly, using properties established in Section 3, Section 4 proves a lower bound of 15 on $C(12, 6, 3)$ and Section 5 shows that the size-15 covers fall into just two classes, each given by an element-minimal covering family.

2 An Upper Bound on $\mathbf{C(12, 6, 3)}$

We represent a (v, k, t) cover as an incidence matrix whose rows are the blocks and whose columns are the elements of the universe. Figure 1 displays what turns out to be a minimum, element-minimal $(12, 6, 3)$ covering family. The (i, j) entry is a '$\bullet$' if the ith block contains the jth element of the universe. (The symbols T and E stand for 10 and 11.) A subset of elements corresponding to columns between two adjacent vertical lines in a figure is called a *file*; the three files defined in Figure 1, for example, are $\{0, 1, 2, 3\}$, $\{4, 5, 6, 7\}$, and $\{8, 9, \mathrm{T}, \mathrm{E}\}$. A subset of blocks between two adjacent horizontal lines in a figure is called a *rank*. And a subset of blocks *covers* a set if one of its blocks contains the set.

Here are a few more definitions. The *degree* of an element (or a set) in a set of blocks is the number of those blocks that contain the element (or the set). The terms *pairs* and *triples* refer to unordered 2-sets and 3-sets; we sometimes abbreviate the usual set notation, writing triples, for example, in the form *abc*. And we sometimes abuse terminology by talking about a block in a cover and the corresponding block in an induced cover as if they were the same.

So in the family of twelve 6-sets and three 4-sets in Figure 1, each element has degree 7 and each pair of elements, it so happens, has degree 3. That family can't be a $(12, 6, 3)$ cover, because the last three sets aren't 6-sets, but we're about to prove that it nevertheless covers all triples.

Lemma 2.1 *The fifteen blocks in Figure 1 form an element-minimal* $(12, 6, 3)$ *covering family.*

	0	1	2	3	4	5	6	7	8	9	T	E
1	•	•			•	•			•	•		
2	•		•		•		•		•		•	
3	•			•	•			•	•			•
4	•	•					•	•			•	•
5	•		•			•		•		•		•
6	•			•		•	•			•	•	
7			•	•	•	•					•	•
8		•		•	•		•			•		•
9		•	•		•			•		•	•	
10			•	•			•	•	•	•		
11		•		•		•		•	•		•	
12		•	•			•	•		•			•
13	•	•	•	•								
14					•	•	•	•				
15									•	•	•	•

Figure 1: An element-minimal $(12, 6, 3)$ covering family of size 15.

Proof. Element minimality follows from Theorem 3.1, a result due to R. G. Stanton [5] implying that each element must have degree at least 7.

To show that any given triple is covered by the family in the figure, we assume that the triple isn't contained in a single file, since block 13, 14, or 15 would cover it, and we examine the structure of the first twelve blocks, which have the form

$$\begin{array}{|c|c|c|}\hline A & A & A \\ \hline A & B & B \\ \hline B & A & B \\ \hline B & B & A \\ \hline \end{array}, \quad \text{where} \quad A = \begin{array}{|cccc|} 0 & 1 & 2 & 3 \\ \hline \bullet & \bullet & & \\ \bullet & & \bullet & \\ \bullet & & & \bullet \\ \hline \end{array} \quad \text{and} \quad B = \begin{array}{|cccc|} 0 & 1 & 2 & 3 \\ \hline & & \bullet & \bullet \\ & \bullet & & \bullet \\ & \bullet & \bullet & \\ \hline \end{array}.$$

The columns of the 4×3 matrix of A's and B's correspond to the files of Figure 1, and the rows correspond to the first four ranks of the figure. Notice that the matrices A and B are complementary, in that an element of A is a '$\bullet$' if and only if the corresponding element of B is not, and that A covers all three pairs containing 0 while B covers the other three pairs.

We consider three cases, based on the number of distinct residues modulo 4 among the three elements of the triple.

Case 1: *exactly one distinct residue.* The AAA rank covers the triple, using any of the three blocks if the residue is 0, or using block i if the residue is i.

Case 2: *exactly two distinct residues.* If one of the residues is 0, then again the first rank covers the triple. Otherwise: If the three elements occur in just two files, the rank having B's in those two files covers the triple, using block i of the rank if the missing nonzero residue is i; and if the elements are spread among all three files, then the rank having A in the file containing the element with nonrepeated residue and having B's in the other two files covers the triple, using block i of the rank if the nonrepeated residue is i.

