Modeling Sparsity of Planar Topologies for Wireless Multi-hop Networks

Ashikur Rahman, Nawshad Rehan Rasha, Imtiaz Ahmed, and Sirat Samyoun
Department of Computer Science and Engineering (CSE),
Bangladesh University of Engineering and Technology (BUET), Dhaka-1000, Bangladesh
Email: ashikur@cse.buet.ac.bd, rasha.cse@gmail.com, super.imti@gmail.com, siratsamyoun@yahoo.com

***Abstract*—Topology Control (TC) is a fundamental research problem where the goal is to determine a set of wireless links such that the composed topology satisfies some desirable properties. Some of the achievable properties are connectivity, symmetricity, planarity, minimal use of energy, sparseness, bounded maximum node degree etc. A topology is said to be *planar* if there is no link crossing in the topology. Planar topologies are heavily used by several protocols lying on different layers of the protocol stack. Several TC algorithms generating planar topologies have been proposed recently. Although sparseness is a key metric in evaluating performance of such topologies, none of the prior algorithms provide any mathematical model to determine sparseness of the topology. In this paper, we provide a generic analytical model for evaluating sparseness of planar topologies. The derived analytical expressions can be used in determining average node degree, topology size etc., without running simulations or prior to the deployment of real systems. The analytical expressions are validated through extensive simulation experiments.**

I. Introduction

The link in wireless multi-hop networks is a *virtual* concept. We say there exists a link between two (wireless) transceivers if they are located within the transmission range of each other. By weaving all virtual links between every transceiver pairs, the *topology* of a wireless network is derived. As the very basic purpose of any network is to facilitate exchange of information between any two nodes, all wireless links are not necessarily needed as long as the connectivity between all possible pairs is ensured. Therefore, by rearranging the transmission range, one can easily eliminate unnecessary links and reshape the underlying topology as needed. This possible rearrangement of transmission powers is known as *topology control* (TC).

Formally, the topology control is a fundamental research problem where the goal is to determine a set of wireless links such that the composed topology satisfies some desirable properties. Some of the achievable properties investigated by several researchers are connectivity, symmetricity, planarity, minimal use of energy, sparseness, bounded maximum node degree etc. Among those properties, planarity has drawn attention in a significant number of researchers [10], [7], [5], [6].

A topology is said to be *planar* if there is no link crossing in the topology. Planar topologies are heavily used by several protocols lying on different layers of the protocol stack. For instance, routing layers determine shortest paths quickly in linear time when the underlying topology is planar [7]. Several position-based routing algorithms use planar topologies to guarantee successful delivery of a packet. For example, Gabriel graph was used as a planar subgraph in Face routing protocol. Geographic routing protocols like GOAFR [3] or GPSR [6] forwards the packet to the neighbor that is closest to the destination in terms of distance or direction. However, they frequently encounters local minima, where a packet gets stuck due to all neighboring nodes being further away from the target than its current location. As a fall back, they require a planar subgraph (traditionally a Gabriel Graph) which essentially allows the packet to be routed around the *communication voids* in the network. Relative Neighborhood Graph (RNG), another planar graph, was used for efficient broadcasting that minimizes the number of retransmissions [15].

Many protocols generating planar topologies occupy a rich proportion of state-of-the-art topology control algorithms. A series of planar graph structures, also known as *proximity graphs*, borrowed from computation geometry have widely been used for solving different fundamental problems of wireless multi-hop networks. Some examples of those planar graph structures are Euclidean Minimum Spanning Tree (EMST), RNG [17], GG [4], Delaney triangulation (DT) [10] etc, to name a few. Another special class of planar graphs, dubbed as *r-neighborhood graph* [5], has recently been proposed for mobile wireless multi-hop networks. It has been shown that RNG and GG are special instances of r-neighborhood graph. All of these graph structures have been heavily analyzed using graph theoretic approaches. Some of the metrics derived from those analysis include maximum bound on total number of edges, maximum node degree, power stretch factor, distance stretch factor etc. However, it is unclear how to estimate *sparseness* of those topologies using such graph theoretic approaches. One inherent reason lies on the fact that most of the topology control algorithms typically generate highly complex structures which are often difficult to analyze using simple graph theoretic approaches and the existing algorithms were only targeting simple heuristics. Nevertheless, *sparseness* is a key metric in evaluating performance of any topology as it indicates how many (wireless) links are still present in the network after running TC. Sparseness also provides insight to average node degree once it is analyzed on *a per node basis*. Many other performance metrics are directly or indirectly related to the sparseness. For instance, per node sparseness (i.e. the average node degree) provides a crude estimation of the level of relaying burden, contention and interference. In general, the relaying burden is inversely proportional to

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the average node degree whereas contention and interference, experienced by a node, are directly proportional to the average number of neighbors. Moreover, larger (average) node degree means tighter dependency among nodes which is truly undesirable when nodes move very rapidly.

Although sparseness is a key metric in evaluating performance of such topologies, none of these prior planar topology generating algorithms provide any mathematical model to determine sparseness. Therefore, in this paper, we focus on a class of planar topologies called the r-neighborhood graph, and provide a generic analytical model for evaluating sparseness of such topologies. Unlike other research works, we derive our analysis by marrying the probability theory with the graph theory. The derived analytical expressions can be used in determining average node degree, topology size etc., without running simulations or prior to the deployment of real systems.

A careful insight to derive analytical expressions backed-up by the simulation results find that the performance of r-neighborhood graphs, in terms of topology size and sparseness, strongly depends on a number of network parameters (node number, deployment area, and node distribution), and some other transceiver parameters (transmission range, antenna height, and gain). Notably, with the increase in transmission power and node density the sparseness also increases. It turns out that, the r value in r-neighborhood graph also has major impact on the sparseness. More specifically, with the increase in r value the sparseness drastically increases.

