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A New Approach to Nonlinear Partial Differential Equations

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The author's decomposition method for the solution of operator equations which may be nonlinear and/or stochastic is generalized to multidimensional cases.

Current developments in mathematical physics, energy problems, and other areas have given impetus to research on nonlinear partial differential equations and linearization techniques. Unfortunately, such techniques which assume essentially that a nonlinear system is "almost linear" often have little *physical* justification. It has become vital, not only to mathematics but to the areas of application, that further advances be made. Fluid mechanics, soliton physics, quantum field theory, and nonlinear evolution equations are all areas which can benefit from such advances. In some areas, such as turbulence studies and, quite possibly, many or all of the other areas, once possibilities are recognized, stochasticity is also a significant factor in the actual physical behavior. Our approach is an operator-theoretic approach in which general dynamical systems will be viewed as being nonlinear and stochastic and described by (nonlinear stochastic) differential equations, systems of equations, or partial differential equations. *Deterministic* nonlinear equations can be considered as a special case where stochasticity vanishes (just as linear is a trivial special case of nonlinear.)

The author's approach to these problems began with linear *stochastic operator* equations and has evolved since 1976 to nonlinear stochastic operator equations [1]. Consider, for example,

$$\frac{\partial u}{\partial t} + a(t,x)\frac{\partial u}{\partial x} + b(t,x)\frac{\partial^2 u}{\partial x^2} = g(t,x)$$

which is rewritten in terms of operators as

$$L_t u + L_x u = g(t,x)$$

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where $L_t = \partial/\partial t$ and $L_x = a(\partial/\partial x) + b(\partial^2/\partial x^2)$. A similar *stochastic* equation

$$\frac{\partial u(x, t, \omega)}{\partial t} + A(x, t)\, u(x, t, \omega) + B(x, t, \omega)\, u(x, t, \omega) = f(x, t, \omega)$$

where $\omega \in (\Omega, F, \mu)$, a probability space, f is a stochastic process, A is a deterministic coefficient but B is a stochastic process coefficient, can similarly be written

$$Lu + \mathscr{R}u = f$$

where $L = \partial/\partial t + A(x, t)$ is a deterministic operator and $\mathscr{R} = B$ is a *stochastic* operator, or

$$\mathscr{L}u = f$$

where $\mathscr{L}$ is a *stochastic operator* with deterministic and random parts, L being $\langle \mathscr{L} \rangle$ if $\mathscr{R}$ is zero-mean.

Let us consider then the operator equation

$$\mathscr{F}u = g$$

where $\mathscr{F}$ represents a differential operator which may be ordinary or partial, linear or nonlinear, deterministic or stochastic. We suppose $\mathscr{F}$ has linear and nonlinear parts, i.e., $\mathscr{F}u = \mathscr{L}u + \mathscr{N}u$ where $\mathscr{L}$ is a linear (stochastic) operator and $\mathscr{N}$ is a nonlinear (stochastic) operator—the script letters indicate stochasticity. We may, of course, have a nonlinear term which depends upon derivatives of u as well as u but we defer this question.

Since $\mathscr{L}$ may have deterministic and stochastic components, let $\mathscr{L} = L + \mathscr{R}$ where conveniently $L = \langle \mathscr{L} \rangle$ and $\mathscr{R} = \mathscr{L} - L$. This is not a limitation on the method but a convenience in explanation. It is necessary that L be invertible. If the above choice makes this difficult, we choose a simpler L and let $\mathscr{R}$ incorporate the remainder. Let $\mathscr{N}u = Nu + \mathscr{M}u$ where Nu indicates a deterministic part and $\mathscr{M}u$ indicates a stochastic nonlinear term.

$\mathscr{F}$ may involve derivatives with respect to x, y, z, t or mixed derivatives. To avoid difficulties in notation which tend to obscure rather than clarify, we will assume the same probability space for each process and let $Lu = L_x + L_y + L_z + L_t$ where the operators indicate quantities like $\partial^2/\partial x^2$, $\partial/\partial y$, etc., but, for now, no mixed derivatives. Similarly, $\mathscr{R}$ is written as $\mathscr{R}_x + \mathscr{R}_y + \mathscr{R}_z + \mathscr{R}_t$. Mixed derivatives and product nonlinearities such as $u^2 u'^3$, uu'', $f(u, u', \ldots, u^{(m)})$ can also be handled as shown elsewhere [1, 2].

