Time-Average Stochastic Optimization with Non-convex Decision Set and its Convergence

Sucha Supittayapornpong, Michael J. Neely
Department of Electrical Engineering
University of Southern California
Los Angeles, California
Email: supittay@usc.edu, mjneely@usc.edu

***Abstract*—This paper considers *time-average stochastic optimization*, where a time average decision vector, an average of decision vectors chosen in every time step from a time-varying (possibly non-convex) set, minimizes a convex objective function and satisfies convex constraints. This formulation has applications in networking and operations research. In general, time-average stochastic optimization can be solved by a Lyapunov optimization technique. This paper shows that the technique exhibits a transient phase and a steady state phase. When the problem has a unique vector of Lagrange multipliers, the convergence time can be improved. By starting the time average in the steady state, the convergence times become $O(1/\epsilon)$ under a locally-polyhedral assumption and $O(1/\epsilon^{1.5})$ under a locally-non-polyhedral assumption, where ϵ denotes the proximity to the optimal objective cost.**

I. Introduction

Stochastic network optimization can be used to design dynamic algorithms that optimally control communication networks [1]. The technique has several unique properties which do not exist in a traditional convex optimization setting. In particular, the technique allows for a time-varying and possibly non-convex decision set. For example, it can treat a packet switch that makes binary (0/1) scheduling decisions, or a wireless system with varying channels and decision sets.

This paper considers *time-average stochastic optimization*, which is useful for example problems of network utility maximization [2]–[5], energy minimization [6], [7], and quality of information maximization [8].

Time $t \in \{0, 1, 2, \ldots\}$ is slotted. Define Ω to be a finite or countably infinite sample space of random states. Let $\omega(t) \in \Omega$ denote a random state at time t. Random state $\omega(t)$ is assumed to be independent and identically distributed (i.i.d.) across time slots. The steady state probability of $\omega \in \Omega$ is denoted by π_ω. Let I and J be any positive integers. Each slot t, decision vector $x(t) = (x_1(t), \ldots, x_I(t))$ is chosen from a decision set $\mathcal{X}_{\omega(t)}$. For any positive integer T, define $\overline{x}(T)$ as

$$\overline{x}(T) \triangleq \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}\left[x(t)\right].$$

The goal is to make decisions over time to solve:

$$\begin{aligned} \text{Minimize} \quad & \limsup_{T\to\infty} f(\overline{x}(T)) && && (1)\\ \text{Subject to} \quad & \limsup_{T\to\infty} g_j(\overline{x}(T)) \leq 0 && j \in \{1, \ldots, J\} \\ & x(t) \in \mathcal{X}_{\omega(t)} && t \in \{0, 1, 2, \ldots\}. \end{aligned}$$

Here it is assumed that $\mathcal{X}_\omega$ is a compact subset of $\mathbb{R}^I$ for each $\omega \in \Omega$. Assume $\cup_{\omega\in\Omega}\mathcal{X}_\omega$ is bounded, and let $\mathcal{C}$ be a compact set that contains it. The functions f and g_j are convex functions from $\overline{\mathcal{C}}$ to $\mathbb{R}$, where $\overline{\mathcal{A}}$ denotes a convex hull of set $\mathcal{A}$. Results in [1] imply that the optimal point can be achieved with an ergodic policy for which the limiting time average expectation exists.

An example of formulation (1) can be a resource allocation problem of a stochastic wireless uplink network. The goal is to achieve:

$$\begin{aligned} \text{Minimize} \quad & \limsup_{T\to\infty} -\sum_{i=1}^{3} \log\left(\overline{x}_i(T)\right) \\ \text{Subject to} \quad & \limsup_{T\to\infty} -\overline{x}_i(T) \leq -x_i^{(\min)} && i \in \{1, 2, 3\} \\ & x(t) \in \mathcal{X}_{\omega(t)} && t \in \{0, 1, 2, \ldots\}, \end{aligned}$$

where $x_i^{(\min)}$ is the mininum rate for user i. In this example, $\Omega = \{1, 2\}$, $\pi_1 = 0.3, \pi_2 = 0.7$, $\mathcal{X}_1 = \{(0,0,0), (2,1,0), (0,2,2)\}$, $\mathcal{X}_2 = \{(0,0,0), (0,1,2), (1,1,1)\}$.

Solving formulation (1) using the stochastic network optimization framework does not require any statistical knowledge of the random states. However, if the steady state probabilities are known, the optimal objective cost of formulation (1) is identical the optimal cost of the following problem:

$$\begin{aligned} \text{Minimize} \quad & f(x) && && (2)\\ \text{Subject to} \quad & g_j(x) \leq 0 && j \in \{1, \ldots, J\} \\ & x \in \overline{\mathcal{X}}, \end{aligned}$$

where $\overline{\mathcal{X}} \triangleq \sum_{\omega\in\Omega} \pi_\omega \overline{\mathcal{X}}_\omega$. Note that, for any $\alpha, \beta \in \mathbb{R}$ and any sets $\mathcal{A}$ and $\mathcal{B}$, notation $\alpha\mathcal{A} + \beta\mathcal{B} = \{\alpha a + \beta b : a \in \mathcal{A}, b \in \mathcal{B}\}$.

Formulation (2) is convex; however, its optimal solution may not be in any of the sets $\mathcal{X}_\omega$. In fact, determining whether x is a member of $\overline{\mathcal{X}}$ may already be a difficult task. This illustrates that traditional and state-of-the-art techniques for solving convex optimization cannot be applied directly to solve problem (1). Nevertheless, their convergence times are compelling to be mentioned for a purpose of comparison.

The convergence time of an algorithm is usually measured as a function of an $O(\epsilon)$-approximation to the optimal solution. For a convex optimization problem, several techniques utilizing a time average solution [9]–[11] have $O(1/\epsilon^2)$ convergence time. For unconstrained optimization without a more restrictive "strong convexity" property, the optimal first-order method

This material is supported by the NSF Career grant CCF-0747525.

[12], [13] has $O(1/\sqrt{\epsilon})$ convergence time, while the gradient and subgradient methods have respectively $O(1/\epsilon)$ and $O(1/\epsilon^2)$ convergence time [14]. For constrained optimization, two algorithms developed in [15], [16] have $O(1/\epsilon)$ convergence time; however, the results rely on special structures of their formulation. All of these results are for convex optimization problems, which are not formulation (1).

This paper considers a drift-plus-penalty algorithm, developed in [1], that solves formulation (1). The algorithm is shown to have $O(1/\epsilon^2)$ convergence time in [17]. Note that a deterministic version of formulation (1) and its corresponding convergence are studied in [18]. Despite the similar analysis procedures, the analysis in this work is more challenging due to stochastic events and multiple decision sets.

Inspired by the analysis in [19], the drift-plus-penalty algorithm is shown to have a *transient phase* and a *steady state phase*. These phases are analyzed in two cases that depend on the structure of a dual function. The first case is when a dual function satisfies a *locally-polyhedral* assumption and the transient time is $O(1/\epsilon)$. The second case is when the dual function satisfies a *locally-non-polyhedral* assumption and the transient time is $O(1/\epsilon^{1.5})$. Then, under a uniqueness assumption on a vector of Lagrange multipliers, if the time average starts in the steady state phase, a solution converges in $O(1/\epsilon)$ and $O(1/\epsilon^{1.5})$ time slots under the locally-polyhedral and locally-non-polyhedral assumptions respectively.