The major contributions of the paper are summarized as follows: (i) we provide an analytical model for determining sparseness/size of a class of planar topologies dubbed as r-neighborhood graph (RNG and GG are special instances of this class), (ii) for the first time in the literature, we demonstrate how to analytically couple sparseness of planar topologies with the radio transceiver parameters. (iii) finally, we quantitatively explore how several factors such as transceiver and network parameters affect the sparseness of the planar topologies.

II. Related Works

The sparsest possible topology of n nodes is the *global* minimum spanning tree (MST) containing exactly $n-1$ edges. Somewhat closer to global MST is the *local* MST (LMST) proposed by Li *et al.* [9] where each node creates LMST within its neighborhood graph by assigning appropriate weight to an edge based on the necessary transmission power to reach its two ends. After constructing the LMST, each node contributes to the final topology those nodes that are its neighbors in its LMST. Although LMST can be constructed in an energy-efficient distributed manner, no analytical model is known to estimate its size or sparseness. In terms of planarity, another established solution is the Relative Neighborhood Graph (RNG) proposed by Toussaint [16] where a link $\langle s,t\rangle$ is eliminated if the distance $d(s,t)$ is greater than the distance of any other node w from s or t, i.e.: $\exists w \neq s,t : max(d(s,w), d(t,w)) < d(s,t)$.

On the other hand, GG [4], which is a super graph of RNG, eliminates a link $\langle s,t\rangle$ if for any other node w it happens that: $\exists w \neq s,t : d^2(s,w) + d^2(t,w)) \leq d^2(s,t)$.

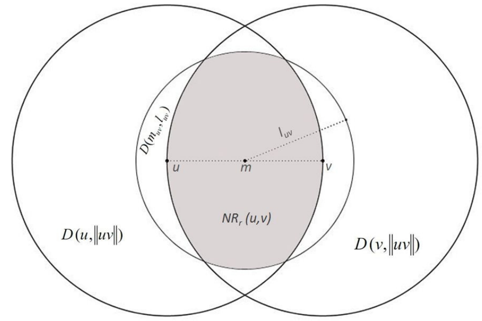


Fig. 1. The r-neighborhood region

Milic and Malek derived analytical models for quantifying dropped edges and face sizes of RNG and GG [11]. Inspired by their work, we propose a framework for more generic planar topologies dubbed as r-neighborhood graphs proposed by Jeng et al [5]. However Milic and Malek limit themselves only to sparseness. Here we provide analytical expressions for some additional performance metrics such as average node degree and topology sizes. There exists a volume of research constructing *minimum-energy path-preserving* (MEPP) topologies [14], [8] etc., neither of these topologies are planar nor there exist any analytical models to determine the sparseness/topology size. In [13], Rahman and Abu-Ghazaleh provide a generic framework for determining sparseness of only *MEPP topologies*. However, the authors provide no insight to determine sparseness of *planar topologies*. Thus, by introducing a general framework for modeling sparseness of *planar* topologies, we seek to fill a notable gap in the literature.

III. Background of the problem

In this section we present as background some of the definitions and the distributed algorithm for constructing r-neighborhood graph structures.

A. The r-Neighborhood region

The *r-neighborhood graph* [5] is based on a concept of a *region* dubbed as *r-neighborhood region* between any node pair u and v located on a two dimensional space. This region is basically the intersecting area of three circles:–(i) the circle centered at u with radius $\|uv\|$, (ii) the circle centered at v with radius $\|uv\|$, and finally, (iii) the circle centered at the middle point m_{uv} on the line segment uv with radius $l_{uv} = \frac{\sqrt{1+2r^2}}{2}$, where $0 \leq r \leq 1$. In Fig. 1, the shaded region is the r-neighborhood region between node pair (u,v), which is the intersecting area of three open disks as defined above. Let us assume that $D(x,d)$ denotes a circle centered at point x with radius d. Then the r-neighborhood region, denoted by $NR_r(u,v)$, is formally defined as:

$$NR_r(u,v) = D(u,\|uv\|) \cap D(v,\|uv\|) \cap D(m_{uv}, l_{uv})$$

In the following theorem we provide an important property of the r-neighborhood region. We prove the theorem in the appendix.

Theorem 1: For any two points u and v separated by distance $\|uv\|$, the area of r-neighborhood region is:

$$\mathcal{A}_{NR_r(u,v)} = \|uv\|^2 \left(\pi + \frac{\alpha^2}{2}\delta - 2\beta + \frac{\alpha^2}{4}\sin(2\delta) - \sin(2\beta) \right)$$

where, $0 \le r \le 1$, $\alpha = \sqrt{1+2r^2}$, $\delta = \sin^{-1}\frac{(3-\alpha^2)}{2\alpha}$, and $\beta = \sin^{-1}\left(\frac{5-\alpha^2}{4}\right)$

B. The r-Neighborhood graph

The r-neighborhood graph [5], denoted as $NG_r(V)$ over a set of nodes V, is a *proximity graph* where two vertices u and v are connected by an edge if and only if there is no node $w \in V$ is located in the r-neighborhood region $NR_r(u,v)$ of node pair (u,v). The $NG_r(V)$ at three different values of r is shown in Fig. 2. Note that, many links are removed from the initial graph shown in Fig. 2(a) while forming r-neighborhood graph structures and as we increase the value of r, the r-neighborhood graph becomes more and more sparse.

