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Fixed points for (ψ, ϕ)-weak contractions

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ABSTRACT

The aim of this paper is to extend a very recent result proved by Dorić (2009) [4], as well as other theorems given by Rhoades (2001) [2] and Dutta and Choudhury (2008) [3].

1. Introduction and preliminaries

Let (X, d) be a metric space. A mapping $T : X \to X$ is a *contraction* if there exists a constant $k \in (0, 1)$ such that

$$d(Tx, Ty) \leq kd(x, y)$$

holds for any $x, y \in X$. If X is complete, then every contraction has a unique fixed point and that point can be obtained as a limit of repeated iteration of the mapping at any point of X (the Banach contraction principle). Obviously, every contraction is a continuous function. A mapping $T : X \to X$ is a *ϕ-weak contraction* if for each $x, y \in X$, there exists a function $\phi : [0, \infty) \to [0, \infty)$ such that ϕ is positive on $(0, \infty)$, $\phi(0) = 0$, and

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)).$$

The concept of the weak contraction was defined by Alber and Guerre-Delabriere [1] in 1997. They defined such mappings for single-valued maps on Hilbert spaces and proved the existence of fixed points. Rhoades [2] showed that most results of [1] are true for any Banach space. Also Rhoades proved the following generalization of the Banach contraction principle.

Theorem 1. *Let (X, d) be a nonempty complete metric space and let $T : X \to X$ be a ϕ-weak contraction on X. If ϕ is a continuous and nondecreasing function with $\phi(t) > 0$ for all $t > 0$ and $\phi(0) = 0$, then T has a unique fixed point.*

Every contraction is a *ϕ-weak contraction* if we take $\phi(t) = kt$, where $0 < k < 1$.

Dutta and Choudhury [3] proved the following generalization of Theorem 1.

Theorem 2. *Let (X, d) be a nonempty complete metric space and let $T : X \to X$ be a self-mapping satisfying the inequality*

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \phi(d(x, y)),$$

where $\psi, \phi : [0, \infty) \to [0, \infty)$ are both continuous and monotone nondecreasing functions with $\psi(t) = \phi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

Recently, Dorić [4] generalized Theorem 2.

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Theorem 3. *Let (X, d) be a nonempty complete metric space and let $T : X \to X$ be a self-mapping satisfying the inequality*

$$\psi(d(Tx, Ty)) \le \psi(M(x, y)) - \phi(M(x, y)),$$

for any $x, y \in X$, where M is given by

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), (d(x, Ty) + d(Tx, y))/2\},$$

and

(a) $\psi : [0, \infty) \to [0, \infty)$ *is a continuous monotone nondecreasing function with* $\psi(t) = 0$ *if and only if* $t = 0$,
(b) $\phi : [0, \infty) \to [0, \infty)$ *is a lower semi-continuous function with* $\phi(t) = 0$ *if and only if* $t = 0$.

Then T has a unique fixed point.

For other related results we refer the reader to [5,6]. The aim of this work is to show that some of the control conditions of Theorem 3 are not necessary.

2. Main results

Theorem 4. *Let (X, d) be a nonempty complete metric space and $T : X \to X$ be a mapping satisfying for all $x, y \in X$*

$$\psi(d(Tx, Ty)) \le \psi(M(x, y)) - \phi(M(x, y)), \tag{1}$$

where

(a) $\psi : [0, \infty) \to [0, \infty)$ *is a monotone nondecreasing function with* $\psi(t) = 0$ *if and only if* $t = 0$,
(b) $\phi : [0, \infty) \to [0, \infty)$ *is a function with* $\phi(t) = 0$ *if and only if* $t = 0$, *and* $\liminf_{n\to\infty} \phi(a_n) > 0$ *if* $\lim_{n\to\infty} a_n = a > 0$,
(c) $\phi(a) > \psi(a) - \psi(a-)$ *for any* $a > 0$, *where* $\psi(a-)$ *is the left limit of* ψ *at* a.

Then T has a unique fixed point.