Case 3: *three distinct residues.* If one of the residues is 0, then the rank having A in the file containing that residue and having B's in the other two files covers the triple, using block i of the rank, where i is either the residue of the other element in the A file, if such an element exists, or is the missing residue if no such element exists. Otherwise, no residue is 0, and at least one of the files contains exactly one element, say with residue i, so the rank having A in that file and B's in the other two files covers the triple, using block i of the rank.

That establishes the lemma. ∎

Incidentally, a simple generalization of the construction above produces a $(6r, 3r, r+1)$ covering family for arbitrary $r > 0$. The A and B matrices, instead of being 3×4, are $\frac{1}{2}\binom{2r}{r} \times 2r$, but the resulting covers, of size $2 \cdot \binom{2r}{r} + 3$, are not in general minimum—in fact they are poor covers asymptotically. Nevertheless for $r = 3$ the construction gives $C(18, 9, 4) \leq 43$, the best current bound [1], cutting in half the gap between the previous best 52 and the lower bound 34.

3 Properties of a Minimum $(11, 5, 2)$ Cover

The previous section established an upper bound of 15 on $C(12, 6, 3)$. A fairly easy lower bound of 14 comes from the inequality

$$C(v,k,t) \;\geq\; \left\lceil \frac{v}{k} \left\lceil \frac{v-1}{k-1} \cdots \left\lceil \frac{v-t+1}{k-t+1} \right\rceil \cdots \right\rceil \right\rceil, \tag{1}$$

which is due to J. Schönheim [4] and is called the Schönheim bound. That bound follows from iterating the well-known inequality

$$C(v,k,t) \;\geq\; \left\lceil \frac{v}{k}\, C(v-1, k-1, t-1) \right\rceil, \tag{2}$$

which in turn follows from a simple counting argument: The number of element-in-block occurrences in a minimum (v, k, t) cover is exactly $k\,C(v, k, t)$, and it also equals the sum, over all v elements, of the size of the element-induced $(v-1, k-1, t-1)$ cover. Since that sum is at least $v\,C(v-1, k-1, t-1)$, we get $k\,C(v, k, t) \geq v\,C(v-1, k-1, t-1)$, which implies inequality (2).

For our case, the inequality says $C(12, 6, 3) \geq \left\lceil \frac{12}{6} C(11, 5, 2) \right\rceil$, which gives the lower bound of 14. But there's more to be gleaned from a minimum $(11, 5, 2)$ cover than that it's of size 7; its structural properties will allow us, in Section 4, to raise the lower bound to 15.

The main property of a minimum $(11, 5, 2)$ cover, due to R. G. Stanton [5] but stated in a slightly different form here, is that it is, in essence, unique:

Theorem 3.1 (Stanton) *Any minimum $(11, 5, 2)$ cover is isomorphic to some completion of the element-minimal covering family in Figure 2.*

	1	2	3	4	5	6	7	8	9	T	E
1	•	•	•	•	•						
2	•	•				•	•	•			
3	•	•							•	•	•
4			•				•	•		•	•
5				•	•	•				•	•
6				•	•		•	•	•		
7			•			•			•		

Figure 2: The minimum, element-minimal $(11, 5, 2)$ covering family.

Remarks. The covering family of Figure 2 is what R. C. Mullin [3] calls a star design. Of its possible completions, only two covers are nonisomorphic. The covering family is element minimal because each element has degree 3, which the is minimum, since an element's induced $(10, 4, 1)$ covering family must have size at least $\lceil 10/4 \rceil = 3$.

Either explicit or implicit in Stanton's paper are several properties of minimum $(11, 5, 2)$ covers that we'll find useful; they are stated without proof and are all evident from Figure 2. (They concern covers rather than covering families.)

Property 3.2 *No element has degree 5 or more, two elements have degree 4, and nine elements have degree 3.*

Property 3.3 *At least one of the blocks contains elements just of degree 3.*

That property makes it easy to see why there are only two nonisomorphic covers: Assume without loss of generality that block 1 contains only elements of degree 3. Then block 7, the triple, must be completed with two elements from 78TE, either both from the same file (78 or TE) or, nonisomorphically, split across the two files (7T, 7E, 8T, or 8E); the two cases are nonisomorphic because only in the former is there a pair of degree 4.