C. Some known results of r-Neighborhood graphs

In [5], the authors have analyzed r-neighborhood graphs $NG_r(V)$ using graph theoretic approaches. They have shown that all r-neighborhood graphs are *planar* graphs and is a generalized structure of both the GG and the RNG. In particular, for $r = 0$ the $NG_r(v)$ becomes GG and for $r = 1$ the $NG_r(V)$ becomes RNG. This relation ship between three graph structures is formally shown below:

$$RNG(V) \subseteq NG_r(V) \subseteq GG(V)$$

Two other important properties that were derived are power stretch factor $\rho(NG_r(v))$ and maximum node degree $d_{max}(NG_r(V))$. Both were defined as functions of r, and their definitions are as follows:

$$\rho(NG_r(V)) \le 1 + r^{\alpha}(n-2)$$

where n is the number of vertices, and,

$$d_{max}(NG_r(V)) = \frac{\pi}{sin^{-1}(r/2)}$$

D. Our contribution

Although the power stretch factor and the maximum node degree is known for r-neighborhood graphs, the average node degree, sparseness and topology size are unknown. Therefore, we develop analytical expressions to measure these new performance metrics and comment on their various aspects. Table I summarizes a complete picture of all performance measures including the new metrics, what is known and what is unknown at this point.

E. Algorithm for constructing r-neighborhood graphs

In this section we present as background the distributed algorithm for constructing r-neighborhood graph. The algorithm is presented with some inessential changes to make the notation and presentation more suitable for better understanding of the proposed analytical models. Let, W be the set of all nodes in the deployment area. For each $s \in W$, our goal is to find two sets of nodes:–(i) the *initNeighbor* set, and, (ii) *remainingNeighbor* set. The first set $initNeighbor$ is the set of nodes in the initial topology and the second set $remainingNeighbor$ is the set of nodes in the r-neighborhood graph. By taking union of the $remainingNeighbor$ sets of all nodes, the r-neighborhood graph $NG_r(V)$ is constructed. The distributed algorithm starts by broadcasting a "HELLO" message from each node $s \in W$ as shown in line 1 of the Algorithm 1 **DiscoverNeighbor**. While s collects the replies from its neighbors, it learns their identity and location. Each replying node v is handled in Algorithm 2 **HandleReply(v)**. In the algorithm, at first for each $x \in$ *remainingNeighbor* set, we check whether v is located in the r-neighborhood region of s and x (line 2). If that is the case, then we update the *remainingNeighbor* set by deleting node x in line 3. When this checking is done we insert v to the *remainingNeighbor* set (line 6). There might be a node in the *initNeighbor* set of s which is located in the r-neighborhood region of s and v. This checking is done in line 8. If such a node x is found then node v is deleted from the *remainingNeighbor* set (line 9).

TABLE I
PERFORMANCE METRICS OF $NG_r(V)$ AT A GLANCE)

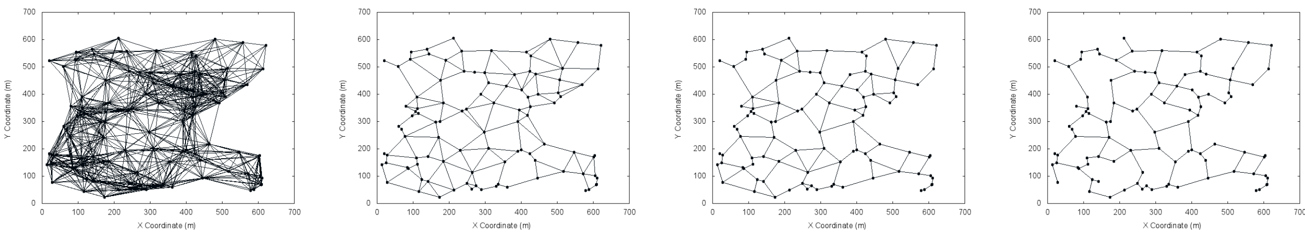
Performance metrics	Notation	Analytical expression
Power stretch factor	$\rho(NG_r(V))$	$1 + r^{\alpha}(n-2)$
Maximum node degree	$d_{max}(NG_r(V))$	$\frac{\pi}{sin^{-1}(r/2)}$
Average node degree	$d_{avg}(NG_r(V))$	**Unknown**
Sparseness	$\mathcal{F}_e$	**Unknown**
Topology size	$S(G_{NG_r(V)})$	**Unknown**

Algorithm 1 DiscoverNeighbor

1: $/ * broadcast \text{``}HELLO\text{''} message * /$
2: **for** each v that Replies **do**
3:     $handleReply(v)$
4: **end for**

Algorithm 2 HandleReply(v)

1: **for** each x in $remainingNeighbor$ **do**
2:     **if** $Loc(v) \in NR_r(s,x)$ **then**
3:         $remainingNeighbor = remainingNeighbor \setminus \{x\}$
4:     **end if**
5: **end for**
6: $remainingNeighbor = remainingNeighbor \cup \{v\}$
7: **for** each x in $initNeighbor$ **do**
8:     **if** $Loc(x) \in NR_r(s,v)$ **then**
9:         $remainingNeighbor = remainingNeighbor \setminus \{v\}$
10:         **goto** *marker*.
11:     **end if**
12: **end for**
13: *marker*: $initNeighbor = initNeighbor \cup \{v\}$



(a) Initial deployment (b) $r = 0$ (c) $r = 0.5$ (d) $r = 1$

Fig. 2. The initial topology and the r-neighborhood graph for three different levels of r

Finally, we insert node v in the *initNeighbor* set in line 14.

IV. Analytical Model

In this section, we develop mathematical expressions for determining *sparseness*, the *average node degree*, and *topology size*.

Consider a multi-hop wireless network with n nodes uniformly distributed over a rectangular region with area A. The average node density is $\mu = \frac{n}{A}$. The maximum transmission radius of each node is R. We assume that transmission (TX) range is homogeneous and same for every nodes.

