ACADEMIC PRESS
J. Math. Anal. Appl. 272 (2002) 368–379
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.academicpress.com

Formulation of Euler–Lagrange equations for fractional variational problems

Om P. Agrawal

Southern Illinois University, Carbondale, IL 62901, USA

Received 27 July 2001
Submitted by M.A. Noor

Abstract

This paper presents extensions to traditional calculus of variations for systems containing fractional derivatives. The fractional derivative is described in the Riemann–Liouville sense. Specifically, we consider two problems, the simplest fractional variational problem and the fractional variational problem of Lagrange. Results of the first problem are extended to problems containing multiple fractional derivatives and unknown functions. For the second problem, we also present a Lagrange type multiplier rule. For both problems, we develop the Euler–Lagrange type necessary conditions which must be satisfied for the given functional to be extremum. Two problems are considered to demonstrate the application of the formulation. The formulation presented and the resulting equations are very similar to those that appear in the field of classical calculus of variations.

Keywords: Fractional derivative; Fractional calculus; Fractional calculus of variations; Fractional optimal control; Fractional variational problems

1. Introduction

The field of calculus of variations is of significant importance in various disciplines such as science, engineering, and pure and applied mathematics (see, for

E-mail address: om@engr.siu.edu.

PII: S0022-247X(02)00180-4

example, [1–6]. Reference [7] presents a Bliss-type multiplier rule for constrained variational problems with delay. Calculus of variations has been the starting point for various approximate numerical schemes such as Ritz, finite difference, and finite element methods (see [2,8]).

Functional minimization problems naturally occur in engineering and science where minimization of functionals, such as, Lagrangian, strain, potential, and total energy, etc. give the laws governing the systems behavior. In optimal control theory, minimization of certain functionals give control functions for optimum performance of the system.

Although many laws of the nature can be obtained using certain functionals and the theory of calculus of variations, not all laws can be obtained this way. For example, almost all systems contain internal damping, yet the traditional energy based approach cannot be used to obtain equations describing the behavior of a nonconservative system (see [9,10]). Recently, Refs. [9,10] presented a new approach to mechanics that allows one to obtain the equations for a nonconservative system using certain functionals. In these references, fractional derivative terms were introduced in functionals to obtain nonconservative terms in the desired differential equations.

Fractional derivatives, or more precisely derivatives of arbitrary orders, have played a significant role in engineering, science, and pure and applied mathematics in recent years. As [11] point out, there is hardly a field or science or engineering that has remained untouched by this field. Reference [12] provide an encyclopedic treatment of this subject. Additional background, survey, and application of this field in science, engineering, and mathematics can be found, among others, in [11–17].

Recent investigations have shown that many physical systems can be represented more accurately using fractional derivative formulations (see, for example, [15,18]). Given this, one can imagine obtaining these formulations by minimizing certain functionals. These functionals will naturally contain fractional derivative terms, and mathematical tools analogous to calculus of variations will be needed to minimize these functional. However, very little work has been done in the area of fractional calculus of variations [9,10].

This paper provides some new results in the area of fractional calculus of variations. A *fractional calculus of variations problem* is a problem in which either the objective functional or the constraint equations or both contain at least one fractional derivative term. In this paper we will develop necessary conditions for two problems from this field, first, minimization of a functional subjected to specified boundary conditions, and second, minimization of a functional subjected to constrains and specified boundary conditions. Both functional and the constraints will be allowed to have fractional derivative terms.