Property 3.4 *Exactly four pairs of elements have degree 3 or more. Those pairs are disjoint, and at most one of the pairs has degree 4.*

It's time for a few more definitions, which apply to minimum $(11, 5, 2)$ covers. A *couple* is one of the pairs of degree 3 or 4, and an element of a couple is a *spouse* of the other element. A block is *short* if it contains at most one couple—it corresponds to the 3-set of the element-minimal covering family. A block is *full* if it's not short. An element is *free* if it has degree 4—it corresponds to an element of the short block added upon completion. An element is *short* if it's contained in a short block but is not free—it has degree 3. And a *triad* comprises the three other elements in a full block containing a specified *marking* couple.

Property 3.5 *The eleven elements are partitioned uniquely into three short elements and four couples.*

Property 3.6 *Two of the eight nonshort elements are free.*

Property 3.7 *A short block comprises three short elements and two free elements.*

Property 3.8 *A free element occurs in three full blocks and in the short block.*

Property 3.9 *Any pair containing a nonshort element occurs in some full block.*

Property 3.10 *The seven blocks consist of one short block and six full blocks.*

Property 3.11 *A full block contains exactly two couples.*

Property 3.12 *Two full blocks meet in exactly a couple or exactly a short element.*

Property 3.13 *The short block meets a full block in at least a short element.*

Property 3.14 *The short block meets a triad in at least a short element.*

Property 3.15 *A couple has degree 3 in the full blocks.*

Property 3.16 *A triad consists of one short element and one couple.*

Property 3.17 *A couple marks three triads. For each such triad: In the three full blocks not containing the marking couple, the triad's couple occurs exactly twice and its short element occurs exactly once, in three separate blocks.*

For example in Figure 2 the couple 12 marks the triads 345, 678, and 9TE; in the other three full blocks, triad 345's couple 45 occurs twice, in blocks 5 and 6, and its short element 3 occurs once, in block 4.

Property 3.18 *A block is short if it contains exactly one element of a couple.*

Property 3.19 *A block is short if it contains two short elements.*

Property 3.20 *If a pair has degree 2, one of the two blocks is short,*

Property 3.21 *If a pair has degree 2, one of the two elements is free (it has degree 4).*

Property 3.22 *If a pair occurs in three blocks, it's a couple.*

Property 3.23 *If a pair occurs in two full blocks, it's a couple.*

4 The Matching Lower Bound on C(12, 6, 3)

The lower-bound proof follows two simple facts about $(12, 6, 3)$ covers of size 14.

Proposition 4.1 *Each element has degree 7.*

Proof. Each element has degree at least 7, since any induced $(11, 5, 2)$ cover has at least seven blocks; but there are only $14 \cdot 6 = 84 = 12 \cdot 7$ total occurrences to go around, so the twelve elements each have degree 7. ∎

Proposition 4.2 *No pair of elements has degree 5 or more.*

Proof. Each element has degree 7, so each induced $(11, 5, 2)$ cover is minimum, and since by Property 3.2 no element has degree 5 or more in such a cover, no pair has degree 5 or more in the $(12, 6, 3)$ cover (of size 14). ∎

Lemma 4.3 *The covering number $C(12, 6, 3) \geq 15$.*

Proof. Assume a size-14 cover. Element 0 has degree 7, so by Theorem 3.1 we may take the first seven blocks as below, with the two free elements of block 7

	0	1	2	3	4	5	6	7	8	9	T	E
1	•	•	•	•	•	•						
2	•	•	•				•	•	•			
3	•	•	•							•	•	•
4	•			•				•	•		•	•
5	•				•	•	•				•	•
6	•				•	•		•	•	•		
7	•			•			•			•		