A simple Langevin equation is written

$$Lu = g$$

where $L = (d/dt) + \beta$ and g is a white noise process. Langevin equations as used for modeling complex nonlinear phenomena in physics of the form $\dot{\Psi} = f(\Psi) + \xi$ are represented by $Lu + Nu = g$, however, we will not make any Markovian or white noise restrictions. All processes will be physical processes without restriction to being Gaussian or stationary. In the *Kdv* equation, for example, $\mathscr{F}u$ would become $L_t u + L_x u + Nu$ where Nu is of the form uu_x (again a product nonlinearity). In equations of the Satsuma–Kaup type for soliton behavior, we also have such products uu_x, uu_{xxx}, $u_x u_{xx}$. Stochastic transport equations will fit nicely into our format since $\nabla^2 = L_x + L_y + L_z$ and stochastic behavior in coefficients or inputs are easily included. For example, instead of $Ly(\bar{z}, t) = \xi(\bar{z}, t, \omega)$ where ξ is a random source and $L = (\partial/\partial t) - D\nabla^2$ or $(\partial/\partial t) - \Lambda_{xyz}$, we can include nonlinear terms or stochastic behavior in the operator. In the double sine-Gordon equation we have $u_{tt} - u_{xx} - \sin u - \sin zu = 0$ or $L_t + L_x + N(u)$ where $N(u)$ includes the trigonometric nonlinearities. We can allow trigonometric, polynomial, exponential, or products, sums of products, etc. Or, as in the LAX theorem, $N(u) = f(u, u_x, u_{xx}, \ldots)$. We remark that the Itô equation

$$dy = f(t, y)\, dt + g(t, y)\, dz$$

where z is the Wiener process can be written

$$\frac{dy}{dt} = f(t, y) + g(t, y)\, u(t)$$

where we can write $dz/dt = u$ since we do not insist that z is a Wiener process and this equation can, consequently, be put into the author's standard form [1, 3]. The nondifferentiability of the Wiener process is, of course, a *mathematical* property. We are interested in *physical* solutions. In the Itô integral $\int f\, dz$ the z process is not of bounded variation, however, a Lipschitz condition on z is reasonable for physical processes so the integral will be a well-defined Riemann–Stieltjes integral. Physically reasonable models and mathematically tractable models are not necessarily the same. This point of view offers interesting and different mathematics for systems characterized by linear or nonlinear stochastic operator equations where the operator may be an ordinary or partial differential operator, an integral operator, or other, where the operator itself is stochastic, e.g., in the case of the differential operator of nth order, one or more coefficient processes may be stochastic processes.

Thus, generalizing further the work of Adomian and Malakian [1, 4], we write

$$[\mathscr{L}_x + \mathscr{L}_y + \mathscr{L}_z + \mathscr{L}_t]u + \mathscr{N}u = g(x, y, z, t) \tag{1}$$

where $\mathcal{N}(u)$ indicates any nonlinear term possibly involving derivatives of u, products, etc. Since $\mathcal{L} = L + \mathcal{R}$, we have

$$(L_x + L_y + L_z + L_t)u + (\mathcal{R}_x + \mathcal{R}_y + \mathcal{R}_z + \mathcal{R}_t)u + \mathcal{N}u = g. \tag{2}$$

We emphasize that $\mathcal{R}_x$, $\mathcal{R}_y$, $\mathcal{R}_z$, and R_t, as well as the g, are not necessarily random. They may, or they may simply be a part of an entirely deterministic operator and be chosen only to make the remaining part easily invertible. We solve for L_xu, L_yu, L_zu, L_tu in turn and then assuming inverses L_x^{-1}, L_y^{-1}, L_z^{-1}, L_t^{-1} exist

$$\begin{aligned}
L_x^{-1}L_xu &= L_x^{-1}[g - L_yu - L_zu - L_tu] \\
&\quad - L_x^{-1}[\mathcal{R}_x + \mathcal{R}_y + \mathcal{R}_z + \mathcal{R}_t]u - L_x^{-1}\mathcal{N}u \\
L_y^{-1}L_yu &= L_y^{-1}[g - L_xu - L_zu - L_tu] \\
&\quad - L_y^{-1}[\mathcal{R}_x + \mathcal{R}_y + \mathcal{R}_z + \mathcal{R}_t]u - L_y^{-1}\mathcal{N}u \\
L_z^{-1}L_zu &= L_z^{-1}[g - L_xu - L_yu - L_tu] \\
&\quad - L_z^{-1}[\mathcal{R}_x + \mathcal{R}_y + \mathcal{R}_z + \mathcal{R}_t]u - L_z^{-1}\mathcal{N}u \\
L_t^{-1}L_tu &= L_t^{-1}[g - L_xu - L_yu - L_zu] \\
&\quad - L_t^{-1}[\mathcal{R}_x + \mathcal{R}_y + \mathcal{R}_z + \mathcal{R}_t]u - L_t^{-1}\mathcal{N}u
\end{aligned} \tag{3}$$