2. The simplest fractional variational problem

Several definitions of a fractional derivative have been proposed. These definitions include Riemann–Liouville, Grunwald–Letnikov, Weyl, Caputo, Marchaud, and Riesz fractional derivatives [11,13,16,17]. Here, we formulate the problem in terms of the left and the right Riemann–Liouville fractional derivatives, which are defined as [16]

The left Riemann–Liouville fractional derivative

$$ {}_aD_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_a^x (x-\tau)^{n-\alpha-1} f(\tau)\,d\tau, \tag{1}$$

and

The right Riemann–Liouville fractional derivative

$$ {}_xD_b^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dx}\right)^n \int_x^b (x-\tau)^{n-\alpha-1} f(\tau)\,d\tau, \tag{2}$$

where α is the order of the derivative such that $n-1 \leqslant \alpha < n$. If α is an integer, these derivatives are defined in the usual sense, i.e.,

$$ {}_aD_x^\alpha f(x) = \left(\frac{d}{dx}\right)^\alpha, \quad {}_xD_b^\alpha f(x) = \left(-\frac{d}{dx}\right)^\alpha, \quad \alpha = 1, 2, \ldots. \tag{3}$$

These derivatives will be denoted as the LRLFD and the RRLFD, respectively. Note that in the literature the Riemann–Liouville fractional derivative generally means the LRLFD. From physical point of view, if x is considered as a time scale, the RRLFD represents an operation performed on the future state of the process $f(x)$. This derivative has generally been neglected with the assumption that the present state of a process does not depend on the results of its future development. However, the derivations to follow will show that both derivatives naturally occur in a problem of fractional calculus of variations.

Using the above definitions, the first simplest fractional calculus of variations problem can be defined as follows: *Let $F(x, y, u, v)$ be a function with continuous first and second* (*partial*) *derivatives with respect to all its arguments. Then, among all functions $y(x)$ which have continuous LRLFD of order α and RRLFD of order β for $a \leqslant x \leqslant b$ and satisfy the boundary conditions*

$$ y(a) = y_a, \qquad y(b) = y_b \tag{4}$$

find the function for which the functional

$$ J[y] = \int_a^b F\big(x, y, {}_aD_x^\alpha y, {}_xD_b^\beta y\big)\,dx \tag{5}$$

is an extremum, where $0 < \alpha, \beta \leqslant 1$. The continuity requirement on F can be given more precisely. However, these assumptions are made for simplicity. Note that (1) we have included both the LRLFD and the RRLFD for generality. (2) We first consider $0 < \alpha, \beta \leqslant 1$. The case of $\alpha, \beta \in R^+$ will be consider shortly. (3) When $\alpha = \beta = 1$, the above problem reduces to the simplest variational problem.

To develop the necessary conditions for the extremum, assume that $y^*(x)$ is the desired function. Let $\epsilon \in R$, and define a family of curve

$$y(x) = y^*(x) + \epsilon\eta(x) \tag{6}$$

which satisfy the boundary conditions; i.e., we require that

$$\eta(a) = \eta(b) = 0. \tag{7}$$

Since ${}_aD_x^\alpha$ and ${}_xD_b^\beta$ are linear operators, it follows that

$${}_aD_x^\alpha y(x) = {}_aD_x^\alpha y^*(x) + \epsilon\, {}_aD_x^\alpha \eta(x), \tag{8a}$$

$${}_xD_b^\beta y(x) = {}_xD_b^\beta y^*(x) + \epsilon\, {}_xD_b^\beta \eta(x) \tag{8b}$$

Substituting Eqs. (6) and (8) into Eq. (5), we find that for each $\eta(x)$

$$J = J[\epsilon] = \int_a^b F\bigl(x, y^* + \epsilon\eta, {}_aD_x^\alpha y^* + \epsilon\, {}_aD_x^\alpha \eta, {}_xD_b^\beta y^* + \epsilon\, {}_xD_b^\beta \eta\bigr)\, dx \tag{9}$$

is a function of ϵ only. Note that $J[\epsilon]$ is extremum at $\epsilon = 0$. Differentiating Eq. (9) with respect to ϵ, we obtain

$$\frac{dJ}{d\epsilon} = \int_a^b \left[\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial\, {}_aD_x^\alpha y}\, {}_aD_x^\alpha \eta + \frac{\partial F}{\partial\, {}_xD_b^\beta y}\, {}_xD_b^\beta \eta\right] dx. \tag{10}$$