Even though this characterization of the time complexity is important in its own right, it can also be used to improve the convergence speed of a system if enough prior samples of random states are available. This can be done by using those samples in the transient phase and only implementing decisions in the steady state phase. Recent work in this direction is in [20], which considers methods to improve transient times that can likely be used in conjunction with results in the current paper.

The paper is organized as follows. Section II constructs an algorithm solving problem (1). The behavior and properties of the algorithm are analyzed in Section III. Section IV analyzes the transient phase and the steady state phase under the locally-polyhedral assumption. Results under the locally-non-polyhedral assumption are provided in Section V. Simulations are performed in Section VI.

II. Time-Average Stochastic Optimization

A solution to problem (1) can be obtained through an auxiliary problem, which is formulated in such a way that its optimal solution is also an optimal solution to the time-average problem. To formulate this auxiliary problem, an additional set and mild assumptions are defined. First of all, it can be shown that $\overline{\mathcal{X}}$ is compact.

Assumption 1: There exists a vector $\hat{x}$ in the interior of $\overline{\mathcal{X}}$ that satisfies $g_j(\hat{x}) < 0$ for all $j \in \{1, \dots, J\}$.

In convex optimization, Assumption 1 is a *Slater condition*, which is a sufficient condition for strong duality [21].

Define the *extended set* $\mathcal{Y}$ that is a compact and convex subset of $\mathbb{R}^I$ and contains $\overline{\mathcal{X}}$. Set $\mathcal{Y}$ can be $\overline{\mathcal{X}}$, but it can be defined as a hyper-rectangle set to simplify a later algorithm. Define $\|\cdot\|$ as the Euclidean norm.

Assumption 2: Functions f and g_j for $j \in \{1, \dots, J\}$ are convex and *Lipschitz continuous* on the extended set $\mathcal{Y}$, so there are constants $M_f > 0$ and $M_{gj} > 0$ for $j \in \{1, \dots, J\}$ that for any $x, y \in \mathcal{Y}$:

$$|f(x) - f(y)| \leq M_f \|x - y\| \tag{3}$$
$$|g_j(x) - g_j(y)| \leq M_{gj} \|x - y\|. \tag{4}$$

We assume that Assumptions 1 and 2 always hold in this paper.

A. Auxiliary formulation

For function $a(x(t))$ of vector $x(t)$, define an average of function values as

$$\overline{a(x)} \triangleq \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}\left[a(x(t))\right].$$

Recall that problem (1) can be achieved with an ergodic policy for which the limiting time average expectation exists. The time average stochastic optimization (1) is solved by considering an *auxiliary formulation*, which is formulated in terms of well defined limiting expectations for simplicity.

$$\begin{array}{lll} \text{Minimize} & \overline{f(y)} & (5) \\ \text{Subject to} & \overline{g_j(y)} \leq 0 & j \in \{1, \dots, J\} \\ & \lim_{T\to\infty} \overline{x}_i(T) = \lim_{T\to\infty} \overline{y}_i(T) & i \in \{1, \dots, I\} \\ & (x(t), y(t)) \in \mathcal{X}_{\omega(t)} \times \mathcal{Y} & t \in \{0, 1, 2, \dots\}. \end{array}$$

This formulation introduces the auxiliary vector $y(t)$. The second constraint ties $\lim_{T\to\infty} \overline{x}(T)$ and $\lim_{T\to\infty} \overline{y}(T)$ together, so the original objective function and constraints of problem (1) are preserved in problem (5). Let $f^{(\text{opt})}$ be the optimal objective cost of problem (1).

Theorem 1: The time-average stochastic problem (1) and the auxiliary problem (5) have the same optimal cost, $f^{(\text{opt})}$.

Proof: Please see the full proof in [22]. ■

B. Lyapunov optimization

The auxiliary problem (5) can be solved by the Lyapunov optimization technique [1]. Define $W_j(t)$ and $Z_i(t)$ to be *virtual queues* of the first and second constraints of problem (5) with update dynamics:

$$W_j(t+1) = \left[W_j(t) + g_j(y(t))\right]_+ \quad j \in \{1, \dots, J\} \tag{6}$$
$$Z_i(t+1) = Z_i(t) + x_i(t) - y_i(t) \quad i \in \{1, \dots, I\}, \tag{7}$$

where operator $[\cdot]_+$ is the projection to a corresponding non-negative orthant.

For ease of notations, let $W(t) \triangleq (W_1(t), \dots, W_J(t))$, $Z(t) \triangleq (Z_1(t), \dots, Z_I(t))$, and $g(y) \triangleq (g_1(y), \dots, g_J(y))$ respectively be the vectors of virtual queues $W_j(t)$, $Z_i(t)$, and functions $g_j(y)$.

Define Lyapunov function (8) and Lyapunov drift (9) as

$$L(t) \triangleq \frac{1}{2}\left[\|W(t)\|^2 + \|Z(t)\|^2\right] \tag{8}$$
$$\Delta(t) \triangleq L(t+1) - L(t). \tag{9}$$

Let notation $A^\top$ denote the transpose of vector A. Define $C \triangleq \sup_{x\in\mathcal{C},y\in\mathcal{Y}} \left[\|g(y)\|^2 + \|x-y\|^2\right]/2$, which is finite.

Lemma 1: For every $t \in \{0,1,2,\ldots\}$, the Lyapunov drift is upper bounded by

$$\Delta(t) \leq C + W(t)^\top g(y(t)) + Z(t)^\top [x(t) - y(t)]. \quad (10)$$

Proof: The proof is similar to the one in [18] and [22]. ■

Let $V > 0$ be any positive real number representing a parameter of an algorithm solving problem (5). The drift-plus-penalty term is defined as $\Delta(t) + Vf(y(t))$. Applying Lemma 1, the drift-plus-penalty term is bounded for every time t by

$$\Delta(t)+Vf(y(t)) \leq C+W(t)^\top g(y(t))+Z(t)^\top [x(t)-y(t)] + Vf(y(t)). \quad (11)$$

C. Drift-plus-penalty algorithm

Let W^0 and Z^0 be the initial condition of $W(0)$ and $Z(0)$ respectively. Every time step, the Lyapunov optimization technique observes the current realization of random state $\omega(t)$ before choosing decisions $x(t) \in \mathcal{X}_{\omega(t)}$ and $y(t) \in \mathcal{Y}$ that minimize the right-hand-side of (11). The drift-plus-penalty algorithm is summarized in Algorithm 1.

```
Initialize V, W(0) = W^0, Z(0) = Z^0.
for t ∈ {0,1,2,...} do
    Observe ω(t)
    x(t) = arginf_{x∈X_ω(t)} Z(t)^T x
    y(t) = arginf_{y∈Y} [V f(y) + W(t)^T g(y) − Z(t)^T y]
    W(t+1) = [W(t) + g(y(t))]_+
    Z(t+1) = Z(t) + x(t) − y(t)
end
```

Algorithm 1: Drift-plus-penalty algorithm solving (5).

III. Behaviors of Drift-Plus-Penalty Algorithm

Starting from $(W(0), Z(0))$, Algorithm 1 reaches the steady state when vector $(W(t), Z(t))$ concentrates around a specific set (defined in Section III-A). The transient phase is the period before this concentration. Note that this behavior is different from the deterministic case in [18], where $(W(t), Z(t))$ in the steady state is contained in a specific set.