a linear combination of these solutions is necessary. Therefore, adding and dividing by four, we write

$$\begin{aligned}
u = u_0 &- \tfrac{1}{4}\{(L_x^{-1}L_y + L_y^{-1}L_x) + (L_x^{-1}L_z + L_z^{-1}L_x) \\
&+ (L_x^{-1}L_t + L_t^{-1}L_x) + (L_y^{-1}L_z + L_z^{-1}L_y) \\
&+ (L_t^{-1}L_y + L_y^{-1}L_t) + (L_z^{-1}L_t + L_t^{-1}L_z)\}u \\
&- \tfrac{1}{4}[L_x^{-1} + L_y^{-1} + L_z^{-1} + L_t^{-1}][\mathcal{R}_x + \mathcal{R}_y + \mathcal{R}_z + \mathcal{R}_t]u \\
&- \tfrac{1}{4}[L_x^{-1} + L_y^{-1} + L_z^{-1} + L_t^{-1}]\, Nu
\end{aligned} \tag{4}$$

where the term u_0 includes

$$\tfrac{1}{4}[L_x^{-1} + L_y^{-1} + L_z^{-1} + L_t^{-1}]\, g$$

as well as terms arising from the initial conditions which depend on the number of integrations involved in the inverse operators. Thus, $L_x^{-1}L_xu = u(x, y, z, t) - \Theta_x$ where $L_x\Theta_x = 0$. Thus, $L_x^{-1}L_xu = u(x, y, z, t) - u(0, y, z, t)$ if L_x involves a single differentiation. $L_x^{-1}L_xu = u(x, y, z, t) - u(0, y, z, t) - x\,\partial u(0, y, z, t)/\partial x$ for a second-order operator, etc. Similarly $L_y^{-1}L_zu = u - \Theta_y$ where $\Theta_y = u(x, 0, z, t)$ for a single differentiation in L_y, etc. Thus, we have the partial homogeneous solutions Θ_x, Θ_y, Θ_z, Θ_t analogous to the one-dimensional problems considered in earlier work where we wrote $L_t^{-1}L_t u(t) = \int_0^t (\partial u/\partial t)\, dt = u(t) - u(0)$ when $L_t \equiv d/dt$. Thus,

$$u_0 = \tfrac{1}{4}[\Theta_x + \Theta_y + \Theta_z + \Theta_t] + \tfrac{1}{4}[L_x^{-1} + L_y^{-1} + L_z^{-1} + L_t^{-1}]\, g. \tag{5}$$

We now write Nu, the nonlinear term, as $Nu = \sum A_n$ where the A_n are the author's previously defined [1] polynomials and assume our usual decomposition of u into $\sum_{n=0}^{\infty} u_n$, or equivalently, of $\mathscr{F}^{-1}g$ into $\sum_{n=0}^{\infty} \mathscr{F}_n^{-1} g$ to determine the individual components.

For n-dimensional problems we can write in a more condensed form

$$\begin{aligned} u = u_0 &- (1/m) \sum_{\substack{j=i+1 \\ (i \neq j)}}^{m} \sum_{i=1}^{m-1} [L_{x_i}^{-1} L_{x_j} + L_{x_j}^{-1} L_{x_i}] u \\ &- (1/m) \left[\sum_{i=1}^{m} L_{x_i}^{-1} \sum_{i=1}^{m} \mathscr{R}_{x_i} \right] u \\ &- (1/m) \left[\sum_{i=1}^{m} L_{x_i}^{-1} \right] \sum_{n=0}^{\infty} A_n \end{aligned} \tag{6}$$

where

$$u_0 = (1/m) \left\{ \sum_{i=1}^{m} \Theta_i + \sum_{i=1}^{m} L_{x_i}^{-1} g \right\}.$$