Equation (10) is also called the variations of $J[y]$ at $y(x)$ along $\eta(x)$. A necessary condition for $J[\epsilon]$ to have an extremum is that $dJ/d\epsilon$ must be zero, and this should be true for all admissible $\eta(x)$. This leads to the condition that for $J[y]$ to have an extremum for $y = y^*(x)$ is that

$$\int_a^b \left[\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial\, {}_aD_x^\alpha y}\, {}_aD_x^\alpha \eta + \frac{\partial F}{\partial\, {}_xD_b^\beta y}\, {}_xD_b^\beta \eta\right] dx = 0 \tag{11}$$

for all admissible $\eta(x)$. Using the formula for fractional integration by parts, the second integral in Eq. (11) can be written as [9,12]

$$\int_a^b \frac{\partial F}{\partial\, {}_aD_x^\alpha y}\, {}_aD_x^\alpha \eta\, dx = \int_a^b {}_xD_b^\alpha \left(\frac{\partial F}{\partial\, {}_aD_x^\alpha y}\right)\eta\, dx \tag{12}$$

provided that $\partial F/\partial_a D_x^\alpha y$ or η is zero at $x=a$ and $x=b$. Using Eq. (7), this condition is satisfied, and it follows that Eq. (12) is valid. Similarly, the third integral in Eq. (11) can be written as

$$\int_a^b \frac{\partial F}{\partial_x D_b^\beta y}\, {}_x D_b^\alpha \eta\, dx = \int_a^b {}_a D_x^\beta \left(\frac{\partial F}{\partial_x D_b^\alpha y}\right) \eta\, dx. \tag{13}$$

Substituting Eqs. (12) and (13) into Eq. (11), we get

$$\int_a^b \left[\frac{\partial F}{\partial y} + {}_x D_b^\alpha \frac{\partial F}{\partial_a D_x^\alpha y} + {}_a D_x^\beta \frac{\partial F}{\partial_x D_b^\beta y}\right] \eta\, dx = 0. \tag{14}$$

Since $\eta(x)$ is arbitrary, it follows from a well established result in calculus of variations that [2]

$$\frac{\partial F}{\partial y} + {}_x D_b^\alpha \frac{\partial F}{\partial_a D_x^\alpha y} + {}_a D_x^\beta \frac{\partial F}{\partial_x D_b^\beta y} = 0. \tag{15}$$

Equation (15) is the Euler–Lagrange equation for the fractional calculus of variations problem. Thus, we have

Theorem 1. *Let $J[y]$ be a functional of the form*

$$\int_a^b F\left(x, y, {}_a D_x^\alpha y, {}_x D_b^\beta y\right) dx,$$

defined on the set of functions $y(x)$ which have continuous LRLFD of order α and RRLFD of order β in $[a,b]$ and satisfy the boundary conditions $y(a)=y_a$ and $y(b)=y_b$. Then a necessary condition for $J[y]$ to have an extremum for a given function $y(x)$ is that $y(x)$ satisfy following Euler–Lagrange equation:

$$\frac{\partial F}{\partial y} + {}_x D_b^\alpha \frac{\partial F}{\partial_a D_x^\alpha y} + {}_a D_x^\beta \frac{\partial F}{\partial_x D_b^\beta y} = 0.$$

Note that for fractional calculus of variation problems the resulting Euler–Lagrange equation contains both the LRLFD and the RRLFD. This is expected since the optimum function must satisfy both terminal conditions. Further, for $\alpha=\beta=1$, we have ${}_a D_x^\alpha = d/dx$ and ${}_x D_b^\beta = -d/dx$, and Eq. (15) reduces to the standard Euler–Lagrange equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y^{(1)}} = 0, \tag{16}$$

where $y^{(1)} = dy/dx$.