A. Embedded Formulation

A convex optimization problem, called *embedded formulation*, is considered. This idea is inspired by [19].

$$\begin{aligned} \text{Minimize} \quad & f(y) & & (12)\\ \text{Subject to} \quad & g_j(y) \leq 0 & j \in \{1,\ldots,J\} & \\ & y = \sum_{\omega\in\Omega} \pi_\omega x^\omega & & \\ & y \in \mathcal{Y}, \quad x^\omega \in \overline{\mathcal{X}}_\omega & \omega \in \Omega. & \end{aligned}$$

Note that formulation (12) contains multiple sets $\overline{\mathcal{X}}_\omega$ and is more complex than its deterministic version in [18].

This formulation has a dual problem, whose properties are used in convergence analysis. Let $w \in \mathbb{R}^J_+$ and $z \in \mathbb{R}^I$ be the vectors of dual variables associated with the first and second constraints of problem (12). The Lagrangian is defined as

$$\Gamma\big(\{x^\omega\}_{\omega\in\Omega}, y, w, z\big) = \sum_{\omega\in\Omega} \pi_\omega \left[f(y) + w^\top g(y) + z^\top (x^\omega - y)\right].$$

The dual function of problem (12) is

$$\begin{aligned} d(w,z) &= \inf_{y\in\mathcal{Y},\ x^\omega\in\overline{\mathcal{X}}_\omega:\forall\omega\in\Omega} \Gamma\big(\{x^\omega\}_{\omega\in\Omega}, y, w, z\big) \\ &= \sum_{\omega\in\Omega} \pi_\omega d_\omega(w,z), \end{aligned} \quad (13)$$

where $d_\omega(w,z)$ is defined in (14) and all of the minimizing solutions y take the same value.

$$d_\omega(w,z) \triangleq \inf_{y\in\mathcal{Y},\ x\in\overline{\mathcal{X}}_\omega} \left[f(y) + w^\top g(y) + z^\top (x - y)\right]. \quad (14)$$

Define the solution to the infimum in (14) as

$$y^*(w,z) \triangleq \operatorname*{arginf}_{y\in\mathcal{Y}} \left[f(y) + w^\top g(y) - z^\top y\right], \quad (15)$$

$$x^{*\omega}(z) \triangleq \operatorname*{arginf}_{x\in\overline{\mathcal{X}}_\omega} z^\top x. \quad (16)$$

Finally, the dual problem of formulation (12) is

$$\begin{aligned} \text{Maximize} \quad & d(w,z) & (17)\\ \text{Subject to} \quad & (w,z) \in \mathbb{R}^J_+ \times \mathbb{R}^I. & \end{aligned}$$

Problem (17) has an optimal solution that may not be unique. A set of these optimal solutions, which are vectors of Lagrange multipliers, can be used to analyze the transient time. However, to simplify the proofs and notations, a uniqueness assumption is defined below. Define $\lambda \triangleq (w,z)$ as a concatenation vector of w and z.

Assumption 3: Dual problem (17) has a unique vector of Lagrange multipliers denoted by $\lambda^* \triangleq (w^*, z^*)$.

This assumption is assumed throughout Section IV and Section V. Note that this is a mild assumption when practical systems are considered, e.g., [5], [19]. Furthermore, simulation results in [18] evince that this assumption may not be needed.

To prove the main result of this section, a useful property of $d_\omega(w,z)$ is derived. Define $h(x,y) \triangleq (g(y), x - y)$.

Lemma 2: For any $\lambda = (w,z) \in \mathbb{R}^J_+ \times \mathbb{R}^I$ and $\omega \in \Omega$, it holds that

$$d_\omega(\lambda^*) \leq d_\omega(\lambda) + h(x^{*\omega}(z), y^*(w,z))^\top [\lambda^* - \lambda]. \quad (18)$$

Proof: From (14), it follows, for any $\lambda = (w,z) \in \mathbb{R}^J_+ \times \mathbb{R}^I$ and $(x,y) \in \overline{\mathcal{X}}_\omega \times \mathcal{Y}$, that

$$\begin{aligned} d_\omega(\lambda^*) &\leq f(y) + h(x,y)^\top \lambda^* \\ &= f(y) + h(x,y)^\top \lambda + h(x,y)^\top [\lambda^* - \lambda] \end{aligned}$$

Setting $(x,y) = (x^{*\omega}(w,z), y^{*\omega}(z))$, as defined in (15) and (16), and using (14) proves the lemma. ■

The following lemma ties the virtual queues of Algorithm 1 to the Lagrange multipliers. Given the generated $W(t)$ and $Z(t)$ of Algorithm 1, define $Q(t)\triangleq(W(t), Z(t))$ as a concatenation of these vectors. The queue dynamics (6) and (7) are equivalent to

$$Q(t+1) = \mathcal{P}[Q(t) + h(x(t), y(t))]$$

where $\mathcal{P}[(W, Z)]$ denotes the projection of the concatenated vector (W, Z) onto the set $\mathbb{R}_+^J \times \mathbb{R}^I$.

Lemma 3: The following holds for every $t \in \{0, 1, 2, \ldots\}$:

$$\begin{aligned}\mathbb{E}\left[\|Q(t+1) - V\lambda^*\|^2 \middle| Q(t)\right] \\ \leq \|Q(t) - V\lambda^*\|^2 + 2C + 2V[d(Q(t)/V) - d(\lambda^*)].\end{aligned}$$

Proof: The non-expansive projection [21] implies that

$$\begin{aligned}&\|Q(t+1) - V\lambda^*\|^2 \leq \|Q(t) + h(x(t), y(t)) - V\lambda^*\|^2 \\ &= \|Q(t) - V\lambda^*\|^2 + \|h(x(t), y(t))\|^2 \\ &\quad + 2h(x(t), y(t))^\top [Q(t) - V\lambda^*] \\ &\leq \|Q(t) - V\lambda^*\|^2 + 2C + 2h(x(t), y(t))^\top [Q(t) - V\lambda^*]\end{aligned} \tag{19}$$

From (14), when $\lambda = Q(t)/V$, we have

$$d_{\omega(t)}(Q(t)/V) = \inf_{y \in \mathcal{Y},\ x \in \overline{\mathcal{X}}_{\omega(t)}} \left[f(y) + \frac{W(t)}{V}^\top g(y) + \frac{Z(t)}{V}^\top (x - y)\right],$$

so $y^*(W(t)/V, Z(t)/V) = y(t)$ and $x^{*\omega(t)}(Z(t)/V) = x(t)$ where $(y^*(W(t)/V, Z(t)/V), x^{*\omega(t)}(Z(t)/V))$ is defined in (15) and (16), and $(y(t), x(t))$ is the decision from Algorithm 1. Therefore, property (18) implies that

$$h(x(t), y(t))^\top [Q(t) - V\lambda^*] \leq V\left[d_{\omega(t)}(Q(t)/V) - d_{\omega(t)}(\lambda^*)\right].$$

Applying the above inequality on the last term of (19) gives

$$\begin{aligned}\|Q(t+1) - V\lambda^*\|^2 \leq \|Q(t) - V\lambda^*\|^2 + 2C \\ + 2V\left[d_{\omega(t)}(Q(t)/V) - d_{\omega(t)}(\lambda^*)\right].\end{aligned}$$

Taking a conditional expectation given $Q(t)$ proves the lemma:

$$\begin{aligned}\mathbb{E}\left[\|Q(t+1) - V\lambda^*\|^2 \middle| Q(t)\right] \leq \|Q(t) - V\lambda^*\|^2 + 2C \\ + 2V \sum_{\omega \in \Omega} \pi_\omega [d_\omega(Q(t)/V) - d_\omega(\lambda^*)].\end{aligned}$$

■

The analysis of transient and steady state phases in Sections IV and V will utilize Lemma 3. The convergence results in the steady state require the following results.