Thus, u_0 can be easily calculated. The following components of the decomposition follow in terms of u_0 without problems of statistical separability when stochasticity is involved. Thus,

$$\begin{aligned} u_1 = &-(1/m) \sum_{\substack{j=i+1 \\ (i \neq j)}}^{m} \sum_{i=1}^{m-1} [L_{x_i}^{-1} L_{x_j} + L_{x_j}^{-1} L_{x_i}]\, u_0 \\ &- (1/m) \left[\sum_{i=1}^{m} L_{x_i}^{-1} \right] \left[\sum_{i=1}^{m} \mathscr{R}_{x_i} \right] u_0 \\ &- (1/m) \left[\sum_{i=1}^{m} L_{x_i}^{-1} \right] A_0 \\ &\vdots \\ u_n = &-(1/m) \sum_{\substack{i=1 \\ (i \neq j)}}^{m} \sum_{j=1}^{m-1} [L_{x_i}^{-1} L_{x_j} + L_{x_j}^{-1} L_{x_i}]\, u_{n-1} \\ &- (1/m) \left[\sum_{i=1}^{m} L_{x_i}^{-1} \right] \left[\sum_{i=1}^{m} \mathscr{R}_{x_i} \right] u_{n-1} \\ &- (1/m) \left[\sum_{i=1}^{m} L_{x_i}^{-1} \right] A_{n-1} \end{aligned}$$

and the complete solution is $u=\sum_{n=0}^{\infty} u_n$ and our n term approximation ϕ_n is given by

$$\phi_n = \sum_{i=0}^{n-1} u_i .$$

For the particular problem here,

$$\begin{aligned} u_0 &= \tfrac{1}{4}[\Theta_x+\Theta_y+\Theta_z+\Theta_t] + \tfrac{1}{4}[L_x^{-1}+L_y^{-1}+L_z^{-1}+L_t^{-1}]\,g \\ &\vdots \\ u_n &= -\tfrac{1}{4}\{(L_x^{-1}L_y+L_y^{-1}L_x)+(L_x^{-1}L_z+L_z^{-1}L_x) \\ &\quad + (L_x^{-1}L_t+L_t^{-1}L_x)+(L_y^{-1}L_z+L_z^{-1}L_y) \\ &\quad + (L_t^{-1}L_y+L_y^{-1}L_t)+(L_z^{-1}L_t+L_t^{-1}L_z)\,u_{n-1} \\ &\quad - \tfrac{1}{4}[L_x^{-1}+L_y^{-1}+L_z^{-1}+L_t^{-1}][\mathscr{R}_x+\mathscr{R}_y+\mathscr{R}_z+\mathscr{R}_t]\,u_{n-1} \\ &\quad - \tfrac{1}{4}[L_x^{-1}+L_y^{-1}+L_z^{-1}+L_t^{-1}]\,A_{n-1}\}. \end{aligned} \tag{7}$$

In the one-dimensional ($m=1$) case,

$$\begin{aligned} u_0 &= \Theta_t + L_t^{-1}g \\ u_1 &= -L_t^{-1}\mathscr{R}_t u_0 - L_t^{-1}A_0 \end{aligned}$$

etc. For simplicity in writing, define

$$\frac{L_x^{-1}+L_y^{-1}+L_z^{-1}+L_t^{-1}}{4} \equiv L^{-1}$$

and

$$\begin{aligned} &\tfrac{1}{4}[(L_x^{-1}L_y+L_y^{-1}L_x)+(L_x^{-1}L_z+L_z^{-1}L_x) \\ &\qquad + (L_x^{-1}L_t+L_t^{-1}L_x)+(L_y^{-1}L_z+L_z^{-1}L_y) \\ &\qquad + (L_t^{-1}L_y+L_y^{-1}L_t)+(L_z^{-1}L_t+L_t^{-1}L_z)] \\ &\equiv G \end{aligned}$$

and

$$\frac{\mathscr{R}_x+\mathscr{R}_y+\mathscr{R}_z+\mathscr{R}_t}{4} = \mathscr{R}$$

