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A Combinatorial Theorem in Plane Geometry

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Let S be a subset of the Euclidean plane. We shall say that a subset A of S *dominates* S if for each $x \in S$ there is an $y \in A$ such that the entire segment xy lies within S. In a conversation, Professor Victor Klee asked for the smallest number $f(n)$ such that every set bounded by a simple closed n-gon is dominated by a set of $f(n)$ points. In a picturesque language, $f(n)$ can be interpreted as the minimum number of guards required to supervise any art gallery with n walls. Figure 1 shows that $f(12) \geqslant 4$; evidently, its pattern generalizes to yield $f(n) \geqslant [n/3]$. We shall prove the reversed inequality in the setting of graph theory.

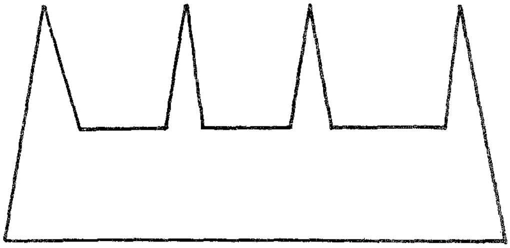

FIGURE 1

For this purpose, we define an *n-triangulation* to be a planar graph G with n vertices such that one of its faces is bounded by an n-gon and each of the remaining faces is bounded by a triangle. An edge of G will be called *inner* if it does not bound the n-gon. A k-triangulation will be called a *fan* if one of its vertices meets all of its $k - 3$ inner edges.

THEOREM. *Every n-triangulation can be partitioned into m fans where* $m \leqslant [n/3]$.

Proof. By induction on n. The cases $n = 3, 4, 5$ are trivial as each n-triangulation with $n \leqslant 5$ is a fan.

Now, let G be an n-triangulation with vertices $1, 2, \ldots, n$ in their cyclic order. Let k be the smallest integer such that $k \geqslant 4$ and G has an edge $(j, j+k)$. First of all, let us note that $k \leqslant 6$. Indeed, let t be maximal with $1 \leqslant t \leqslant k-1$ and such that j is adjacent to $j+t$. To complete the triangle with sides $(j, j+t)$ and $(j, j+k)$, G must include the edge $(j+t, j+k)$. By the minimality of k, we must have $t \leqslant 3$, $k-t \leqslant 3$ and the desired inequality follows.

The edge $(j, j+k)$ cuts G into a $(k+1)$-triangulation G_1 and an $(n-k+1)$-triangulation G_2. It is easy to verify that we have one of the following four cases (or perhaps a mirror image of (2) or (4)).

(1) G_1 is a fan.

(2) $k = 5$ and the inner edges of G_1 are $(j, j+2)$, $(j, j+3)$, $(j+3, j+5)$.

(3) $k = 6$ and the inner edges of G_1 are $(j, j+2)$, $(j, j+3)$, $(j+3, j+6)$, $(j+4, j+6)$.

(4) $k = 6$ and the inner edges of G_1 are $(j, j+3)$, $(j+1, j+3)$, $(j+3, j+6)$, $(j+4, j+6)$.

In Case 1, by the induction hypothesis, G_2 can be partitioned into m fans with $m \leqslant [(n-k+1)/3]$. Augmenting this partition by the fan G_1, we obtain a partition of G into $m+1$ fans where $m+1 \leqslant [n/3]$.

In Case 2, consider the $(n-3)$-triangulation G_0 obtained from G_2 by adjoining the triangle $(j, j+3, j+5)$. In a partition of G_0 into m-fans, let F be the fan containing $(j, j+3, j+5)$. If F is centered at j, we can enlarge it by the triangles $(j, j+2, j+3)$ and $(j, j+1, j+2)$; adding then a new fan $(j+3, j+4, j+5)$ we obtain a partition of G into $m+1$ fans. If F is centered at $j+5$, we can enlarge it by $(j+5, j+3, j+4)$ and add a new fan $(j+2, j+1, j)$, $(j+2, j, j+3)$. (If F is centered at $j+3$ then it consists of a single triangle and is centered at j and $j+5$ as well.)

In Case 3, consider G_0 obtained from G_2 by adjoining $(j, j+3, j+6)$. In a partition of G_0 into m fans, let F be the fan containing $(j, j+3, j+6)$. If F is centered at j, enlarge it by $(j, j+2, j+3)$, $(j, j+1, j+2)$ and add a new fan $(j+6, j+3, j+4)$, $(j+6, j+4, j+5)$. If F is centered at $j+6$, enlarge it by $(j+6, j+3, j+4)$, $(j+6, j+4, j+5)$ and add a new fan $(j, j+2, j+3)$, $(j, j+1, j+2)$.

In Case 4, consider G_0 obtained from G_2 by adjoining $(j, j+3, j+6)$ and $(j+3, j+6, j+4)$. In a partition of G_0 into m fans, let F be the fan containing $(j+3, j+6, j+4)$. If F is centered at $j+3$ and contains

$(j+3, j+6, j)$, enlarge it by $(j+3, j, j+1)$, $(j+3, j+1, j+2)$ and add the fan $(j+4, j+5, j+6)$. If F is centered at $j+6$ or $j+4$, enlarge it by $(j+6, j+4, j+5)$ and add the fan $(j+3, j, j+1)$, $(j+3, j+1, j+2)$.

The proof is finished.

Obviously, the inequality $f(n) \leqslant [n/3]$ is a corollary to our theorem. Indeed, one can triangulate S and partition it into m fans with $m \leqslant [n/3]$. Each fan is dominated by a single point (which can be chosen from the interior of S). Note also that the bound $[n/3]$ in our theorem cannot be improved (otherwise we would have $f(n) < [n/3]$ for some n which has been shown to be false.) The definition of $f(n)$ can be generalized in various ways (to more than two dimensions, to plane regions with a given number of holes etc.). I don't know the values of these generalized functions.