B. T-slot convergence

For any positive integer T and any starting time t_0, define the T-slot average starting at t_0 as

$$\overline{x}(t_0, T)\triangleq\frac{1}{T}\sum_{t=t_0}^{t_0+T-1} x(t).$$

This average leads to the following convergence bounds.

Theorem 2: Let $\{Q(t)\}_{t=0}^\infty$ be a sequence generated by Algorithm 1. For any positive integer T and any starting time t_0, the objective cost converges as

$$\begin{aligned}\mathbb{E}\left[f(\overline{x}(t_0, T))\right] - f^{(\text{opt})} \leq \frac{M_f}{T}\mathbb{E}\left[\|Z(t_0 + T) - Z(t_0)\|\right] \\ + \frac{1}{2TV}\mathbb{E}\left[\|Q(t_0)\|^2 - \|Q(t_0 + T)\|^2\right] + \frac{C}{V},\end{aligned} \tag{20}$$

and the constraint violation for every $j \in \{1, \ldots, J\}$ is

$$\begin{aligned}\mathbb{E}\left[g_j(\overline{x}(t_0, T))\right] \leq \frac{1}{T}\mathbb{E}\left[W_j(t_0 + T) - W_j(t_0)\right] \\ + \frac{M_{gj}}{T}\mathbb{E}\left[\|Z(t_0 + T) - Z(t_0)\|\right].\end{aligned} \tag{21}$$

Proof: Please see the full proof in [22]. ■

To interprete Theorem 2, the following concentration bound is provided. It is proven in [23].

C. Concentration bound

Theorem 3: Let $K(t)$ be a real random process over $t \in \{0, 1, 2, \ldots\}$ satisfying

$$|K(t+1) - K(t)| \leq \delta \quad \text{and}$$
$$\mathbb{E}\left[K(t+1) - K(t) | K(t)\right] \leq \begin{cases} \delta & , K(t) < \gamma \\ -\beta & , K(t) \geq \gamma, \end{cases}$$

for some positive real-valued δ, γ, and $0 < \beta \leq \delta$.

Suppose $K(0) = k_0$ (with probability 1) for some $k_0 \in \mathbb{R}$. Then for every time $t \in \{0, 1, 2, \ldots\}$, the following holds:

$$\mathbb{E}\left[e^{rK(t)}\right] \leq D + \left(e^{rk_0} - D\right)\rho^t$$

where $0 < \rho < 1$ and constants r, ρ, and D are:

$$r\triangleq\frac{\beta}{(\delta^2 + \delta\beta/3)}, \qquad \rho\triangleq 1 - \frac{r\beta}{2}$$
$$D\triangleq\frac{(e^{r\delta} - \rho)e^{r\gamma}}{1 - \rho}.$$

In this paper, random process $K(t)$ is defined to be the distance between $Q(t)$ and the vector of Lagrange multipliers, so $K(t)\triangleq\|Q(t) - V\lambda^*\|$ for every $t \in \{0, 1, 2, \ldots\}$.

Lemma 4: It holds for every $t \in \{0, 1, 2, \ldots\}$ that

$$|K(t+1) - K(t)| \leq \sqrt{2C}$$
$$\mathbb{E}\left[K(t+1) - K(t) | K(t)\right] \leq \sqrt{2C}.$$

Proof: The first part is proven in two cases.

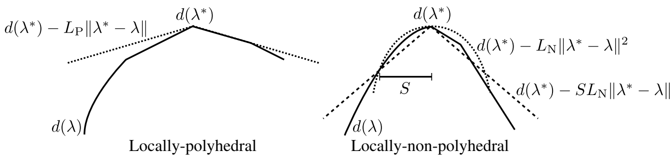

Fig. 1. Illustrations of locally-polyhedral and locally-non-polyhedral dual functions

i) If $K(t+1) \geq K(t)$, the non-expansive projection implies

$$\begin{aligned}|K(t+1) - K(t)| &= K(t+1) - K(t)\\ &\leq \|Q(t) + h(x(t),y(t)) - V\lambda^*\| - \|Q(t) - V\lambda^*\|\\ &\leq \|h(x(t),y(t))\| \leq \sqrt{2C}.\end{aligned}$$

ii) If $K(t+1) < K(t)$, then

$$\begin{aligned}|K(t+1) - K(t)| &= K(t) - K(t+1)\\ &\leq \|Q(t) - Q(t+1)\| + \|Q(t+1) - V\lambda^*\| - K(t+1)\\ &\leq \|h(x(t),y(t))\| \leq \sqrt{2C}.\end{aligned}$$

Therefore, $|K(t+1) - K(t)| \leq \sqrt{2C}$. Using $K(t+1) - K(t) \leq |K(t+1) - K(t)|$ proves the second part. ∎

Lemma 4 prepares $K(t)$ for Theorem 3. The only constants left to be specified are β and γ, which depend on properties of dual function (13).

IV. Locally-polyhedral dual function

This section analyzes the transient time and a convergence result in the steady state. Dual function (13) in this section is assumed to satisfy a locally-polyhedral property, introduced in [19]. This property is illustrated in Figure 1. It holds when f and each g_j for $j \in \{1, \ldots, J\}$ are either linear or piece-wise linear.

Assumption 4: Let λ^* be the unique Lagrange multiplier vector. There exists $L_{\mathrm{P}} > 0$ such that dual function (13) satisfies, for any $\lambda \in \mathbb{R}^J_+ \times \mathbb{R}^I$,

$$d(\lambda^*) \geq d(\lambda) + L_{\mathrm{P}}\|\lambda - \lambda^*\|. \tag{22}$$

Note that, by concavity of the dual function, if inequality (22) holds locally about λ^*, it must also hold globally. The subscript P denotes "polyhedral."

A. Transient time

The progress of $Q(t)$ at each step can be analyzed. Define

$$B_{\mathrm{P}} \triangleq \max\left[\frac{L_{\mathrm{P}}}{2}, \frac{2C}{L_{\mathrm{P}}}\right]. \tag{23}$$

Lemma 5: Under Assumptions 3 and 4, whenever $\|Q(t) - V\lambda^*\| \geq B_{\mathrm{P}}$, the following holds

$$\mathbb{E}\left[\|Q(t+1) - V\lambda^*\| | Q(t)\right] \leq \|Q(t) - V\lambda^*\| - \frac{L_{\mathrm{P}}}{2}.$$

Proof: If condition

$$2C + 2V[d(Q(t)/V) - d(\lambda^*)] \leq -L_{\mathrm{P}}\|Q(t) - V\lambda^*\| + L_{\mathrm{P}}^2/4 \tag{24}$$

is true, Lemma 3 implies that

$$\begin{aligned}&\mathbb{E}\left[\|Q(t+1) - V\lambda^*\|^2 | Q(t)\right]\\ &\quad\leq \|Q(t) - V\lambda^*\|^2 - L_{\mathrm{P}}\|Q(t) - V\lambda^*\| + L_{\mathrm{P}}^2/4\\ &\quad= [\|Q(t) - V\lambda^*\| - L_{\mathrm{P}}/2]^2.\end{aligned}$$