(then for $m=4$),

$$u = u_0 - Gu - L^{-1}\mathscr{R}u - L^{-1}Nu.$$

In a one-dimensional case, Gu vanishes and the $\frac{1}{4}$ or $1/m$ factor is, of course, equal to one and we have

$$u = u_0 - L^{-1}\mathscr{R}u - L^{-1}Nu$$

which is precisely the basis of earlier solutions of nonlinear ordinary differential equations. For m dimensions, we write

$$L^{-1} = (1/m)\sum_{i=1}^{m} L_{x_i}^{-1}$$

$$\mathscr{R} = (1/m)\sum_{i=1}^{m} \mathscr{R}_{x_i}.$$

Now

$$u = u_0 - (1/m)\sum_{j=i+1}^{m}\sum_{i=1}^{m-1}[L_{x_i}^{-1}L_{x_j} + L_{x_j}^{-1}L_{x_i}]u - L^{-1}\mathscr{R}u - L^{-1}Nu \tag{8}$$

which reduces to

$$u = u_0 - L^{-1}\mathscr{R}u - L^{-1}Nu$$

as previously written for ordinary differential equations when $m = 1$ since the second term vanishes.

Parametrization and the A_n Polynomials

A parametrization [5] of Eq. (8) into

$$u = u_0 - (1/m)\lambda\sum_{j=i+1}^{m}\sum_{i=1}^{m-1}[L_{x_i}^{-1}L_{x_j} + L_{x_j}^{-1}L_{x_i}]u - \lambda L^{-1}\mathscr{R}u - \lambda L^{-1}Nu \tag{9}$$

and $u = \mathscr{F}^{-1}g = \sum_n u_n$ into

$$u = \sum_n \lambda^n \mathscr{F}_n^{-1} g = \sum_n \lambda^n u_n \tag{10}$$

has been *convenient* in determining the components of u and also in finding the A_n polynomials originally. We will later set $\lambda = 1$ so $u = \sum_n \lambda^n$. *The λ is not a perturbation parameter.* It is simply an identifier helping us to collect terms in a way which will result in each u_i depending only on u_{i-1}, $u_{i-2}, \ldots, u_0$.

Now $\mathscr{N}(u)$ is a nonlinear function and $u = u(\lambda)$. We assume $\mathscr{N}(u)$ is analytic and write it as $\sum A_n \lambda^n$ if $\mathscr{N}(u) = Nu$, i.e., if N is deterministic. (If the nonlinear *stochastic* term $\mathscr{M}u$ appears, we simply carry that along or a second "stochastically analytic" expansion $\sum B_n\lambda^n$.)

Now (9) becomes

$$\sum_n \lambda^n \mathscr{F}_n^{-1} g = u_0 - (1/m) \sum_{j=i+1}^{m} \sum_{i=1}^{m-1} [L_{x_i}^{-1} L_{x_j} + L_{x_j}^{-1} L_{x_i}] \sum_n \lambda^n \mathscr{F}_n^{-1} g$$
$$- \lambda L^{-1} \mathscr{R} \sum_n \lambda^n \mathscr{F}_n^{-1} g - \lambda L^{-1} \sum_n \lambda^n A_n. \tag{11}$$

Equating powers of λ

$$\mathscr{F}_0^{-1} g = u_0$$
$$\mathscr{F}_1^{-1} g = -(1/m) \sum_{j=i+1}^{m} \sum_{i=1}^{m-1} [L_{x_i}^{-1} L_{x_j} + L_{x_j}^{-1} L_{x_i}](\mathscr{F}_0^{-1} g)$$
$$- L^{-1} \mathscr{R}(\mathscr{F}_0^{-1} g) - L^{-1} A_0 \tag{12}$$
$$\vdots$$
$$\mathscr{F}_n^{-1} g = -(1/m) \sum_{j=i+1}^{m} \sum_{i=1}^{m-1} [L_{x_i}^{-1} L_{x_j} + L_{x_j}^{-1} L_{x_i}](\mathscr{F}_{n-1}^{-1} g)$$
$$- L^{-1} \mathscr{R}(\mathscr{F}_{n-1}^{-1} g) - L^{-1} A_{n-1}.$$

Hence all terms are calculable. If there is both a deterministic and a stochastic term which is nonlinear, i.e., $\mathscr{N}u = Nu + \mathscr{M}u$, we also have $-L^{-1}B_{n-1}$ calculated the same way but involving randomness. If randomness is involved anywhere in any part of the equation, we will then calculate the statistical measures—the expectation and covariance of the solution process.