coming from 78TE. Element 0 occurs no more, and elements 1 and 2 occur four more times each. But by Proposition 4.2 the pair 12 has degree at most 4, so the remaining seven blocks are three $1xxxxx$'s, three $2xxxxx$'s, and one $12xxxx$, where $x \neq 0$, 1, or 2. Now in the 1-cover the pair 02 occurs in blocks 1, 2, and 3 and thus by Property 3.22 is a couple. Furthermore the block $12xxxx$ is short, by Property 3.18, so by Property 3.10 all three blocks 1, 2, and 3 are full, hence the couple 02 marks the triads 345, 678, and 9TE. Similarly, in the 2-cover the couple 01 marks the same triads. Moreover since the block corresponding to $12xxxx$ is short in each of the two covers, elements 1 and 2, in each other's cover, belong to couples and thus are not short, by Property 3.5. Thus the 1- and 2-covers share a short element a (in fact they share at least two), and since element a is in a triad abc among 345, 678, and 9TE that occurs in both covers, the three $1xxxxx$'s must, by Property 3.17, contain the couple bc twice, and so must the three $2xxxxx$'s. Hence given its occurrence in block 1, 2, or 3 the pair bc has degree at least 5, contradicting Proposition 4.2. ∎

Lemmas 2.1 and 4.3 prove that size 15 is the minimum:

Theorem 4.4 *The covering number $C(12, 6, 3) = 15$.*

5 The Two Classes of Minimum Covers

A final few definitions for minimum $(12,6,3)$ covers: An *a-element* has degree a. An *abc-triple* consists of an a-element, a b-element, and a c-element. A *duplicate (triplicate) triple* has degree at least 2 (at least 3). An *ab-couple*, in the induced cover of a 7-element, is a couple consisting of an a-element and a b-element (counted in the $(12,6,3)$ cover). The *excess* of an element is seven fewer than its degree. The excess of a set of elements is the sum of their excesses. And an element or a block is a-short if it is short in the induced cover of the 7-element a.

Proposition 5.1 *A minimum* $(12,6,3)$ *cover contains six or more 7-elements.*

Proof. Since each element has degree at least 7, and since there are only $15 \cdot 6 = 90$ total occurrences, at most $90 - 12 \cdot 7 = 6$ elements have degree more than 7, so at least six have degree 7. ∎

Lemma 5.2 *Any 7-element in a minimum* $(12,6,3)$ *cover has degree 3 in the induced cover of another 7-element.*

Proof. Assume to the contrary that 0 and 1 are 7-elements, and that 1 has degree 4 in the 0-cover. (It has degree 3 or 4 by Property 3.2.) Since 1 is not short, let 2 be 1's spouse; let the couple 12 mark the triads 345, 678, and 9TE in blocks 1, 2, and 3; let 3, 6, and 9 be the corresponding short elements; and let block 7 be short. Now consider the 1-cover. The pair 02 is a couple, by Property 3.22, and none of blocks 1, 2, or 3 is short, by Property 3.13, so it must be block 7—the other one containing the free element 0—that's short, by Property 3.8. Couple 02 marks the triads 345, 678, and 9TE in blocks 1, 2, and 3; and at least two of the 0-short elements 3, 6, and 9, say 3 and 6, are also 1-short, and thus the files 345 and 678 in blocks 4, 5, and 6 are duplicated in blocks 8, 9, and 10, by Property 3.17. The first ten blocks thus start out

	0	1	2	3	4	5	6	7	8	9	T	E
1	•	•	•	•	•	•						
2	•	•	•				•	•	•			
3	•	•	•							•	•	•
4	•			•				•	•		•	•
5	•				•	•	•				•	•
6	•				•	•		•	•	•		
7	•	•	?	•	?	?	•	?	?	•	?	?
8		•		•				•	•	?	?	?
9		•			•	•	•			?	?	?
10		•			•	•		•	•	?	?	?

with one more element in block 7 along with the uncertain region in blocks 8, 9, and 10 to be filled in. Now by Proposition 5.1 there are at least four 7-elements besides 0 and 1. But none of 4, 5, 7, or 8 is a 7-element, because its spouse in the 0-cover would have degree at least 5 in its own induced cover, violating Property 3.2; nor is 3 or 6 a 7-element, because the pair 01 in the 3- or 6-cover would violate Property 3.21, as both 0 and 1 have degree 3 in each. Thus 2, 9, T, and E are 7-elements. But 9 can't be 1-short, since then it would by Property 3.17 have degree 1 among blocks 8, 9, and 10, and thus in the 9-cover the pair 01 would again violate Property 3.21; so either T or E is 1-short, say T. Then among 9, T, and E, only T occurs in block 10, and only 9 and E occur in blocks 8 and 9, by Property 3.17. Finally, in the E-cover the pair 0T is a couple that marks the triads 129, 378, and 456 (in blocks 3, 4, and 5); but block 9 isn't short, because it misses the triad 378 completely, contrary to Property 3.14, yet it's also not full, as it contains the complete triad 456, violating Property 3.12. That contradiction finishes the lemma. ∎

Corollary 5.3 *In a minimum* $(12, 6, 3)$ *cover, if a 7-element x is in the short block of a y-cover (y is a 7-element), then x is y-short.*

Proof. By the preceding lemma, x is not free, hence by Property 3.7 it is short.