Applying Jensen's inequality [21] on the left-hand-side yields

$$\mathbb{E}\left[\|Q(t+1) - V\lambda^*\| | Q(t)\right]^2 \leq [\|Q(t) - V\lambda^*\| - L_{\mathrm{P}}/2]^2.$$

When $\|Q(t) - V\lambda^*\| \geq L_{\mathrm{P}}/2$, it follows that

$$\mathbb{E}\left[\|Q(t+1) - V\lambda^*\| | Q(t)\right] \leq \|Q(t) - V\lambda^*\| - L_{\mathrm{P}}/2. \tag{25}$$

It requires to show that condition (24) holds whenever $\|Q(t) - V\lambda^*\| \geq B_{\mathrm{P}}$. Assumption 4 implies that

$$2C + 2V[d(Q(t)/V) - d(\lambda^*)] \leq 2C - 2L_{\mathrm{P}}\|Q(t) - V\lambda^*\|.$$

From the definition of B_{P} in (23), $\|Q(t) - V\lambda^*\| \geq B_{\mathrm{P}}$ implies $L_{\mathrm{P}}\|Q(t) - V\lambda^*\| \geq 2C$ and

$$2C + 2V[d(Q(t)/V) - d(\lambda^*)] \leq -L_{\mathrm{P}}\|Q(t) - V\lambda^*\|,$$

which implies (24). ∎

Lemma 5 implies that, in expectation, $Q(t)$ proceeds closer to $V\lambda^*$ in the next step when the distance between them is at least B_{P}. This implication means that $Q(t)$ concentrates around $V\lambda^*$ in the steady state.

B. Convergence time in a steady state

Define constants $r_{\mathrm{P}}, \rho_{\mathrm{P}}, D_{\mathrm{P}}, U_{\mathrm{P}}, U'_{\mathrm{P}}$ as

$$r_{\mathrm{P}} \triangleq \frac{3L_{\mathrm{P}}}{12C + L_{\mathrm{P}}\sqrt{2C}}, \qquad \rho_{\mathrm{P}} \triangleq 1 - \frac{r_{\mathrm{P}}L_{\mathrm{P}}}{4}, \tag{26}$$

$$D_{\mathrm{P}} \triangleq \frac{e^{r_{\mathrm{P}}B_{\mathrm{P}}}\left(e^{r_{\mathrm{P}}\sqrt{2C}} - \rho_{\mathrm{P}}\right)}{1 - \rho_{\mathrm{P}}} \tag{27}$$

$$U_{\mathrm{P}} \triangleq \frac{\log(D_{\mathrm{P}} + 1)}{r_{\mathrm{P}}}, \qquad U'_{\mathrm{P}} \triangleq \frac{2(D_{\mathrm{P}} + 1)}{r_{\mathrm{P}}^2}. \tag{28}$$

Given the initial condition $Q^0 \triangleq (W^0, Z^0)$, define

$$T_{\mathrm{P}} \triangleq \left\lceil \frac{r_{\mathrm{P}}\|Q^0 - V\lambda^*\|}{\log(1/\rho_{\mathrm{P}})} \right\rceil, \tag{29}$$

where constants r_{P} and ρ_{P} are defined in (26). The value T_{P} is $O(V)$. The next lemma shows T_{P} can be interpreted as the transient time, so that desirable "steady state" bounds hold after this time.

Lemma 6: Suppose Assumptions 3 and 4 hold. Given the initial condition $Q^0 \triangleq (W^0, Z^0)$, for any time $t \geq T_{\mathrm{P}}$ when T_{P} is defined in (29), the following holds

$$\mathbb{E}\left[\|Q(t) - V\lambda^*\|\right] \leq U_{\mathrm{P}} \tag{30}$$

$$\mathbb{E}\left[\|Q(t) - V\lambda^*\|^2\right] \leq U'_{\mathrm{P}}, \tag{31}$$

where constants U_{P} and U'_{P} are defined in (28).

Proof: Recall that $K(t) \triangleq \|Q(t) - V\lambda^*\|$. From Lemmas 4 and 5, constants in Theorem 3 are $\delta = \sqrt{2C}, \gamma = B_{\mathrm{P}}$, and $\beta = L_{\mathrm{P}}/2$. Theorem 3 implies, for any $t \geq 0$, that

$$\mathbb{E}\left[e^{r_{\mathrm{P}}K(t)}\right] \leq D_{\mathrm{P}} + \left(e^{r_{\mathrm{P}}k_0} - D_{\mathrm{P}}\right)\rho_{\mathrm{P}}^t \leq D_{\mathrm{P}} + e^{r_{\mathrm{P}}k_0}\rho_{\mathrm{P}}^t \tag{32}$$

where $k_0 = K(0) = \left\|Q^0 - V\lambda^*\right\|$, and constants $r_{\mathrm{P}}, \rho_{\mathrm{P}}, D_{\mathrm{P}}$ are defined in (26) and (27). We then show that

$$e^{r_{\mathrm{P}}k_0}\rho_{\mathrm{P}}^t \le 1 \qquad \forall t \ge T_{\mathrm{P}}. \tag{33}$$

Inequality $e^{r_{\mathrm{P}}k_0}\rho_{\mathrm{P}}^t \le 1$ is equivalent to $t \ge \frac{r_{\mathrm{P}}\left\|Q^0-V\lambda^*\right\|}{\log(1/\rho_{\mathrm{P}})}$ by arithmetic and the fact that $\log(1/\rho_{\mathrm{P}}) > 0$. From the definition of T_{P} in (29), it holds that $T_{\mathrm{P}} \ge \frac{r_{\mathrm{P}}\left\|Q^0-V\lambda^*\right\|}{\log(1/\rho_{\mathrm{P}})}$, and the result (33) follows.

From (33), inequality (32) becomes

$$\mathbb{E}\left[e^{r_{\mathrm{P}}K(t)}\right] \le D_{\mathrm{P}} + 1 \qquad \forall t \ge T_{\mathrm{P}}. \tag{34}$$

Jensen's inequality implies that $e^{r_{\mathrm{P}}\mathbb{E}[K(t)]} \le \mathbb{E}\left[e^{r_{\mathrm{P}}K(t)}\right]$, and we have $e^{r_{\mathrm{P}}\mathbb{E}[K(t)]} \le D_{\mathrm{P}} + 1$. Taking logarithm and dividing by r_{P} proves (30).

Chernoff bound (see for example in [24]) implies that, for any $m \in \mathbb{R}_+$,

$$\mathbb{P}\left[K(t) \ge m\right] \le e^{-r_{\mathrm{P}}m}\mathbb{E}\left[e^{r_{\mathrm{P}}K(t)}\right] \le e^{-r_{\mathrm{P}}m}(D_{\mathrm{P}}+1) \;\; \forall t \ge T_{\mathrm{P}} \tag{35}$$

where the last inequality uses (34). Since $K(t)^2$ is always non-negative, it can be shown that $\mathbb{E}\left[K(t)^2\right] = 2\int_0^\infty m\mathbb{P}\left[K(t) \ge m\right]dm$ by the integration by parts. Using (35), we have

$$\mathbb{E}\left[K(t)^2\right] \le 2(D_{\mathrm{P}}+1)\int_0^\infty me^{-r_{\mathrm{P}}m}dm.$$

Performing the integration by parts proves (31). ∎

The above lemma implies that, in the steady state, the expected distance and square distance between $Q(t)$ and the vector of Lagrange multipliers are bounded by constants that do not depend on V. This phenomenon leads to an improved convergence time when the average is performed in the steady state. A useful result is derived before the main theorem.