Sparseness. The number of links removed from the initial topology determines the sparseness of a r-neighborhood subgraph. Here we focus on *per node* sparseness which is the average fraction of links removed from a node's neighborhood. When a link is removed, the node at the other end of the link also gets removed from a node's neighbor set. Thus, mathematically, (per node) sparseness is defined as follows:

$$\begin{aligned} \text{Sparseness} &= \text{Average fraction of nodes eliminated} \\ &= \frac{\text{Average number of nodes removed}}{\text{Number of nodes in a node's TX area}} \quad (1) \end{aligned}$$

As the node distribution is assumed to be uniform, it is easy to determine the number of nodes present in a node's transmission (TX) area if the node density is known apriori. To see how, let us observe an arbitrary node s within the deployment area. The average number of nodes located in the communication region of node s is:

$$N_R = \text{Node density} \times \text{Transmission area} = \mu \times \pi R^2 = \frac{\pi n R^2}{A}$$

To find the *average number of nodes removed* in Equation 1, at first we need to find $P_N(x)$, the probability that there exists a neighbor t at distance x from s. Clearly, $P_N(x) = 0 \; for \; x > R$. For $x \leq R$, consider a small area strip defined by dx at the perimeter of the circle with radius x and centered at s as shown in Fig. 3. Consider a small angle $d\theta$ measured from an arbitrary but fixed axis. The length of the arc $l = xd\theta$ and the area of the small region dA within this small strip can be approximated as $dA = ldx = xdxd\theta$. Therefore the area of the entire small strip denoted by A_{strip} becomes:

$$A_{strip} = \int_0^{2\pi} dA = \int_0^{2\pi} \ell dx = \int_0^{2\pi} xdxd\theta = 2\pi xdx$$

Thus $P_N(x)$ becomes:

$$\begin{aligned} P_N(x) &= \text{Area of the strip} \times \text{Node density} = A_{strip} \times \mu \\ &= 2\pi xdx \times \mu = 2\pi\mu xdx \quad (2) \end{aligned}$$

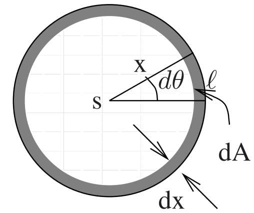


Fig. 3. Illustrating a circular strip at distance x

Once we find the probability of a node's existence at distance x, the next thing is to find the probability that a node s would *prune* such a node from its neighborset. According to the definition of r-neighborhood graph, the node t gets pruned from node s's neighborhood if there exists a node in the r-neighborhood region. Let $P_p(x)$ be the probability that there exists a node in the r-neighborhood region $NR_r(s,t)$ between node pair (s,t). The probability $P_E(x)$ of eliminating any node t from the neighbor set of s is the probability that there exists a neighbor t at distance x from s, and there is a node in the r-neighborhood region between (s,t). So $P_E(x)$ is:

$$P_E(x) = P_N(x) \times P_p(x) = \frac{2\pi nxdx}{A} \times P_p(x) = 2\pi\mu xdx \times P_p(x)$$

The expected number of neighbors eliminated by s from its neighbor set is found by integrating $P_E(x)$ over the transmission radius R within which s possibly can communicate:

$$\mathcal{T}_e = \int_0^R 2\pi\mu x \times P_P(x)dx \quad (3)$$

If we assume a disc communication area for s with radius R then the expected number of nodes within s's maximum communication range becomes $\pi R^2 \times \mu = \pi\mu R^2$. Therefore, if we divide $\mathcal{T}_e$ by $\pi\mu R^2$, we get the *average fraction of neighbors eliminated*, $\mathcal{F}_e$, which we define as *sparseness*,

$$Sparseness = \mathcal{F}_e = \frac{\mathcal{T}_e}{\pi\mu R^2} \quad (4)$$

Average node degree. The average node degree (d_{avg}) is the expected number of neighbors retained after pruning. Therefore, if we subtract $\mathcal{T}_e$ from the expected number of neighbors within s's communication range then we get d_{avg}:

$$\begin{aligned} d_{avg} &= \pi\mu R^2 - \mathcal{T}_e = \pi\mu R^2 \left(1 - \frac{\mathcal{T}_e}{\pi\mu R^2}\right) \\ &= \pi\mu R^2 (1 - \mathcal{F}_e) = \pi\mu R^2 (1 - Sparseness) \quad (5) \end{aligned}$$

Topology size. Finally, if we multiply d_{avg} by the total number of nodes n, we obtain twice the number of links retained after running the topology control algorithm (an edge contributes to exactly two node's degree counts). Thus, the size of the r-neighborhood graph $G_{NG_r(V)}$ becomes,

$$\begin{aligned} S(G_{NG_r(V)}) &= \frac{n \times d_{avg}}{2} \\ &= \frac{\pi \mu n R^2}{2} (1 - Sparseness) \end{aligned} \tag{6}$$

V. Sparseness of r-neighborhood graphs

In this section we derive the exact expression of sparseness. Let us revisit Equation 4. To solve the equation we need to determine the quantity, pruning probability $P_p(x)$. Recall that, $P_p(x)$ is the probability that a node t located at distance x from a node s gets pruned. As, node t gets pruned from node s's neighborhood if there exists at least one node in the r-neighborhood region, $P_p(x)$ becomes the probability that there exists a node in the r-neighborhood region $NR_r(s,t)$ between node pair (s,t). The distance x between s and t plays an important role in determining the value of $P_p(x)$. For large x, the size of the r-neighborhood region $NR_r(s,t)$ is also large and the probability of a node's existence within this region also becomes large. At first, let us find the probability that a certain number of nodes k is located within the r-neighborhood region of the node pair (s,t). The probability P_T that a node is placed in this r-neighborhood region $NR_r(s,t)$ within the deployment area A is:

$$P_T = \frac{\text{Area of r-neighborhood region}}{\text{Total deployment area}} = \frac{\mathcal{A}_{NR_r(s,t)}}{A}.$$

According to Theorem 1 the area of r-neighborhood region:

$$\mathcal{A}_{NR_r(s,t)} = \|st\|^2 \left(\pi + \frac{\alpha^2}{2}\delta - 2\beta + \frac{\alpha^2}{4}\sin(2\delta) - \sin(2\beta)\right)$$