Lemma 5.4 *In a minimum* $(12, 6, 3)$ *cover, if a 7-element x is y-short, then conversely y is x-short.*

Proof. In the y-cover, x occurs in two full blocks meeting in just x, by Property 3.12, hence in the x-cover those blocks meet in just y; so y is x-short, by Properties 3.12 and 3.13. ∎

Lemma 5.5 *In a minimum* $(12, 6, 3)$ *cover, if a block is both x- and y-short, then for any other 7-element z in the block, it's also z-short.*

Proof. By the previous corollary and lemma, both x and y are z-short, hence by Property 3.19 the block, too, is z-short. ∎

Next we consider minimum covers without duplicate 777-triples. Part of those covers' structure, derived in the next paragraph, is used both in Lemmas 5.6 and 5.7 and is shown in files 012 and 345 and blocks 1 through 7 of Figure 3.

The excess of the twelve elements in a minimum cover is six. If the cover has no duplicate 777-triple, then the induced cover of any 7-element, say element 0, contains no 77-couple, so the excess of each of the four couples is at least 1, hence there is a couple with excess exactly 1. Let its 7-element be 1 and its 8-element be 2, so that 012 is a triplicate 778-triple. In the 0-cover let the couple 12 mark the triads 345, 678, and 9TE in blocks 1, 2, and 3, let blocks 4, 5, and 6 also be full, and let block 7 be short. Similarly, in the 1-cover the couple 02 marks in blocks 1, 2, and 3 the same three triads; let blocks 8, 9, and 10 be full and block 11 be short. Since there are at least four 7-elements among 3 through E, at least one of the triads contains exactly two 7-elements,

	0	1	2	3	4	5	6	7	8	9	T	E
1	•	•	•	•	•	•						
2	•	•	•				•	•	•			
3	•	•	•							•	•	•
4	•			•				•	•		•	•
5	•				•	•	•				•	•
6	•				•	•		•	•	•		
7	•		?	•		?	•	?	?	•	?	?
8		•			•			•	•		•	•
9		•		•		•	•				•	•
10		•		•		•		•	•	•		
11		•	•		•	?	•	?	?	•	?	?
12			?			?	?	?	?	?	?	?
13			•	•	•		•	•	•			
14			•	•	•					•	•	•
15			?			?	?	?	?	?	?	?

Figure 3: An unfinishable partial $(12, 6, 3)$ cover.

say elements 3 and 4. (No triad may contain three 7-elements, since each couple has positive excess.) Furthermore, each of 3 and 4 is either 0-short or 1-short but not both, since, of the triad 345, (a) element 5 can't be 0- or 1-short, as that would create a couple 34 without positive excess, and (b) element 3, say, can't be short in both covers, as 5 would then have degree at least 5 in the 4-cover, contrary to Property 3.2. Thus let 3 be 0-short and let 4 be 1-short. So by block 11 elements 3 and 4 have occurred exactly five times, exactly once together, hence each must occur exactly twice more, and thus together, say in blocks 13 and 14. Now the 8-element 2 doesn't occur in blocks 4 through 6 or 8 through 10, so it must occur at least once in blocks 7 or 11, say 11. It must also occur at least once in blocks 13 and 14; but then it must occur in *both* blocks 13 and 14, since otherwise in the 4-cover the pair 23 would violate Property 3.21. Moreover element 5 can't occur in either block 13 or 14, because the six elements 6 through E must each occur in those two blocks, lest one of the six triples $34x$, for x among 6 through E, be uncovered. Finally, by Property 3.18 block 11 is 4-short, due to the couple 23.