Thus, each $\mathscr{F}_{n+1}^{-1} g$ depends on $\mathscr{F}_n^{-1} g$ and ultimately on $\mathscr{F}_0^{-1} g$. Hence, $\mathscr{F}^{-1}$ the stochastic nonlinear inverse has been determined. The quantities A_n and B_n have been calculated for general classes of nonlinearities [1], and explicit formulas have been developed. Their calculation is as simple as writing down a set of Hermite or Legendre polynomials. They depend, of course, on the particular nonlinearity.

If stochastic quantities are involved, the above series then involves processes and can be averaged for $\langle u \rangle$ or multiplied and averaged to form the correlation $\langle u(t_1)\, \mathring{u}(t_2) \rangle = \mathscr{R}_u(t_1, t_2)$ as discussed in the author's previous work. Thus, the solution statistics (or statistical measures) are obtained when appropriate statistical knowledge of the random quantities is available.

Summarizing, we have decomposed the solution process for the output of a physical system into additive components—the first being the solution of a simplified linear deterministic system which takes account of initial conditions. Each of the other components is then found in terms of a *preceding* component and thus ultimately in terms of the first.

The usual statistical separability problems requiring closure approximations are eliminated with the reasonable assumption of statistical independence of the system *input* and the system itself! Quasimonochromaticity assumptions are unnecessary and processes can be assumed to be general physical processes rather than white noise. White noise is not a physical process. Physical inputs are neither unbounded nor do they have zero correlation times. In any event, the results can be obtained as a special case. If fluctuations are small, the results of perturbation theory are exactly obtained [1] but again this is a special case, as are the diagrammatic methods of physicists.

Just as spectral spreading terms are lost by a quasimonochromatic approximation, when a random or scattering medium is involved, or terms are lost in the use of closure approximations, Boussinesq approximations, replacement of stochastic quantities by their expectations, significant terms may be lost by the usual linearizations, unless of course, the behavior is actually close to linear.

One hopes, therefore, that physically more realistic and accurate results and predictions will be obtained in many physical problems by this method of solution, as well as interesting new mathematics from the study of such operators and relevant analysis.

Examples

Four-Dimensional Linear Partial Differential Equation

Consider the equation

$$(\partial u/\partial x) + (\partial u/\partial y) + (\partial u/\partial z) + (\partial u/\partial t) = 0.$$

Assume initial conditions:

$$\begin{aligned} u(0, y, z, t) &= \Theta_x = f_1(y, z, t) = -x + y + z - t \\ u(x, 0, z, t) &= \Theta_y = f_2(x, z, t) = x + z - t \\ u(x, y, 0, t) &= \Theta_z = f_3(x, y, t) = x - y - t \\ u(x, y, z, 0) &= \Theta_t = f_4(x, y, z) = x - y + z. \end{aligned}$$

(We note $u(0, 0, 0, 0) = 0$.) In our usual notation we write

$$L_x + L_y + L_z + L_t = \mathscr{L}_{x,y,z,t}$$

and

$$\mathscr{L}u = 0.$$

By the decomposition method, $u(x, y, z, t)$ is given by

$$u = \sum_{n=0}^{\infty} (-1)^n (\tfrac{1}{4})^n \left\{ \sum_{j=i+1}^{4} \sum_{i=1}^{3} (L_{x_i}^{-1} L_{x_j} + L_{x_j}^{-1} L_{x_i}) \right\}^n \cdot \tfrac{1}{4} \left(\sum_{m=1}^{4} \Theta_m \right).$$

The last sum $\sum_{m=1}^{4} \Theta_m = 3x - 3y + 3z - 3t$ and the double summation within the curly brackets is

$$(L_x^{-1} L_y + L_y^{-1} L_x) + (L_x^{-1} L_z + L_z^{-1} L_x) + (L_x^{-1} L_t + L_t^{-1} L_x)$$
$$+ (L_y^{-1} L_z + L_z^{-1} L_y) + (L_y^{-1} L_t + L_t^{-1} L_y) + (L_z^{-1} L_t + L_t^{-1} L_z).$$