Therefore, P_T becomes:

$$P_T = \frac{\|st\|^2 \left(\pi + \frac{\alpha^2}{2}\delta - 2\beta + \frac{\alpha^2}{4}\sin(2\delta) - \sin(2\beta)\right)}{A}$$

The distance between s and t is x, i.e., $x = \|st\|$. Let $\gamma = \left(\pi + \frac{\alpha^2}{2}\delta - 2\beta + \frac{\alpha^2}{4}\sin(2\delta) - \sin(2\beta)\right)$. So P_T becomes:

$$P_T = \frac{\gamma \|st\|^2}{A} = \frac{\gamma x^2}{A} \tag{7}$$

The probability $P_m(NR_r(s,t))$ that exactly m nodes are located within r-neighborhood region $NR_r(s,t)$ is:

$$P_m(NR_r(s,t)) = \binom{n-2}{m} P_T^m \times (1 - P_T)^{n-2-m}$$

Note that, here we have used $n-2$ rather than n because we exclude s and t. For large n and small P_T, the binomial distribution can be approximated using Poisson distribution with mean nP_T [1]. Thus,

$$P_m(NR_r(s,t)) = \frac{(nP_T)^m \times e^{-nP_T}}{m!}$$

The probability that there exists at least one node within r-neighborhood region $NR_r(s,t)$ is:

$$\begin{aligned} P_p(x) &= \sum_{k=1}^{n} P_m(NR_r(s,t)) = \sum_{k=1}^{\infty} \frac{(nP_T)^m \times e^{-nP_T}}{m!} \\ &= e^{-nP_T} \sum_{k=1}^{\infty} \frac{(nP_T)^m}{m!} = e^{-nP_T}\left(\sum_{k=0}^{\infty} \frac{(nP_T)^m}{m!} - 1\right) \\ &= e^{-nP_T}(e^{-nP_T} - 1) = 1 - e^{-nP_T} \end{aligned} \tag{8}$$

By substituting the value of P_T from Equation 7 into Equation 8, we get:

$$P_p(x) = 1 - e^{\frac{-n\gamma x^2}{A}} = 1 - e^{-\gamma\mu x^2} \left[\text{As, } \mu = \frac{n}{A}\right] \tag{9}$$

By replacing $P_p(x)$ in Equation 3 we get:

$$\begin{aligned} \mathcal{T}_e &= \int_0^R 2\pi\mu x \times P_p(x) dx = \int_0^R 2\pi\mu x \times (1 - e^{-\gamma\mu x^2}) dx \\ &= \pi\mu R^2 - \pi\mu \int_0^R 2x e^{-\gamma\mu x^2} dx \end{aligned} \tag{10}$$

Let $\gamma\mu x^2 = z$. $\therefore 2xdx = \frac{dz}{\gamma\mu}$. Using these values we get:

$$\begin{aligned} \mathcal{T}_e &= \pi\mu R^2 - \frac{\pi}{\gamma}\int_0^{\gamma\mu R^2} e^{-z} dz \\ &= \pi\mu R^2 + \frac{\pi}{\gamma}\left(e^{-\gamma\mu R^2} - 1\right) \end{aligned} \tag{11}$$

Sparseness, $\mathcal{F}_e = \mathcal{T}_e / \pi\mu R^2$. Therefore:

$$\begin{aligned} Sparseness = \mathcal{F}_e &= \frac{\pi\mu R^2 + \frac{\pi}{\gamma}\left(e^{-\gamma\mu R^2} - 1\right)}{\pi\mu R^2} \\ &= \frac{\gamma\mu R^2 + \mathrm{e}^{-\gamma\mu R^2} - 1}{\gamma\mu R^2} \end{aligned} \tag{12}$$

Substituting the value of sparseness on Equation 5 we get:

$$\begin{aligned} d_{avg} &= \pi\mu R^2 (1 - Sparseness) \\ &= \pi\mu R^2 \left(1 - \frac{\gamma\mu R^2 + \mathrm{e}^{-\gamma\mu R^2} - 1}{\gamma\mu R^2}\right) \\ &= \frac{\pi}{\gamma}\left(1 - \mathrm{e}^{-\gamma\mu R^2}\right) \end{aligned} \tag{13}$$

Finally, substituting sparseness on Equation 6 we get:

$$\begin{aligned} S(G_{NG_r(V)}) &= \frac{\pi\mu n R^2}{2}(1 - Sparseness) \\ &= \frac{\pi\mu n R^2}{2}\left(1 - \frac{\gamma\mu R^2 + \mathrm{e}^{-\gamma\mu R^2} - 1}{\gamma\mu R^2}\right) \\ &= \frac{\pi n}{2\gamma}\left(1 - \mathrm{e}^{-\gamma\mu R^2}\right) = \frac{n}{2} \times d_{avg} \end{aligned} \tag{14}$$

VI. Different r-neighborhood graphs

Let us find the expression of $\mathcal{F}_e$ for two extreme values of r. Recall that, r assumes a value between $0 \le r \le 1$.

$$\begin{aligned} r &= 0, \alpha = \sqrt{(1+2r^2)} = \sqrt{(1+2.0^2)} = 1 \\ \delta &= \sin^{-1}\frac{(3-\alpha^2)}{2\alpha} = \sin^{-1}\frac{(3-1^2)}{2\times 1} = \sin^{-1}(1) = \frac{\pi}{2} \\ \beta &= \sin^{-1}\left(\frac{5-\alpha^2}{4}\right) = \sin^{-1}\left(\frac{5-1^2}{4}\right) = \sin^{-1}(1) = \frac{\pi}{2} \\ \gamma &= \left(\pi + \frac{\alpha^2}{2}\delta - 2\beta + \frac{\alpha^2}{4}\sin(2\delta) - \sin(2\beta)\right) \\ &= \left(\pi + \frac{1^2}{2}\frac{\pi}{2} - 2\frac{\pi}{2} + \frac{1^2}{4}\sin(2\frac{\pi}{2}) - \sin(2\frac{\pi}{2})\right) = \frac{\pi}{4} \end{aligned}$$