Lemma 5.6 *If a minimum* $(12, 6, 3)$ *cover has no duplicate 777-triple then no element is both* x*- and* y*-short for 7-elements* x *and* y *of a triplicate 778-triple.*

Proof. Assume to the contrary that 6 is both 0- and 1-short, as shown in the first eleven blocks of the file 678 of Figure 3. (For now, ignore the rest of that file, along with blocks 8 through 15 of file 9TE.) In the 4-cover, the couple 05 marks the triads 123, 6TE, and 789, and by Property 3.23 the pair 78 is a couple, since it occurs in the full blocks 6 and 8 (block 11 is short). Furthermore, by

	0	1	2	3	4	5	6	7	8	9	T	E
1	•	•	•	•	•	•						
2	•	•	•				•	•	•			
3	•	•	•							•	•	•
4	•			•				•	•		•	•
5	•				•	•	•				•	•
6	•				•	•		•	•	•		
7	•			•			•			•		
8		•			•		•		•	•		•
9		•		•		•		•		•		•
10		•		•		•	•		•		•	
11		•			•			•			•	
12			•			•	•	•		•	•	
13			•	•	•				•	•	•	
14			•	•	•		•	•				•
15			•			•			•			•

Figure 4: A canonical element-minimal $(12, 6, 3)$ covering family.

Property 3.12 the full blocks 6 and 8 can't both contain element 9, hence 9 must be missing from block 8, which by Property 3.17 makes 9 the 1-short element in the triad 9TE, so that the file 9TE in blocks 8 through 11 is as shown in the figure. Therefore back in the 4-cover, Property 3.23 shows that the pair TE is a couple, along with 78; and in blocks 13 and 14 each couple occurs with the short element of the other's 05-marked triad (the triads are 6TE and 789), since if, say, 78 occurred with the short element 9 from its own triad Property 3.12 would be violated (recall that each element 6 through E has degree 1 in those two blocks). Thus we may take blocks 13 and 14 as in the figure. Finally, since each pair 78 and TE has degree at least 5, Property 3.2 ensures that no element in 78TE is a 7-element; and neither 2 nor 5 is a 7-element, since in the 1-cover each couple (02 and 35 in particular) has positive excess. So 6 and 9 must be 7-elements. But in the 6-cover blocks 2 and 13 meet in a triple, as do blocks 5 and 9, contradicting Property 3.12 and completing the lemma. ∎

Lemma 5.7 *A minimum* $(12, 6, 3)$ *cover without a duplicate 777-triple is isomorphic to some completion of the covering family in Figure 1.*

Proof. The covering family in Figure 1, after permuting the elements by (267T8)(13954) and the blocks by (7 8 9 10 12 14 11 13), becomes the one in Figure 4, so to prove the lemma it suffices to use the covering family of Figure 4, which more clearly shows the $(11, 5, 2)$ structure of the 0-, 1-, and 2-covers.

By Lemma 5.6 and the paragraph preceding it, we may take 012 to be a 778-triplicate triple for which none of the 0-short elements, say 3, 6, and 9, is

	0	1	2	3	4	5	6	7	8	9	T	E
1	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$						
2	$\bullet$	$\bullet$	$\bullet$				$\bullet$	$\bullet$	$\bullet$			
3	$\bullet$	$\bullet$	$\bullet$							$\bullet$	$\bullet$	$\bullet$
4	$\bullet$			$\bullet$				$\bullet$	$\bullet$		$\bullet$	$\bullet$
5	$\bullet$				$\bullet$	$\bullet$	$\bullet$				$\bullet$	$\bullet$
6	$\bullet$				$\bullet$	$\bullet$		$\bullet$	$\bullet$	$\bullet$		
7	$\bullet$		?	$\bullet$		?	$\bullet$	?	?	$\bullet$	?	?
8		$\bullet$			$\bullet$		$\bullet$		$\bullet$	$\bullet$		$\bullet$
9		$\bullet$		$\bullet$		$\bullet$		$\bullet$		$\bullet$		$\bullet$
10		$\bullet$		$\bullet$		$\bullet$	$\bullet$		$\bullet$		$\bullet$	
11		$\bullet$	$\bullet$		$\bullet$	?	?	$\bullet$	?	?	$\bullet$	?
12			?			?	?	?	?	?	?	?
13			$\bullet$	$\bullet$	$\bullet$				$\bullet$	$\bullet$	$\bullet$	
14			$\bullet$	$\bullet$	$\bullet$		$\bullet$	$\bullet$				$\bullet$
15			?			?	?	?	?	?	?	?