We seek to find our approximate solution ϕ_n

$$\phi_n = u_0 + u_1 + \cdots + u_{n-1}$$

as far as we wish to calculate it. We have

$$u_0 = \frac{1}{4} \sum_{m=1}^{4} \Theta_m = \frac{1}{4} (3x - 3y + 3z - 3t)$$
$$= \frac{3}{4} (x - y + z - t)$$

$$u_1 = -\frac{3}{16} \{ \cdot \} (x - y + z - t)$$

where the double summation in the curly brackets $\{\cdot\}$ is easily evaluated. Thus,

$$(L_x^{-1} L_y)(x - y + z - t) = L_x^{-1}(-1) = \int_0^x (-1)\, dx = -x$$

$$(L_y^{-1} L_x)(x - y + z - t) = (L_y^{-1})(1) = \int_0^y dy = y.$$

Hence $(L_x^{-1} L_y + L_y^{-1} L_x)(x - y + z - t) = -x + y$. Similarly evaluating the other five quantities in the double summation,

$$u_1 = -\frac{3}{16} [-x + y + x + z - x + t + y - z - t - y - z + t]$$

$$u_1 = -\frac{3}{16} [-x + y - z + t]$$

$$= \frac{3}{16} [x - y + z - t].$$

(Thus, a two-term approximation $\phi_2 = u_0 + u_1$ would be given by

$$\phi_2 = \frac{3}{4}(x-y+z-t) + \frac{3}{16}(x-y+z-t) = \frac{15}{16}(x-y+z-t).)$$

Continuing in the same manner, we get

$$u_n = \frac{3}{4^{n+1}}(x-y+z-t)$$

and since $u = \sum_{n=0}^{\infty} u_n$

$$u = k(x-y+z-t)$$

where

$$k = \sum_{n=0}^{\infty} 3/4^{n+1}.$$

The series for k is given by

$$3\left(\frac{1}{4^1} + \frac{1}{4^2} + \frac{1}{4^3} + \frac{1}{4^4} + \frac{1}{4^5} + \frac{1}{4^6} + \cdots\right)$$

or

$$3\left(\frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \frac{1}{2^8} + \cdots\right) = \frac{3}{4}\left(\frac{1}{2^0} + \frac{1}{2^2} + \frac{1}{2^4} + \cdots\right).$$

The series for

$$k = \frac{3}{4}\left(\frac{1}{2^0} + \frac{1}{2^2} + \frac{1}{2^4} + \cdots\right) = 0.75 + 0.1875 + 0.046875$$
$$+ 0.01171875 + 0.0029296815 + 0.0007324219 + \cdots.$$

The sum of the first five terms is 0.9990 and for six terms the sum is 0.9998 to four places. Thus, the correct solution is

$$u = x - y + z - t$$

to within 0.02% error. Of course it's trivial to verify that this answer is indeed the correct solution.

Nonlinear Partial Differential Equation

Let $\nabla^2 = (\partial^2/\partial x^2) + (\partial^2/\partial y^2) + (\partial^2/\partial z^2) = L_x + L_y + L_z$ (where the notation L_x symbolizes a linear (deterministic) differential operator $\partial^2/\partial x^2$, etc.) and $N(p)$, or Np for convenience, symbolizes the nonlinear (deterministic) operator acting on p to give $\sinh p$.

We will consider the nonlinear equation $\nabla^2 p = k^2 \sinh p$ which we write as[1]

$$[L_x + L_y + L_z]\, p = k^2 Np$$

where $Np = \sinh p$. Hence

$$\begin{aligned} L_x p &= k^2 Np - L_y p - L_z p \\ L_y p &= k^2 Np - L_x p - L_z p \\ L_z p &= k^2 Np - L_x p - L_y p. \end{aligned}$$

Assuming the inverses $L_x^{-1}, L_y^{-1}, L_z^{-1}$ exist[2]

$$\begin{aligned} L_x^{-1} L_x p &= L_x^{-1} k^2 Np - L_x^{-1} L_y p - L_x^{-1} L_z p \\ L_y^{-1} L_y p &= L_y^{-1} k^2 Np - L_y^{-1} L_x p - L_y^{-1} L_z p \\ L_z^{-1} L_z p &= L_z^{-1} k^2 Np - L_z^{-1} L_x p - L_z^{-1} L_y p. \end{aligned} \tag{13}$$

But

$$\begin{aligned} L_x^{-1} L_x p &= p - p(0, y, z) - x \frac{\partial p}{\partial x}(0, y, z) \\ L_y^{-1} L_y p &= p - p(x, 0, z) - y \frac{\partial p}{\partial y}(x, 0, z) \\ L_z^{-1} L_z p &= p - p(x, y, 0) - z \frac{\partial p}{\partial z}(x, y, 0). \end{aligned} \tag{14}$$

Consequently, summing (13) with the substitution (14), and dividing by three,

[1] In the first author's usual form this is $Fp = Lp - Np = g$ where the inhomogeneous term is zero in this case. If $g \neq 0$, Eq. (5) has an additional term $\frac{1}{3}[L_x^{-1} + L_y^{-1} + L_z^{-1}]\, g(x, y, z)$.