Plugging in the value of γ into Equation 12 we get:

$$\mathcal{F}_e = \frac{\pi\mu R^2 + 4\mathrm{e}^{\frac{-\mu\pi R^2}{4}} - 4}{\pi\mu R^2} \tag{15}$$

As, for $r = 0$, the r-neighborhood graph becomes GG [5], Equation 15 gives us the sparseness expression for GG graphs. Similarly for $r = 1$,

$$\alpha = \sqrt{3}, \delta = 0, \beta = \frac{\pi}{6}, \gamma = \frac{4\pi - 3\sqrt{3}}{6}$$

Plugging in the value of γ into Equation 12 we get:

$$\mathcal{F}_e = \frac{(4\pi - 3\sqrt{3})\mu R^2 + 6\left(\mathrm{e}^{-\frac{1}{6}(4\pi - 3\sqrt{3})\mu R^2} - 1\right)}{(4\pi - 3\sqrt{3})\mu R^2} \tag{16}$$

As, for $r = 1$, the r-neighborhood graph becomes RNG [5], Equation 16 gives us the sparseness expression for RNG.

VII. Simulation Results

In this section, we present simulation results to verify the accuracy of our analytical model. We also explore the effect of network and transceiver parameters on performance metrics.

A. Simulation environment and performance metrics

To evaluate the performance, we simulate randomly deployed networks of 100–500 nodes over a 625m × 625m square region. The maximum transmission radius is limited between 125m to 225m. We consider three different values of $r = 0, \frac{1}{2}$, and 1. Three performance metrics *sparseness*, *average node degree* and *topology size* are measured. For detailed definition and analytical expressions for these performance metrics please see Section IV and Section V.

B. Experimental Results

r-Neighborhood regions. The area of $NR_r(u, v)$ increases with the the increase of r which indicates that the probability that at least one node exists in this $NR_r(u, v)$ also increases. Thus, one should expect more and more elimination of links when r value is increased.

Effect of node density on sparseness. Node density were varied by varying number of nodes between 100–500 while keeping the deployment region constant at 625m × 625m square area. A higher fraction of neighbors is eliminated in more dense networks for all transmission ranges. With larger node densities, it is highly probable that at least one node exists in the r-neighborhood region of a link, and the link gets pruned by the algorithm. Also the fraction of eliminated neighbors is much higher in Fig. 4(b) compared to 4(a) because the r value increases from 0 to $\frac{1}{2}$. The sparseness increases when we increase r values because the size of the r-neighborhood region also increases (cf. see Fig. 4). Figure 4(a), 4(b) and 4(c) show that for all scenarios, the results of analytical expressions are very close to the simulation results; the difference is very small, maximal being around 4.5%. The small inaccuracy arises from the nodes located close to the boundaries of the deployment region, for which the communication area is restricted, and thus they have fewer neighbors. We ignored this "boundary effects" to simplify the analytical models.

Effect of TX range on sparseness. To see the effect of transmission range on sparseness, we measure $\mathcal{F}_e$ with different node densities. The transmission range is varied between 125m to 225m with an increment of 25m at each step. Figure 5 shows the result for three different values of r. When transmission range is increased, $\mathcal{F}_e$ exponentially increases at the beginning and linearly increases at the end. Also, as expected, the sparseness increases when we increase r values.

Average node Degree. Fig. 6 shows the plot of Equation 13 under two transmission ranges $R = 225$m and $R = 125$m. With the increase of r, the average node degree $d_{avg}(NG_r(V))$ decays exponentially indicating that the r-neighborhood graph becomes sparser when we increase r.

Topology size. Based on Equation 6, the topology size can be derived by multiplying $d_{avg}(NG_r(V))$ with a constant $\frac{n}{2}$. Therefore, its plot would have the similar shape of Fig. 6 but magnified by a factor of $\frac{n}{2}$. As it is clearly evident from the discussion, we omit the plots due to page limitations.

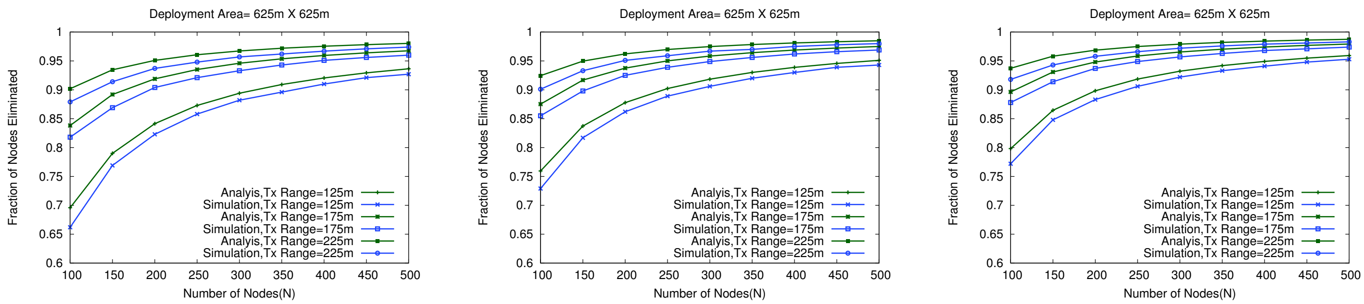
C. Comparison with traditional topologies