Figure 5: A finishable partial $(12, 6, 3)$ cover.

among the 1-short elements, say 4, 7, and T. Thus the first eleven blocks, along with files 012 and 345, in which 3 and 4 are 7-elements, are as in Figure 5. Furthermore in the 4-cover the couple 05 marks the triads 123, 6TE, and 789, and by Property 3.23 the pairs 6E and 89 are couples, since each occurs in two full blocks. Thus in blocks 13 and 14 each of those couples occurs with the short element from the other couple's triad, and we've reached the state of Figure 5.

It remains to show that, as in Figure 4, block 12 is exactly 25679T and block 15 contains 258E. Let a and b be two 7-elements among 6 through E. Eight of the $\binom{6}{2}$ possible cases—a is in 679T and b is in 8E—are disallowed, because ab would be a 77-couple in the induced cover of the 7-element 0, 1, 3, or 4, yielding a forbidden duplicate 777-triple. For the remaining cases—ab is 8E or is chosen from 679T—a and b have occurred, outside of blocks 12 and 15, exactly six times, exactly twice together (neither a nor b corresponds to a '?' of blocks 7 or 11 of Figure 5, as a or b would then be free in the 0- or 1-cover, violating Lemma 5.2), so in blocks 12 and 15 they must each occur just once, together. If ab is 8E then the pair must occur, say in block 15, with both 2 and 5, since neither triple 258 nor 25E occurs elsewhere; thus block 15 contains 258E, as required. Also as required, block 12 must be exactly 25679T, since the other six elements have used up their seven occurrences elsewhere. Finally, if ab is chosen from 679T then each of a and b is 0- or 1-short. Now block 7 is 3-short, by Property 3.11, since it contains at most one of the 3-cover's couples 15, 24, 7E, and 8T. So block 7 is 0- and 3-short, and, as we've seen, block 11 is 1- and 4-short, hence by Lemma 5.5 the short block of each of a and b is 7 or 11, whichever is relevant. Thus a and b must occur, say in block 12, together with

	0	1	2	3	4	5	6	7	8	9	T	E
1	•	•	•	•	•	•						
2	•	•	•				•	•	•			
3	•	•	•							•	•	•
4	•			•				•	•		•	•
5	•				•	•	•				•	•
6	•				•	•		•	•	•		
7	•			•	•	•	•			•		
8		•		•			•		•	•		•
9		•			•	•		•		•		•
10		•			•	•	•		•		•	
11		•		•	•	•		•			•	
12			•	•			•	•		•	•	
13			•		•	•			•	•	•	
14			•		•	•	•	•				•
15			•	•	•	•			•			•

Figure 6: The other minimum $(12, 6, 3)$ cover.

the other two elements x and y from 679T, since otherwise neither triple abx nor aby would occur outside both a's short block and b's short block, violating Property 3.9 (for example bx, at most one of whose elements is a-short, would be the violating pair in the a-cover). Similarly, 2 and 5 must be in block 12 if the triple $25a$ is to occur outside a's short block. Hence again block 12 is 25679T, and block 15, which must cover 28E and 58E, therefore contains 258E. Both cases yield the desired covers, and the lemma is proved. ∎

Finally we turn to minimum covers that contain a duplicate 777-triple.

Lemma 5.8 *A minimum* $(12, 6, 3)$ *cover containing a duplicate 777-triple is isomorphic to some completion of the covering family in Figure 1 or to the cover of Figure 6.*

Proof. The duplicate 777-triple must also be triplicate, since otherwise if a, b, and c are the three 7-elements, the pair ab has degree 2 in the c-cover and hence by Property 3.21 contains a free element, violating Lemma 5.2. Thus we may take blocks 1, 2, and 3 and file 012 to be as in Figure 6. Since 0, 1, and 2 are 7-elements, every block is in either the 0-, 1-, or 2-cover, and the locations of the short elements in blocks 7, 11, and 15 completely determine the nine remaining full blocks. We distinguish three cases, based on the distribution of the 0-, 1-, and 2-short elements among 3 through E.

Case 1: *no element is 0-, 1-, or 2-short twice.* Any such cover is isomorphic to a completion of the covering family in Figure 4, and therefore Figure 1.