[2] They do in this case; when they don't, we can decompose each of the L_x, L_y, L_z into an invertible operator and a term taken to the right side of the equation.

$$\begin{aligned}
p = \frac{1}{3} \Bigg[& p(0, y, z) + x \frac{\partial p}{\partial x}(0, y, z) \\
& + p(x, 0, z) + y \frac{\partial p}{\partial y}(x, 0, z) \\
& + p(x, y, 0) + z \frac{\partial p}{\partial z}(x, y, 0) \Bigg] \\
& + \frac{1}{3} [(L_x^{-1} L_y + L_y^{-1} L_z) + (L_x^{-1} L_z + L_z^{-1} L_x) \\
& + (L_y^{-1} L_z + L_z^{-1} L_y)] \, p \\
& + \frac{1}{3} [L_x^{-1} + L_y^{-1} + L_z^{-1}] \sum A_n
\end{aligned} \tag{15}$$

where $N(u)$ has been replaced by Adomian's A_n polynomials [1]. We take immediately the first term of our approximation

$$\begin{aligned}
p_0 = \frac{1}{3} \Bigg[& p(0, y, z) + x \frac{\partial p}{\partial x}(0, y, z) \\
& + p(x, 0, z) + y \frac{\partial p}{\partial y}(x, 0, z) \\
& + p(x, y, 0) + z \frac{\partial p}{\partial z}(x, y, 0) \Bigg]
\end{aligned} \tag{16}$$

and the following terms given by

$$\begin{aligned}
p_1 = & \frac{1}{3} [(L_x^{-1} L_y + L_y^{-1} L_z) + (L_x^{-1} L_z + L_z^{-1} L_x) \\
& + (L_y^{-1} L_z + L_z^{-1} L_y)] \, p_0 \\
& + \frac{1}{3} [L_x^{-1} + L_y^{-1} - L_z^{-1}] \, A_0 \\
\vdots & \\
P_n = & \frac{1}{3} [(L_x^{-1} L_y + L_y^{-1} L_z) + (L_x^{-1} L_z + L_z^{-1} L_x) \\
& + (L_y^{-1} L_z + L_z^{-1} L_y)] \, p_{n-1} \\
& + \frac{1}{3} [L_x^{-1} + L_y^{-1} + L_z^{-1}] \, A_{n-1}.
\end{aligned} \tag{17}$$

Now we must evaluate the A_n which has been discussed adequately elsewhere [1, 5]. We use the fact that $(d/dx)(\sinh x) = \cosh x$ and $(d/dx) \cosh x = \sinh x$ and the formulas for A_n given in the above papers,

$$\begin{aligned} H_0(p_0) &= \sinh p_0 \\ H_1(p_0) &= \cosh p_0 \\ H_2(p_0) &= \sinh p_0 \\ &\vdots \\ H_n(p_0) &= \sinh p_0 \qquad \text{for even } n \\ H_n(p_0) &= \cosh p_0 \qquad \text{for odd } n. \end{aligned}$$

i.e.,

Then

$$\begin{aligned} A_0 &= \sinh p_0 \\ A_1 &= p_1 \cosh p_0 \\ A_2 &= p_2 \cosh p_0 + \frac{1}{2} p_1^2 \sinh p_0 \\ A_3 &= p_3 \cosh p_0 + p_1 p_2 \sinh p_0 + \frac{1}{6} p_1^3 \cosh p_0 \\ A_4 &= p_4 \cosh p_0 + \left[\frac{1}{2} p_2^2 + p_1 p_3\right] \sinh p_0 \\ &\quad + \frac{1}{2} p_1^2 p_2 \cosh p_0 + \frac{1}{24} p_1^4 \sinh p_0 \end{aligned} \tag{18}$$

etc. The A_n are easily written down by the procedures given in the references for as many terms as desired. From (17), (18), and (19) we have the complete decomposition $p = \sum_{n=0}^{\infty} p_i$ and hence the solution.

Many more examples and applications to problems of physics and engineering will appear in forthcoming books [6–8].

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