r-Neighborhood graphs can be compared to some traditional topologies such as Minimum Spanning Tree (MST), Local Minimum Spanning Tree (LMST), RNG and GG in terms of sparsity. Prior work [12] has shown that $MST \subseteq RNG \subseteq GG$. Cartigny et al. [2] show that $LMST \subseteq RNG$. It is also known that LMST contains MST (see [12]). For r-neighborhood graph NG_r, the authors in [5] show that $RNG \subseteq NG_r \subseteq GG$. Combining all these results the following conclusion can be made about the topology size of r-neighborhood graph:

$$|MST| \le |LMST| \le |RNG| \le |NG_r| \le |GG|$$

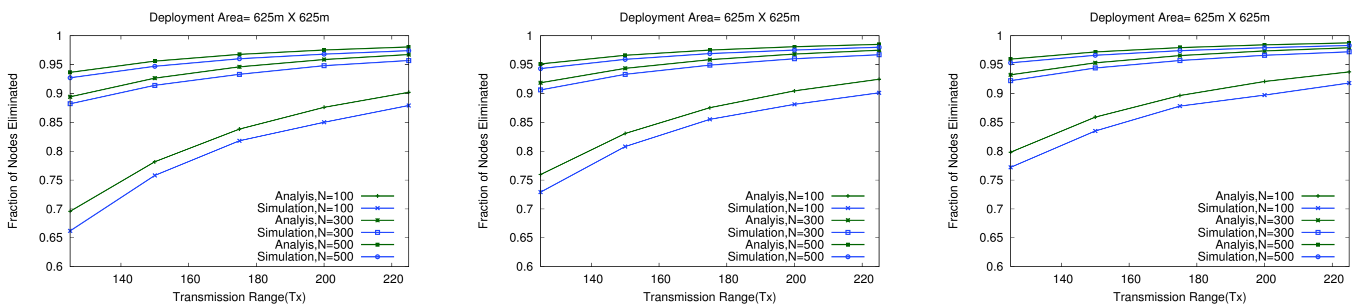
VIII. Conclusion

r-neighborhood graphs constitute an important class of *planar* topologies. The algorithm generating such topologies is very appealing and practically implementable due to their simplicity, distributed property, and strictly local behavior. We provided analytical models to determine the structural densities for this class of topologies. Using the proposed models, the



(a) $r = 0$ (b) $r = 1/2$ (c) $r = 1$

Fig. 4. Number of nodes were varied



(a) $r = 0$ (b) $r = 1/2$ (c) $r = 1$

Fig. 5. Transmission ranges were varied

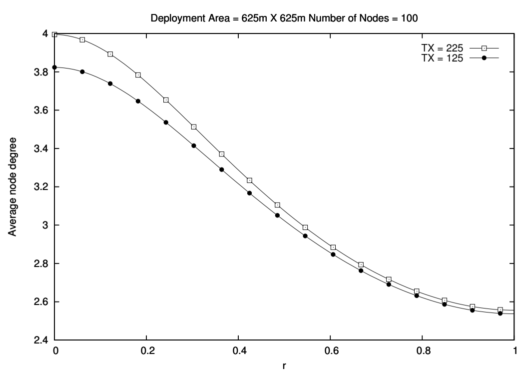


Fig. 6. Effect of r on average node degree

network designer can easily estimate the sparseness, average node degree and network size for the desired topology, perhaps prior to the network deployment. In future we plan to extend the model to accommodate other kind of node distributions.

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APPENDIX

Theorem 1: For any two points U and V separated by distance $\|UV\|$, the area of r-neighborhood region is:

$$\mathcal{A}_{NR_r(U,V)} = \|UV\|^2\left(\pi + \frac{\alpha^2}{2}\delta - 2\beta + \frac{\alpha^2}{4}\sin(2\delta) - \sin(2\beta)\right)$$

where, $0 \le r \le 1$, $\alpha = \sqrt{1+2r^2}$, $\delta = \sin^{-1}\frac{(3-\alpha^2)}{2\alpha}$, and $\beta = \sin^{-1}\left(\frac{5-\alpha^2}{4}\right)$

Proof:

Let us consider Fig. 7. Without loss of generality let us assume that U and V are located at $(0,0)$ and $(\|UV\|, 0)$

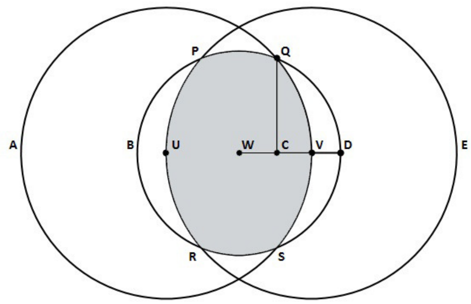

Fig. 7. area calculation for r-neighborhood graph

respectively. There are three circles in the figure:-(i) the circle $AQVS$ centered at $U(0,0)$ with radius $\|UV\|$, (ii) the circle $PERU$ centered at $V(\|UV\|,0)$ with radius $\|UV\|$, and finally (iii) the circle $BPQDSR$ centered at $W\left(\frac{\|UV\|}{2},0\right)$ with radius $\frac{\alpha\|UV\|}{2}$ where $\alpha=\sqrt{1+2r^2}$ and $0\leq r\leq 1$. The coordinate of point D becomes $\left(\frac{(1+\alpha)\|UV\|}{2},0\right)$. The shaded region in Fig. 7 is the r-neighborhood region $NR_r(U,V)$ between node pair U and V. The area of this shaded region can be expressed as follows:

$$\mathcal{A}_{NR_r(U,V)}=\mathcal{A}_{BPQDSR}-4\times\mathcal{A}_{QDV}$$

$$\begin{aligned}\mathcal{A}_{NR_r(U,V)}&=\pi\frac{\|WD\|^2}{2}-4\times(\mathcal{A}_{QCD}-\mathcal{A}_{QCV})\\&=\pi\left(\frac{\alpha\|UV\|}{2}\right)^2-4\times(\mathcal{A}_{QCD}-\mathcal{A}_{QCV})\end{aligned}$$