Case 2: *an element is 0-, 1-, and 2-short.* Assume it's element 3. Then none of the twelve triples $3yz$, for y in 45 and z in 6789TE, occurs outside

	0	1	2	3	4	5	6	7	8	9	T	E
1	●	●	●	●	●	●						
2	●	●	●				●	●	●			
3	●	●	●							●	●	●
4	●			●				●	●		●	●
5	●				●	●	●				●	●
6	●				●	●		●	●	●		
7	●			●	?	?	●	?	?	●	?	?
8		●		●			●		●	●		●
9		●			●	●		●		●		●
10		●			●	●	●		●		●	
11		●		●	?	?	?	●	?	?	●	?
12			●		●		●	●		●	●	
13			●	●		●			●	●	●	
14			●	●		●	●	●				●
15			●	●	●	?	?	?	●	?	?	●

Figure 7: Another unfinishable partial $(12, 6, 3)$ cover.

the 0-, 1-, and 2-short blocks 7, 11, and 15, and at most four such triples can occur in such a block, so exactly four must occur in each. Thus 45 occurs in all three blocks, and we may take the remaining 0-, 1-, and 2-short elements as in Figure 6, which determines the rest. It remains to show that what's in the figure is actually a cover. Any triple containing 0, 1, or 2 is covered, since the 0-, 1-, and 2-covers are completions of the covering family in Figure 2. And it's not hard to show that the remaining $\binom{9}{3} = 84$ triples in 3456789TE are covered; we omit the details.

Case 3: *an element is 0-, 1-, or 2-short exactly twice.* We may assume that element 3 is 0- and 1-short, and that 4 is 2-short, as in file 345 of Figure 7. If 3 were a 7-element, it wouldn't occur in block 15, and block 1 would be 3-short, because it's the only block in the 3-cover that contains both 0 and 1, which by Lemma 5.4 are both 3-short. But then 2 would be 3-short, by Corollary 5.3, and vice versa, by Lemma 5.4, contradicting 3's nonoccurrence in block 15, and showing that 3 is an 8-element and that it occurs in block 15. Furthermore neither 4 nor 5 is a 7-element, since the pair 45 has degree at least 5 and thus the 4- or 5-cover would violate Property 3.2. So any file with an element that's 0-, 1-, or 2-short exactly twice contains no 7-element and has excess at least 3. Now to cover all six triples $34z$, for z in 6789TE, either 34 occurs three times in blocks 7, 11, and 15, in which case file 345 has excess at least 4, or it occurs just twice in those blocks, in which case files 678 and 9TE each have excess at least one; but in both cases, neither file 678 nor 9TE has excess 3, hence neither file has an element that's 0-, 1-, or 2-short twice, and we've reached the state of Figure 7. Finally, since one of the last two files contains two 7-elements, we may assume that the 0-short element 6 is a 7-element. Then in the 6-cover: Element 0 is short, by Lemma 5.4; the pair 39 occurs exactly twice—in blocks 7 and 8—so block 7 is 6-short, as it's the one containing 0; and the pair 45 has degree 2 outside the short block, contradicting Property 3.15. Therefore Case 3 is impossible, completing the lemma. ∎

Theorem 5.9 *Any minimum* $(12, 6, 3)$ *cover is isomorphic to a completion of the element-minimal covering family in Figure 1 or to the element-minimal cover of Figure 6, but not both.*

Proof. The isomorphisms and the covering family's element minimality follow from Lemmas 5.7, 5.8, and 2.1. The cover is element minimal, since (i) each element but 4 and 5 has degree 7, the minimum, and (ii) there exists, for x in 012, for y in 45, for s an x-short element, and for n an x-nonshort element, a nonduplicate triple of the form $01y$ in block 1, xyn in blocks 5, 6, 9, 10, 13, and 14, and yss in blocks 7, 11, and 15. Finally, the pair 45 has degree 10 in the cover, which thus is not isomorphic to a completion of the covering family. ∎

Remarks. Our computer calculations have shown that the covering family of Figure 1 has exactly 67 nonisomorphic completions, giving exactly 68 nonisomorphic minimum $(12, 6, 3)$ covers in all. Also, the relation defined by two minimum covers being completions of the same element-minimal covering family is not in general an equivalence relation, since there exist three minimum $(13, 4, 1)$ covers for which transitivity fails. Still, for the $(12, 6, 3)$ case this classification scheme is much nicer (two classes versus 68) than simple isomorphism.

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