3. The case of $\alpha, \beta \in R^+$ and several functions

We now consider further generalization of the above problem. Specifically, we consider two different cases, first, in which $\alpha_j, \beta_j \in R^+$ $(j = 1, \ldots)$, i.e., one can have multiple positive α and β, and second, in which one has more than one function. In both cases, we consider the end points fixed.

Case 1. Fixed end points and $\alpha_j, \beta_j \in R^+$ $(j = 1, \ldots)$.

Assume that α_j $(j = 1, \ldots, n)$ and β_k $(k = 1, \ldots, m)$ are two sets of real numbers all greater than zero,

$$\alpha_{\max} = \max(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m) \tag{17}$$

is the maximum of all these numbers, and M is an integer such that $M - 1 \leqslant \alpha_{\max} < M$. Assume that $F(x, y, z_1, \ldots, z_{m+n})$ is a function with continuous first and second (partial) derivatives with respect to all its arguments, and consider a functional of the form

$$J[y] = \int_a^b F\big(x, y, {}_aD_x^{\alpha_1} y, \ldots, {}_aD_x^{\alpha_n} y, {}_xD_b^{\beta_1} y, \ldots, {}_xD_b^{\beta_m} y\big)\, dx. \tag{18}$$

The problem can now be defined as follows: *Among all functions $y(x)$ satisfying the conditions*

$$y(a) = y_{a0}, \quad y^{(1)}(a) = y_{a1}, \quad \ldots, \quad y^{(M-1)}(a) = y_{a(M-1)}, \tag{19a}$$

$$y(b) = y_{b0}, \quad y^{(1)}(b) = y_{b1}, \quad \ldots, \quad y^{(M-1)}(b) = y_{b(M-1)}, \tag{19b}$$

find the function for which Eq. (18) *has an extremum*. Here it is implicitly assumed that $y(x)$ meets all the differentiability requirements.

The necessary condition for this problem can be found following the approach presented above. This leads to

Theorem 2. *Let $J[y]$ be a functional of the form given by Eq.* (18) *defined on the set of functions satisfying the boundary conditions given by Eq.* (19). *Then a necessary condition for $J[y]$ to have an extremum for a given function $y(x)$ is that $y(x)$ satisfy the Euler–Lagrange equation*

$$\frac{\partial F}{\partial y} + \sum_{j=1}^{n} {}_xD_b^{\alpha_j} \frac{\partial F}{\partial\, {}_aD_x^{\alpha_j} y} + \sum_{k=1}^{m} {}_aD_x^{\beta_k} \frac{\partial F}{\partial\, {}_xD_b^{\beta_k} y} = 0. \tag{20}$$

As a special case, consider that $\alpha_j = j$ $(j = 1, \ldots, n)$, and that F does not contain the ${}_xD_b^{\beta_k} y$ $(k = 1, \ldots, m)$ terms. In this case, using Eq. (3), we have

$$\frac{\partial F}{\partial y} + \sum_{j=1}^{n} \left(-\frac{d}{dx}\right)^j \frac{\partial F}{\partial y^{(j)}} = 0. \tag{21}$$

Thus, for integral order derivatives, the necessary conditions obtained using fractional calculus of variations approach reduces to that obtained using standard calculus of variations approach.

Case 2. Fixed end points and several functions.

The simplest fractional variational problem discussed in Section 2 can be generalized in a straight forward manner to problems containing several unknown functions. This problem can be defined as follows: Let $F(x, y_1, \dots, y_n, z_1, \dots, z_{2n})$ be a function with continuous first and second (partial) derivatives with respect to all its arguments. For $0 < \alpha, \beta \leqslant 1$, consider the problem of finding necessary conditions for an extremum of a functional of the form

$$J[y_1, \dots, y_n] = \int_a^b F\big(x, y_1, \dots, y_n, {}_aD_x^\alpha y_1, \dots, {}_aD_x^\alpha y_n, {}_xD_b^\beta y_1, \dots, {}_xD_b^\beta y_n\big)\, dx, \tag{22}$$

which depends on n continuously differentiable functions $y_1(x), \dots, y_n(x)$ satisfying the boundary conditions

$$y_j(a) = y_{ja}, \quad y_j(b) = y_{jb} \quad (j = 1, \dots, n). \tag{23}$$

Note that no relationship exists among the functions $y_j(x)$ $(j = 1, \dots, n)$. Therefore, the necessary condition for the functional in Eq. (22) to have an extremum can be found by considering the variations of each function one at a time. Thus we have