Proof. We note that there exists the left limit of ψ at a by the monotonicity of ψ. Let $x_0 \in X$ and the sequence $\{x_n\}$ be defined by $x_n = Tx_{n-1}$, $n = 1, 2, \ldots$. If there exists n such that $x_n = x_{n+1}$ then the conclusion holds. Then we can assume that $x_n \ne x_{n+1}$ for any $n \ge 0$. Substituting $x = x_{n-1}$ and $y = x_n$ in (1) we obtain

$$\psi(d(x_n, x_{n+1})) \le \psi(M(x_{n-1}, x_n)) - \phi(M(x_{n-1}, x_n)), \tag{2}$$

which implies $\psi(d(x_n, x_{n+1})) \le \psi(M(x_{n-1}, x_n))$. Using the monotone property of the ψ-function, we get

$$d(x_n, x_{n+1}) \le M(x_{n-1}, x_n). \tag{3}$$

Now from the triangle inequality for d we have

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), (d(x_{n-1}, x_{n+1}) + d(x_n, x_n))/2\} \\ &= \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1})/2\} \\ &\le \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), (d(x_{n-1}, x_n) + d(x_n, x_{n+1}))/2\} \\ &= \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \end{aligned}$$

If $d(x_n, x_{n+1}) > d(x_{n-1}, x_n)$, then $M(x_{n-1}, x_n) = d(x_n, x_{n+1}) > 0$. By (2) it furthermore implies that

$$\psi(d(x_n, x_{n+1})) \le \psi(d(x_n, x_{n+1})) - \phi(d(x_n, x_{n+1})),$$

which is a contradiction. So, we have

$$d(x_n, x_{n+1}) \le M(x_{n-1}, x_n) \le d(x_{n-1}, x_n). \tag{4}$$

Therefore, the sequence $\{d(x_{n+1}, x_n)\}$ is monotone nonincreasing and bounded. Thus, there exists $r \ge 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} M(x_{n-1}, x_n) = r. \tag{5}$$

We suppose that $r > 0$. If there exists n such that $d(x_{n-1}, x_n) = r$, then by (4) we have $d(x_n, x_{n+1}) = M(x_{n-1}, x_n) = r$ and by (2) we get $\psi(r) \le \psi(r) - \phi(r)$. This is a contradiction. If $d(x_{n-1}, x_n) > r$ for all $n \ge 1$, then by (2) and (5) letting $n \to \infty$ we obtain

$$\psi(r+) \le \psi(r+) - \liminf_{n\to\infty} \phi(M(x_{n-1}, x_n))$$

which is also a contradiction. Hence

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{6}$$

Next we prove that $\{x_n\}$ is a Cauchy sequence. Otherwise there exists $\epsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that $n(k)$ is the smallest index for which $n(k) > m(k) > k$ and $d(x_{m(k)}, x_{n(k)}) \ge \epsilon$. This implies that

$d(x_{m(k)}, x_{n(k)-1}) < \epsilon$ for all $k \geq 1$. Using the triangle inequality we have

$$\epsilon \leq d(x_{m(k)}, x_{n(k)}) \leq d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)}) < \epsilon + d(x_{n(k)-1}, x_{n(k)}).$$

Letting $k \to \infty$ and using (6) we obtain

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \epsilon. \tag{7}$$

Again,

$$d(x_{m(k)}, x_{n(k)-1}) \leq d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)}, x_{n(k)-1})$$

and

$$d(x_{m(k)}, x_{n(k)}) \leq d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)}, x_{n(k)-1}).$$

Then we have

$$|d(x_{m(k)}, x_{n(k)-1}) - d(x_{m(k)}, x_{n(k)})| \leq d(x_{n(k)}, x_{n(k)-1}).$$

Letting $k \to \infty$ and using (6) and (7) it follows that

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)-1}) = \epsilon. \tag{8}$$

Similarly, we can prove that

$$\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}) = \lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)-1}) = \lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)}) = \lim_{k\to\infty} d(x_{m(k)}, x_{n(k)+1}) = \epsilon. \tag{9}$$