Lemma 7: For any times t_1 and t_2, it holds that

$$\begin{aligned}\mathbb{E}\left[\|Q(t_1)\|^2 - \|Q(t_2)\|^2\right] &\le \mathbb{E}\left[\|Q(t_1) - V\lambda^*\|^2\right] \\ &\quad + 2\|V\lambda^*\|\mathbb{E}\left[\|Q(t_1) - V\lambda^*\| + \|Q(t_2) - V\lambda^*\|\right].\end{aligned}$$

Proof: It holds for any $Q \in \mathbb{R}_+^J \times \mathbb{R}^I$ that

$$\|Q\|^2 = \|Q - V\lambda^*\|^2 + \|V\lambda^*\|^2 + 2(Q - V\lambda^*)^\top(V\lambda^*).$$

Using the above equality with $Q_1, Q_2 \in \mathbb{R}_+^J \times \mathbb{R}^I$ leads to

$$\begin{aligned}&\|Q_1\|^2 - \|Q_2\|^2 \\ &\quad\le \|Q_1 - V\lambda^*\|^2 - \|Q_2 - V\lambda^*\|^2 + 2(Q_1 - Q_2)^\top(V\lambda^*) \\ &\quad\le \|Q_1 - V\lambda^*\|^2 + 2\|Q_1 - Q_2\|\|V\lambda^*\| \\ &\quad\le \|Q_1 - V\lambda^*\|^2 + 2\|V\lambda^*\|\left[\|Q_1 - V\lambda^*\| + \|Q_2 - V\lambda^*\|\right].\end{aligned}$$

Taking an expectation proves the lemma. ∎

Finally, the convergence in the steady state is analyzed.

Theorem 4: Suppose Assumptions 3 and 4 hold. For any time $t_0 \ge T_{\mathrm{P}}$ and positive integer T, the objective cost converges as

$$\mathbb{E}\left[f(\overline{x}(t_0,T))\right] - f^{(\mathrm{opt})} \le \frac{2M_fU_{\mathrm{P}}}{T} + \frac{U_{\mathrm{P}}' + 4VU_{\mathrm{P}}\|\lambda^*\|}{2TV} + \frac{C}{V} \tag{36}$$

and the constraint violation is upper bounded by

$$\mathbb{E}\left[g_j(\overline{x}(t_0,T))\right] \le \frac{2U_{\mathrm{P}}}{T} + \frac{2M_{gj}U_{\mathrm{P}}}{T}. \tag{37}$$

Proof: From Theorem 2, the objective cost converges as (20). Since $T_{\mathrm{P}} \le t_0 < t_0 + T$, we use results in Lemma 6 to upper bound $\mathbb{E}\left[\|Q(t) - V\lambda^*\|\right]$ and $\mathbb{E}\left[\|Q(t) - V\lambda^*\|^2\right]$ for t_0 and $t_0 + T$. Terms in the right-hand-side of (20) are bounded by

$$\begin{aligned}\mathbb{E}\left[\|Z(t_0+T) - Z(t_0)\|\right] &\le \mathbb{E}\left[\|Q(t_0+T) - Q(t_0)\|\right] \\ &\le \mathbb{E}\left[K(t_0+T) + K(t_0)\right] \le 2U_{\mathrm{P}}.\end{aligned} \tag{38}$$

Lemma 7 implies that

$$\begin{aligned}&\mathbb{E}\left[\|Q(t_0)\|^2 - \|Q(t_0+T)\|^2\right] \\ &\quad\le \mathbb{E}\left[K(t_0)^2 + 2\|V\lambda^*\|\left[K(t_0) + K(t_0+T)\right]\right] \\ &\quad\le U_{\mathrm{P}}' + 4VU_{\mathrm{P}}\|\lambda^*\|.\end{aligned} \tag{39}$$

Substituting bounds (38) and (39) into (20) proves (36).

The constraint violation converges as (21) where $T_{\mathrm{P}} \le t_0 < t_0 + T$. Using Lemma 6, the last term in the right-hand-side of (21) is bounded in (38). The first term is bounded by

$$\begin{aligned}&\mathbb{E}\left[W_j(t_0+T) - W_j(t_0)\right] \\ &\quad\le \mathbb{E}\left[|W_j(t_0+T) - V\lambda^*| + |W_j(t_0) - V\lambda^*|\right] \\ &\quad\le \mathbb{E}\left[K(t_0+T) + K(t_0)\right] \le 2U_{\mathrm{P}}.\end{aligned}$$

Substituting the above bound and (38) into (21) proves (37). ∎

The implication of Theorem 4 is as follows. When the average starts in the steady state, the deviation from the optimal cost is $O(1/T + 1/V)$, and the constraint violation is bounded by $O(1/T)$. By setting $V = 1/\epsilon$ and $T = 1/\epsilon$, both optimal cost and constrain violation achieve $O(\epsilon)$-approximation, and the convergence time is $O(1/\epsilon)$ slots. Note that this setting yields $O(1/\epsilon)$ transient time, since $T_{\mathrm{P}} = O(V) = O(1/\epsilon)$.

V. Locally-non-polyhedral dual function

The dual function (13) in Section V is assumed to satisfy a locally-non-polyhedral property, modified from [19]. This property is illustrated in Figure 1.

Assumption 5: Let λ^* be the unique Lagrange multiplier vector. There exist $S > 0$ and $L_{\mathrm{N}} > 0$ such that, whenever $\lambda \in \mathbb{R}_+^J \times \mathbb{R}^I$ and $\|\lambda - \lambda^*\| \le S$, dual function (13) satisfies

$$d(\lambda^*) \ge d(\lambda) + L_{\mathrm{N}}\|\lambda - \lambda^*\|^2.$$

Note that the subscript N denotes "non-polyhedral."

It can be shown that Assumption 5 implies

$$d(\lambda^*) \ge d(\lambda) + SL_{\mathrm{N}}\|\lambda - \lambda^*\|$$

for all $\lambda \in \mathbb{R}_+^J \times \mathbb{R}^I$ and $\|\lambda - \lambda^*\| > S$.[1]

[1]We would like to thank Hao Yu for noticing this fact.

A. Transient time

The progress of $Q(t)$ at each step can be analyzed. Define

$$B_{\mathrm{N}}(V) \triangleq \max\left[\frac{1}{\sqrt{V}}, \sqrt{V}\left(\frac{1+\sqrt{1+4L_{\mathrm{N}}C}}{2L_{\mathrm{N}}}\right)\right]$$
$$B'_{\mathrm{N}} \triangleq \max\left[\frac{SL_{\mathrm{N}}}{2}, \frac{2C}{SL_{\mathrm{N}}}\right].$$

Lemma 8: Suppose Assumptions 3 and 5 hold. When V is large enough to ensure both $B_{\mathrm{N}}(V) \leq SV$ and $B'_{\mathrm{N}} \leq SV$, the following holds

$$\begin{aligned}&\mathbb{E}\left[\|Q(t+1)-V\lambda^*\| | Q(t)\right] - \|Q(t)-V\lambda^*\| \\ &\leq \begin{cases} -\frac{1}{\sqrt{V}} & \text{, if } B_{\mathrm{N}}(V) \leq \|Q(t)-V\lambda^*\| \leq SV \\ -\frac{SL_{\mathrm{N}}}{2} & \text{, if } \|Q(t)-V\lambda^*\| > SV \end{cases}\end{aligned} \tag{40}$$

Proof: Please see the full proof in [22]. ■

The interpretation of Lemma 8 is similar to Lemma 5 except that $B_{\mathrm{N}}(V)$ and the negative drift (40) are functions of V. Nevertheless, Lemma 8 implies that $Q(t)$ concentrates around $V\lambda^*$ in the steady state.