Theorem 3. *A necessary condition for the curve*

$$y_j = y_j(x) \quad (j = 1, \dots, n), \tag{24}$$

which satisfies the boundary conditions given by Eq. (23) *to be an extremal of the functional given by Eq.* (22) *is that the functions* $y_j(x)$ *satisfy the following Euler–Lagrange equation*:

$$\frac{\partial F}{\partial y_j} + {}_xD_b^\alpha \frac{\partial F}{\partial_a D_x^\alpha y_j} + {}_aD_x^\beta \frac{\partial F}{\partial_x D_b^\beta y_j} = 0 \quad (j = 1, \dots, n). \tag{25}$$

In vector notation, the above condition can be written as

$$\frac{\partial F}{\partial \mathbf{y}} + {}_xD_b^\alpha \frac{\partial F}{\partial_a D_x^\alpha \mathbf{y}} + {}_aD_x^\beta \frac{\partial F}{\partial_x D_b^\beta \mathbf{y}} = 0, \tag{26}$$

where $\mathbf{y} \in R^n$.

The above problem considers several functions but only one LRLFD of order $\alpha \leqslant 1$ and one RRLFD of order $\beta \leqslant 1$. The problem of finding extremum of a functional consisting of multiple functions and multiple LRLFD and RRLFD of order greater than zero can be developed using the discussion presented in cases 1 and 2 above.

4. The problem of Lagrange and the multiplier rule

In this section we consider the following problem: Find the extremum of the functional

$$J[\mathbf{y}] = \int_a^b F\left(x, \mathbf{y}, {}_aD_x^\alpha \mathbf{y}, {}_xD_b^\beta \mathbf{y}\right) dx \tag{27}$$

such that

$$\mathbf{\Phi}(x, \mathbf{y}) = 0 \tag{28}$$

and

$$y_{s1(j)}(a) = y_{s1(j)a}, \quad y_{s2(j)}(b) = y_{s2(j)b} \quad (j = 1, \ldots, n - m), \tag{29}$$

where $\mathbf{y} \in R^n$, $\mathbf{\Phi} \in R^m$, $m < n$, and $s1$ and $s2$ are two sets of n numbers obtained by reordering the numbers 1 to n. It is assumed that the constrained functions $\phi_j(x, \mathbf{y}) = 0$ $(j = 1, \ldots, m)$ are all independent. This problem is essentially the same as that of Lagrange except that in this case the functional contains the LRLFD and the RRLFD. For this reason, we will call this problem as the problem of Lagrange containing fractional derivatives or simply a fractional Lagrange problem. This is a special case, and in a general fractional Lagrange problem, $\mathbf{\Phi}$ may also contain the left and the right fractional derivatives.

To develop the necessary conditions for the problem, note that $\mathbf{y}$ at the two ends are completely known. This follows from the fact that the constraints $\phi_j(x, \mathbf{y}) = 0$ $(j = 1, \ldots, m)$ are all independent and the values of $n - m$ functions $y_j(x)$ $(j = 1, \ldots, n)$ are specified at both ends. Therefore, the values of the rest of the functions at the two ends can be determined using a technique such as Newton–Raphson.