Then we get

$$\lim_{k\to\infty} M(x_{m(k)-1}, x_{n(k)-1}) = \epsilon. \tag{10}$$

If there exists a subsequence $\{k(p)\}$ of $\{k\}$ such that $\epsilon < d(x_{m(k(p))}, x_{n(k(p))})$ for any p, then substituting $x = x_{m(k(p))-1}$, $y = x_{n(k(p))-1}$ in (1) we get

$$\psi(d(x_{m(k(p))}, x_{n(k(p))})) \leq \psi(M(x_{m(k(p))-1}, x_{n(k(p))-1})) - \phi(M(x_{m(k(p))-1}, x_{n(k(p))-1})),$$

for any p. By (7) and (10), letting $p \to \infty$ we obtain

$$\psi(\epsilon+) \leq \psi(\epsilon+) - \liminf_{p\to\infty} \phi(M(x_{m(k(p))-1}, x_{n(k(p))-1})),$$

which is a contradiction. We repeat the procedure if there exists a subsequence $\{k(p)\}$ of $\{k\}$ such that $\epsilon < d(x_{m(k(p))}, x_{n(k(p))+1})$ for any p or $\epsilon < d(x_{m(k(p))+1}, x_{n(k(p))})$ for any p. Therefore we can suppose now that $d(x_{m(k)}, x_{n(k)}) = \epsilon$, $d(x_{m(k)}, x_{n(k)+1}) \leq \epsilon$ and $d(x_{m(k)+1}, x_{n(k)}) \leq \epsilon$ for any $k \geq k_1$. Then $M(x_{m(k)}, x_{n(k)}) = \epsilon$ for $k \geq k_3 = \max\{k_1, k_2\}$, where k_2 is such that $d(x_k, x_{k+1}) < \epsilon$ for all $k \geq k_2$. Substituting $x = x_{m(k)}$, $y = x_{n(k)}$ in (1) we have

$$\psi(d(x_{m(k)+1}, x_{n(k)+1})) \leq \psi(\epsilon) - \phi(\epsilon)$$

for any $k \geq k_3$. Obviously, $d(x_{m(k)+1}, x_{n(k)+1}) < \epsilon$; otherwise we have $\phi(\epsilon) = 0$. Letting $k \to \infty$ we obtain

$$\psi(\epsilon-) \leq \psi(\epsilon) - \phi(\epsilon),$$

which contradicts (c) by the hypothesis. Hence $\{x_n\}$ is a Cauchy sequence. By the completeness of X there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$.

Next we show that z is a fixed point of T. Substituting $x = x_n$, $y = z$ in (1) we have

$$\psi(d(x_{n+1}, Tz)) \leq \psi(M(x_n, z)) - \phi(M(x_n, z)), \tag{11}$$

where

$$M(x_n, z) = \max\{d(x_n, z), d(x_n, x_{n+1}), d(z, Tz), (d(x_{n+1}, z) + d(x_n, Tz))/2\}.$$

Suppose that $z \neq Tz$. Then there exists n_1 such that for any $n \geq n_1$ we have

$$d(x_{n+1}, x_n) < d(z, Tz)/2,\ d(x_n, z) < d(z, Tz)/2,\ d(x_{n+1}, z) < d(z, Tz)/2.$$

Accordingly,

$$\begin{aligned} d(z, Tz) \leq M(z, x_n) &\leq \max\{d(z, Tz)/2, d(z, Tz), d(z, Tz)/2, (d(x_{n+1}, z) + d(x_n, z) + d(z, Tz))/2\} \\ &\leq \max\{d(z, Tz)/2, d(z, Tz), d(z, Tz)/2, (d(z, Tz)/2 + d(z, Tz)/2 + d(z, Tz))/2\} \\ &= d(z, Tz), \end{aligned}$$

that is, $M(z, x_n) = d(z, Tz)$. By (11) we obtain

$$\psi(d(z, Tz)-) \le \psi(d(z, Tz)) - \phi(d(z, Tz)),$$

which contradicts (c) by the hypothesis. Hence $z = Tz$.

If there exists another point $y \in X$ such that $y = Ty$, then substituting $x = z$ in (1) we get

$$\psi(d(z, y)) \le \psi(M(z, y)) - \phi(M(z, y)) = \psi(d(z, y)) - \phi(d(z, y))$$

which is a contradiction. □

If ϕ is a lower semi-continuous function then for $\lim_{n\to\infty} a_n = a > 0$ we have $\liminf_{n\to\infty} \phi(a_n) \ge \phi(a) > 0$. Also, if ψ is a left-continuous function then $\psi(a) - \psi(a-) = 0$ and (c) obviously holds. Therefore our control conditions are weaker than those of Theorem 3.

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