B. Convergence time in a steady state

Define constants $r_{\mathrm{N}}(V), \rho_{\mathrm{N}}(V), D_{\mathrm{N}}(V), U_{\mathrm{N}}(V), U'_{\mathrm{N}}(V)$ as

$$r_{\mathrm{N}}(V) \triangleq \frac{3}{6C\sqrt{V}+\sqrt{2C}}, \qquad \rho_{\mathrm{N}}(V) \triangleq 1 - \frac{r_{\mathrm{N}}(V)}{2\sqrt{V}}, \tag{41}$$

$$D_{\mathrm{N}}(V) \triangleq \frac{e^{r_{\mathrm{N}}(V)B_{\mathrm{N}}(V)}\left(e^{r_{\mathrm{N}}(V)\sqrt{2C}} - \rho_{\mathrm{N}}(V)\right)}{1-\rho_{\mathrm{N}}(V)} \tag{42}$$

$$U_{\mathrm{N}}(V) \triangleq \frac{\log\left(D_{\mathrm{N}}(V)+1\right)}{r_{\mathrm{N}}(V)}, \qquad U'_{\mathrm{N}}(V) \triangleq \frac{2(D_{\mathrm{N}}(V)+1)}{r_{\mathrm{N}}(V)^2}. \tag{43}$$

Given the initial condition $Q^0 \triangleq (W^0, Z^0)$, define the transient time for a locally-non-polyhedral dual function as

$$T_{\mathrm{N}} \triangleq \left\lceil \frac{r_{\mathrm{N}}(V)\|Q^0 - V\lambda^*\|}{\log(1/\rho_{\mathrm{N}}(V))} \right\rceil, \tag{44}$$

where constants $r_{\mathrm{N}}(V)$ and $\rho_{\mathrm{N}}(V)$ are defined in (41). Definition (44) implies that the transient time under the locally-non-polyhedral assumption is $O(V^{1.5})$. This T_{N} can be interpreted as the transient time, so that desirable bounds hold after this time.

Lemma 9: Suppose Assumptions 3 and 5 hold. When V is large enough to ensure $B_{\mathrm{N}}(V) \leq SV$, $B'_{\mathrm{N}} \leq SV$, and $\sqrt{V} \geq 2/SL_{\mathrm{N}}$, for any time $t \geq T_{\mathrm{N}}$, the following holds

$$\mathbb{E}\left[\|Q(t)-V\lambda^*\|\right] \leq U_{\mathrm{N}}(V) \tag{45}$$
$$\mathbb{E}\left[\|Q(t)-V\lambda^*\|^2\right] \leq U'_{\mathrm{N}}(V) \tag{46}$$

where $U_{\mathrm{N}}(V)$ and $U'_{\mathrm{N}}(V)$ are defined in (43).

Proof: Please see the full proof in [22]. ■

The convergence results in the steady state are as follows.

Theorem 5: Suppose Assumptions 3 and 5 hold. When V is large enough to ensure $B_{\mathrm{N}}(V) \leq SV, B'_{\mathrm{N}} \leq SV$, and $\sqrt{V} \geq 2/SL_{\mathrm{N}}$, then for any time $t_0 \geq T_{\mathrm{N}}$ and any positive integer T, the objective cost converges as

$$\begin{aligned}\mathbb{E}\left[f(\overline{x}(t_0,T))\right] - f^{(\mathrm{opt})} \leq & \frac{2M_f U_{\mathrm{N}}(V)}{T} \\ &+ \frac{U'_{\mathrm{N}}(V) + 4VU_{\mathrm{N}}(V)\|\lambda^*\|}{2TV} + \frac{C}{V}\end{aligned} \tag{47}$$

and the constraint violation is upper bounded by

$$\mathbb{E}\left[g_j(\overline{x}(t_0,T))\right] \leq \frac{2U_{\mathrm{N}}(V)}{T} + \frac{2M_{gj}U_{\mathrm{N}}(V)}{T}. \tag{48}$$

Proof: Please see the full proof in [22]. ■

The implication of Theorem 5 is as follows. When the average starts in the steady state, the deviation from the optimal cost is $O(\sqrt{V}/T + 1/V)$, and the constraint violation is bounded by $O(\sqrt{V}/T)$. Note that this can be shown by substituting $B_{\mathrm{N}}(V), r_{\mathrm{N}}(V), \rho_{\mathrm{N}}(V), D_{\mathrm{N}}(V)$ into (47) and (48). By setting $V = 1/\epsilon$ and $T = 1/\epsilon^{1.5}$, this achieves an $O(\epsilon)$-approximation with transient time and convergence time of $O(1/\epsilon^{1.5})$.

VI. Simulation

A. Staggered Time Averages

In order to take advantage of the improved convergence times, computation of time averages must be started in the steady state phase shortly after the transient time. To achieve this performance without knowing the end of the transient phase, time averages can be restarted over successive frames whose frame lengths increase geometrically. For example, if one triggers a restart at times 2^k for integers k, then a restart is guaranteed to occur within a factor of 2 of the time of the actual end of the transient phase.

B. Results

This section illustrates the convergence times of the drift-plus-penalty Algorithm 1 under locally-polyhedron and locally-non-polyhedron assumptions. Let $\Omega = \{0,1,2\}, \mathcal{X}_0 = \{(0,0)\}, \mathcal{X}_1 = \{(-5,0),(0,10)\}, \mathcal{X}_2 = \{(0,-10),(5,0)\}$, and $(\pi_0,\pi_1,\pi_2) = (0.1, 0.6, 0.3)$. A formulation is

$$\begin{aligned}\text{Minimize} \quad & \limsup_{T\to\infty} f(\overline{x}(T)) \\ \text{Subject to} \quad & \limsup_{T\to\infty} \left[-2\overline{x}_1(T) - \overline{x}_2(T)\right] \leq -1.5 \\ & \limsup_{T\to\infty} \left[-\overline{x}_1(T) - 2\overline{x}_2(T)\right] \leq -1.5 \\ & (x_1(t), x_2(t)) \in \mathcal{X}_{\omega(t)}, \quad t \in \{0,1,2,\dots\}\end{aligned} \tag{49}$$

where function f will be given for different cases.

Under locally-polyhedron assumption, let $f(x) = 1.5x_1 + x_2$ be the objective function of problem (49). In this setting, the optimal value is 1.25 where $\lim_{T\to\infty}\overline{x}_1(T) = \lim_{T\to\infty}\overline{x}_2(T) = 0.5$. Figure 2 shows the values of objective and constraint functions of time-average solutions. It is easy to see the improved convergence time $O(1/\epsilon)$ from the staggered time averages compared to the convergence time $O(1/\epsilon^2)$ of Algorithm 1.