Suppose $\mathbf{y}^*(x)$ is the solution to the above problem, and define

$$\mathbf{y}(x) = \mathbf{y}^*(x) + \epsilon\eta(x), \tag{30}$$

where ϵ is a sufficiently small number, and $\eta(x) \in R^n$ is a variation of $\mathbf{y}(x)$ consistent with the constraints, i.e., $\mathbf{y}(x)$ satisfies Eq. (28). From the above discussion, it follows that

$$\eta(a) = \eta(b) = 0. \tag{31}$$

Substituting Eq. (31) into Eq. (28), expanding the resulting vector into Taylor series, and neglecting second and higher order terms in ϵ, we get

$$\frac{\partial \mathbf{\Phi}}{\partial \mathbf{y}} \eta(x) = 0. \tag{32}$$

Equation (32) clearly indicates that not all functions $\eta_j(x)$ $(j = 1, \ldots, n)$ can be independent. Substituting Eq. (30) into Eq. (27), we get a function that is only dependent on ϵ. Extremum of this function requires that its derivative with respect to ϵ must be zero. This leads to

$$\int_a^b \left[\frac{\partial F}{\partial \mathbf{y}} \eta + \frac{\partial F}{\partial_a D_x^\alpha \mathbf{y}} {}_a D_x^\alpha \eta + \frac{\partial F}{\partial_x D_b^\beta \mathbf{y}} {}_x D_b^\beta \eta \right] dx = 0. \tag{33}$$

The left-hand side of Eq. (33) is the directional derivative of J at $\mathbf{y}(x)$ in the direction $\eta(x)$. Using the formula for fractional integration by parts and Eq. (31), it follows that

$$\int_a^b \left[\frac{\partial F}{\partial \mathbf{y}} + {}_x D_b^\alpha \frac{\partial F}{\partial_a D_x^\alpha \mathbf{y}} + {}_a D_x^\beta \frac{\partial F}{\partial_x D_b^\beta \mathbf{y}} \right] \eta \, dx = 0. \tag{34}$$

Here the elements of $\eta(x)$ are not all independent, and therefore its coefficients cannot be set to zero. Equation (15) motivates the following

Definition. An admissible arc $\mathbf{y}^*(x)$ is said to satisfy the *multiplier rule* if there exists a vector of multipliers $\mathbf{l}(x) \in R^m$ continuous on $[a, b]$, and a function

$$\bar{F}\big(x, \mathbf{y}, {}_a D_x^\alpha \mathbf{y}, {}_x D_b^\beta \mathbf{y}, \mathbf{l}\big) = F\big(x, \mathbf{y}, {}_a D_x^\alpha \mathbf{y}, {}_x D_b^\beta \mathbf{y}\big) + \mathbf{l}^T(x) \mathbf{\Phi}(x, \mathbf{y}), \tag{35}$$

such that

$$\frac{\partial \bar{F}}{\partial \mathbf{y}} + {}_x D_b^\alpha \frac{\partial \bar{F}}{\partial_a D_x^\alpha \mathbf{y}} + {}_a D_x^\beta \frac{\partial \bar{F}}{\partial_x D_b^\beta \mathbf{y}} = 0 \tag{36}$$

is satisfied along $\mathbf{y}^*(x)$.

Thus:

Theorem 4. *Every minimizing arc $\mathbf{y}^*(x)$ must satisfy the multiplier rule.*

Proof. To prove this, multiply Eq. (32) with $\mathbf{l}^T(x)$ and add the results to Eq. (34) to get

$$\int_a^b \left[\frac{\partial F}{\partial \mathbf{y}} + {}_x D_b^\alpha \frac{\partial F}{\partial_a D_x^\alpha \mathbf{y}} + {}_a D_x^\beta \frac{\partial F}{\partial_x D_b^\beta \mathbf{y}} + \mathbf{l}^T(x) \frac{\partial \mathbf{\Phi}}{\partial \mathbf{y}} \right] \eta \, dx = 0. \tag{37}$$