Next, we find the area $\mathcal{A}_{QCD}$ which can be derived by integrating the equation of the circle $BPQDSR$ from C to D. To find the coordinate of point C, at first we determine the coordinate of point Q which is the intersecting point of the two circles $AQVS$ and $BPQDSR$. The equation of the circle $AQVS$ with center $(0,0)$ and radius $\|UV\|$ is:

$$x^2+y^2=\|UV\|^2 \tag{17}$$

Equation of circle $BPQDSR$ with center $\left(\frac{\|UV\|}{2},0\right)$ and radius $\frac{\alpha\|UV\|}{2}$ is:

$$\left(x-\frac{\|UV\|}{2}\right)^2+y^2=\left(\frac{\alpha\|UV\|}{2}\right)^2 \tag{18}$$

Subtracting Equation 18 from Equation 17 we get,

$$x=\frac{(5-\alpha^2)\|UV\|}{4}$$

As point C and point Q share the same x coordinate and C lies on the x-axis, the coordinate of point C becomes $\left(\frac{(5-\alpha^2)\|UV\|}{4},0\right)$.

And the area $\mathcal{A}_{QCD}$ is,

$$\mathcal{A}_{QCD}=\int_{\frac{(5-\alpha^2)\|UV\|}{4}}^{\frac{(1+\alpha)\|UV\|}{2}}\left(\sqrt{\frac{\|UV\|^2\alpha^2}{4}-\left(x-\frac{\|UV\|}{2}\right)^2}\right)\mathrm{d}x$$

Let us assume that, $x-\frac{\|UV\|}{2}=\frac{\|UV\|\alpha}{2}\sin\theta$. Therefore, $\mathrm{d}x=\frac{\|UV\|\alpha}{2}\cos\theta\mathrm{d}\theta$. For the upper limit $x=\frac{(1+\alpha)\|UV\|}{2}$ we get:

$$\begin{aligned}\frac{(1+\alpha)\|UV\|}{2}-\frac{\|UV\|}{2}&=\frac{\alpha\|UV\|}{2}\sin\theta\\\Leftrightarrow\sin\theta=1\Leftrightarrow\theta&=\frac{\pi}{2}\end{aligned}$$

Similarly, for the lower limit $x=\frac{(5-\alpha^2)\|UV\|}{4}$ we get:

$$\begin{aligned}\frac{(5-\alpha^2)\|UV\|}{4}-\frac{\|UV\|}{2}&=\frac{\|UV\|\alpha}{2}\sin\theta\\\Leftrightarrow\sin\theta=\frac{(3-\alpha^2)}{2\alpha}&\Rightarrow\theta=\sin^{-1}\frac{(3-\alpha^2)}{2\alpha}\end{aligned}$$

Let, $\sin^{-1}\frac{(3-\alpha^2)}{2\alpha}=\delta$. After all substitutions we get:

$$\begin{aligned}\mathcal{A}_{QCD}&=\int_{\delta}^{\frac{\pi}{2}}\frac{\|UV\|^2\alpha^2}{4}\cos^2\theta\mathrm{d}\theta\\&=\frac{\|UV\|^2\alpha^2}{8}\left[\frac{\pi}{2}-\delta-\frac{1}{2}\sin(2\delta)\right]\end{aligned}$$

Finally, we derive the area $\mathcal{A}_{QCV}$ by integrating the equation of the circle $AQVS$ from C to V. Therefore:

$$\mathcal{A}_{QCV}=\int_{\frac{(5-\alpha^2)\|UV\|}{4}}^{\|UV\|}\left(\sqrt{\|UV\|^2-x^2}\right)\mathrm{d}x$$

Let, $x=\|UV\|\sin\theta$, therefore, $\mathrm{d}x=\|UV\|\cos\theta\mathrm{d}\theta$. For the upper limit, $x=\|UV\|$, the angle θ becomes:

$$\|UV\|=\|UV\|\sin\theta\Rightarrow\theta=\frac{\pi}{2}$$

For the lower limit $x=\frac{(5-\alpha^2)\|UV\|}{4}$, the angle θ becomes:

$$\begin{aligned}\left(\frac{5-\alpha^2}{4}\right)\|UV\|&=\|UV\|\sin\theta\\\Rightarrow\theta&=\sin^{-1}\left(\frac{5-\alpha^2}{4}\right)\end{aligned}$$

Let, $\sin^{-1}\left(\frac{5-\alpha^2}{4}\right)=\beta$. After all substitutions we get:

$$\begin{aligned}\mathcal{A}_{QCV}&=\int_{\beta}^{\frac{\pi}{2}}\|UV\|^2\cos^2\theta\mathrm{d}\theta\\&=\frac{\|UV\|^2}{2}\left[\frac{\pi}{2}-\beta-\frac{1}{2}\sin^{-1}(2\beta)\right]\end{aligned}$$

$$\begin{aligned}\mathcal{A}_{NR_r(U,V)}=&\pi\left(\frac{\alpha\|UV\|}{2}\right)^2-4\times(\mathcal{A}_{QCD}-\mathcal{A}_{QCV})\\=&\pi\left(\frac{\alpha\|UV\|}{2}\right)^2-4\left(\frac{\|UV\|^2\alpha^2}{8}\left[\frac{\pi}{2}-\delta-\frac{1}{2}\sin(2\delta)\right]\right)\\&+4\left(\frac{\|UV\|^2}{2}\left[\frac{\pi}{2}-\beta-\frac{1}{2}\sin^{-1}(2\beta)\right]\right)\\=&\|UV\|^2\left(\pi+\frac{\alpha^2}{2}\delta-2\beta+\frac{\alpha^2}{4}\sin(2\delta)-\sin(2\beta)\right)\end{aligned}$$

$\mathcal{Q.E.D.}$