Under locally-non-polyhedral assumption, let $f(x) = x_1^2 + x_2^2$ be the objective function of problem (49). Note that the

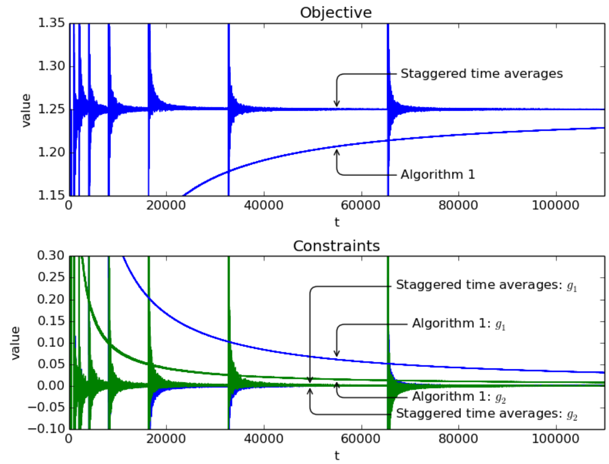

Fig. 2. Processes of solving (49) with $f(x) = 1.5x_1 + x_2$

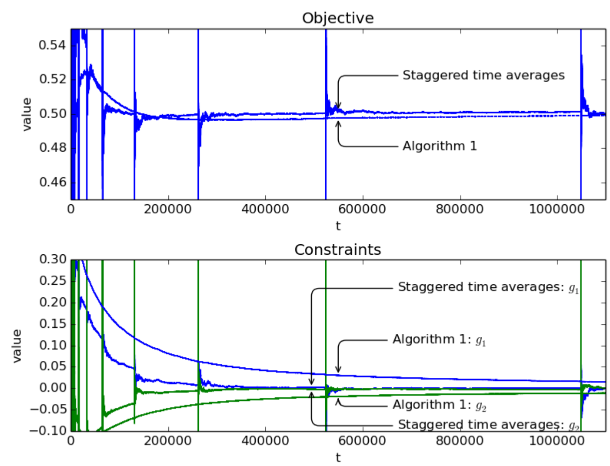

Fig. 3. Processes of solving (49) with $f(x) = x_1^2 + x_2^2$

optimal value of this problem is 0.5 where $\lim_{T\to\infty} \overline{x}_1(T) = \lim_{T\to\infty} \overline{x}_2(T) = 0.5$. Figure 3 shows the values of objective and constraint functions of time-average solutions. It can be seen from the plot of constraints that the staggered time averages converge faster than Algorithm 1. This illustrates the different between convergence times $O(1/\epsilon^{1.5})$ and $O(1/\epsilon^2)$. Additional results of problems without the uniqueness assumption can be found in [22].

VII. Conclusion

We consider the time-average stochastic optimization problem with a non-convex decision set. The problem can be solved using the drift-plus-penalty algorithm, which has convergence time $O(1/\epsilon^2)$. After we analyze the transient and steady state phases of the algorithm, the convergence time can be improved by performing time average in the steady state. We prove that the improved convergence time is $O(1/\epsilon)$ under the locally-polyhedral assumption and is $O(1/\epsilon^{1.5})$ under the locally-non-polyhedral assumption.

References

[1] M. J. Neely, "Stochastic network optimization with application to communication and queueing systems," *Synthesis Lectures on Communication Networks*, vol. 3, no. 1, 2010.

[2] M. J. Neely, E. Modiano, and C. Li, "Fairness and optimal stochastic control for heterogeneous networks," *Networking, IEEE/ACM Transactions on*, vol. 16, no. 2, Apr. 2008.

[3] L. Georgiadis, M. J. Neely, and L. Tassiulas, "Resource allocation and cross-layer control in wireless networks," *Foundations and Trends in Networking*, Apr. 2006.

[4] A. Stolyar, "Greedy primal-dual algorithm for dynamic resource allocation in complex networks," *Queueing Systems*, vol. vol. 54, no. 3, pp. 203-220, 2006.

[5] A. Eryilmaz and R. Srikant, "Fair resource allocation in wireless networks using queue-length-based scheduling and congestion control," *Networking, IEEE/ACM Transactions on*, vol. 15, no. 6, Dec. 2007.

[6] M. J. Neely, "Energy optimal control for time-varying wireless networks," *Information Theory, IEEE Transactions on*, Jul. 2006.

[7] L. Lin, X. Lin, and N. B. Shroff, "Low-complexity and distributed energy minimization in multi-hop wireless networks," *Proc. IEEE INFOCOM*, 2007.

[8] S. Supittayapornpong and M. J. Neely, "Quality of information maximization for wireless networks via a fully separable quadratic policy," *Networking, IEEE/ACM Transactions on*, 2014.

[9] Y. Nesterov, "Primal-dual subgradient methods for convex problems," *Mathematical Programming*, vol. 120, no. 1, 2009.

[10] A. Nedić and A. Ozdaglar, "Approximate primal solutions and rate analysis for dual subgradient methods," *SIAM Journal on Optimization*, vol. 19, no. 4, 2009.

[11] M. J. Neely, "Distributed and secure computation of convex programs over a network of connected processors," *DCDIS Conf., Guelph, Ontario*, Jul. 2005.

[12] Y. Nesterov, *Introductory Lectures on Convex Optimization: A Basic Course (Applied Optimization)*. Springer Netherlands, 2004.

[13] P. Tseng, "On accelerated proximal gradient methods for convex-concave optimization," *submitted to SIAM Journal on Optimization*, 2008.

[14] S. Boyd and L. Vandenberghe, *Convex Optimization*. New York, NY, USA: Cambridge University Press, 2004.

[15] E. Wei and A. Ozdaglar, "On the o(1/k) convergence of asynchronous distributed alternating direction method of multipliers," *arXiv:1307.8254*, Jul. 2013.

[16] A. Beck, A. Nedić, A. Ozdaglar, and M. Teboulle, "An o(1/k) gradient method for network resource allocation problems," *Control of Network Systems, IEEE Transactions on*, vol. 1, no. 1, Mar. 2014.

[17] M. J. Neely, "A simple convergence time analysis of drift-plus-penalty for stochastic optimization and convex program," *arXiv:1412.0791v1 [math.OC]*, Dec. 2014.

[18] S. Supittayapornpong and M. J. Neely, "Time-average optimization with nonconvex decision set and its convergence," in *CDC, 2014 Proceedings IEEE*, Dec. 2014.

[19] L. Huang and M. J. Neely, "Delay reduction via lagrange multipliers in stochastic network optimization," *Automatic Control, IEEE Transactions on*, vol. 56, no. 4, Apr. 2011.

[20] L. Huang, X. Liu, and X. Hao, "The power of online learning in stochastic network optimization," *SIGMETRICS Perform. Eval. Rev.*, vol. 42, no. 1, Jun. 2014.

[21] D. Bertsekas, A. Nedić, and A. Ozdaglar, *Convex Analysis and Optimization*. Athena Scientific, 2003.

[22] S. Supittayapornpong and M. J. Neely, "Time-average stochastic optimization with non-convex decision set and its convergence," *arXiv:1412.4509 [math.OC]*, Jan. 2015.

[23] M. J. Neely, "Energy-aware wireless scheduling with near optimal backlog and convergence time tradeoffs," in *INFOCOM, 2015 Proceedings IEEE*, Apr. 2015.

[24] S. M. Ross, *Stochastic Processes (Wiley Series in Probability and Statistics)*, 2nd ed. Wiley, Feb. 1995.