It can now be shown that

$$\frac{\partial F}{\partial \mathbf{y}} + {}_x D_b^\alpha \frac{\partial F}{\partial_a D_x^\alpha \mathbf{y}} + {}_a D_x^\beta \frac{\partial F}{\partial_x D_b^\beta \mathbf{y}} + \mathbf{l}^T(x) \frac{\partial \mathbf{\Phi}}{\partial \mathbf{y}} = 0. \tag{38}$$

This follows from the fact that $\mathbf{l}(x)$ may be selected such that m of the n equations in Eq. (38) are zero. This is true since $\partial \mathbf{\Phi}/\partial \mathbf{y}$ has a full rank. Rest of the η's can be selected as independent and therefore the other $n-m$ equations in (38) follows by using Eq. (37) and applying a theorem in calculus of variations. Note that Eq. (36) can now be obtained using Eqs. (35) and (38). Equation (38) will be called the Euler–Lagrange equation for constrained fractional variational problems. □

The multiplier rule is also applicable for the case when $\mathbf{\Phi}$ is also a function of the LRLFD and the RRLFD. This can be proved following the discussion given in [1,7]. Multiplier rule for a system containing multiple fractional derivatives can be developed in a similar manner.

5. Examples

In this section, we obtain the Euler–Lagrange equations for an unconstrained and a constrained fractional variational problems.

Example 1. As the first example, consider the following unconstrained fractional variational problem:

$$\text{minimize} \quad J[y] = \frac{1}{2}\int_0^1 \left({}_0D_x^\alpha y\right)^2 dx \tag{39}$$

such that

$$y(0) = 0 \quad \text{and} \quad y(1) = 1. \tag{40}$$

This example with $\alpha = 1$, for which the solution is $y(x) = x$, is often considered in textbooks on variational calculus. It can be shown that for this problem, the Euler–Lagrange equation is

$${}_xD_1^\alpha\left({}_0D_x^\alpha y\right) = 0. \tag{41}$$

It can be shown that for $\alpha > 1/2$, the solution is given as

$$y(x) = (2\alpha - 1)\int_0^x \frac{dt}{[(1-t)(x-t)]^{1-\alpha}}. \tag{42}$$

Example 2. As the second example, consider the following constrained fractional variational problem:

$$\text{minimize} \quad J[y] = \frac{1}{2}\int_0^1 \left[y_1^2 + y_2^2\right] dx \tag{43}$$

such that

$$ {}_0D_x^\alpha y_1 = -y_1 + y_2, \tag{44} $$

$$ y_1(0) = 1. \tag{45} $$

This example with integral order derivative is often considered in textbooks on optimal control. It can be shown that for this problem, the Euler–Lagrange equation is

$$ y_1 + l + {}_xD_1^\alpha l = 0, \tag{46} $$

$$ y_2 - l = 0. \tag{47} $$

Note that in both examples both the LRLFD and the RRLFD occur in the resulting Euler–Lagrange equations even when the problems contain only LRLFDs. Such differential equations have not been studied much in the literature. A method to find solutions for such problems will be presented in a later work.

Remarks. In closing, we would like to make the following two remarks.

1. Here we have assumed that the terminal conditions are fixed and the functions meet all the smoothness requirements. The case of unspecified end conditions, unspecified end points, (the transversality conditions), and piecewise smoothness (the corner conditions) will be considered in a future work.

2. The theorems and their proofs presented here are very similar to those given in standard textbooks on calculus of variations. Thus, many of the concepts of classical calculus of variations can be extended with minor modifications to fractional calculus of variations. Given the fact that many systems are described more accurately using fractional derivative models and that nature attempts to minimize certain functionals, it is hoped that more research will continue in this field.

6. Conclusions

Euler–Lagrange equations have been presented for unconstrained and constrained fractional variational problems. The approach presented and the resulting equations are very similar to those for variational problems containing integral order derivatives. In special cases, when the derivatives are of integral order only, the results of fractional calculus of variations reduce to those obtained from classical calculus of variations. Given the fact that many systems can be modeled more accurately using fractional derivative models, it is hoped that future research will continue in this area.

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