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# New pre-dual space of Morrey space

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ABSTRACT

In this paper, we give new characterization of the classical Morrey space. Complementary global Morrey-type spaces are introduced. It is proved that for particular values of parameters these spaces give new pre-dual space of the classical Morrey space. We also show that our new pre-dual space of the Morrey space coincides with known pre-dual spaces.
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## 1. Introduction

The well-known Morrey spaces $\mathcal{M}_{p,\lambda}$ introduced by Morrey in 1938 [1] in relation to the study of partial differential equations, were widely investigated during last decades, including the study of classical operators of Harmonic Analysis—maximal, singular and potential operators—in generalizations of these spaces (the so-called Morrey-type spaces). In the theory of partial differential equations, along with the weighted Lebesgue spaces, Morrey-type spaces also play an important role. These spaces appeared to be quite useful in the study of the local behavior of the solutions to partial differential equations, a priori estimates and other topics in the theory of PDE.

In [2] local Morrey-type spaces $LM_{p\theta,\omega}$ and global Morrey-type spaces $GM_{p\theta,\omega}$ were defined and some properties of these spaces were studied. Authors investigated the boundedness of the Hardy–Littlewood maximal operator in these spaces. Later on there was intensive study of boundedness of other classical operators such as the fractional maximal operator, the Riesz potential and the Calderón–Zygmund singular integral operator (see, for instance [3], for references).

Later in [4] "so-called" complementary local Morrey-type spaces ${}^{\complement}LM_{p\theta,\omega}$ were introduced and the boundedness of the fractional maximal operator from complementary local Morrey-type space ${}^{\complement}LM_{p\theta,\omega}$ into local Morrey-type space $LM_{p\theta,\omega}$ was investigated. As in the definition of the space ${}^{\complement}LM_{p\theta,\omega}$ was used the complement of ball instead of ball, it was named the complementary local Morrey-type space and no relation between $LM_{p\theta,\omega}$ and ${}^{\complement}LM_{p\theta,\omega}$ was studied.

In [5] it is proved that the space ${}^{\complement}LM_{p'\theta',\widetilde{\omega}}$ is dual space of the space $LM_{p\theta,\omega}$, where $1 \le p,\ \theta < \infty$, $p'$ and $\theta'$ are conjugate exponents of $p$ and $\theta$, respectively, and $\widetilde{\omega}(t) = \omega^{\theta-1}(t)\left(\int_t^\infty \omega^\theta(s)ds\right)^{-1}$ (see Theorem 3.6 below).

Our goal in this paper is to introduce global complementary Morrey-type space and show that this new space is pre-dual of the classical Morrey space.

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The paper is organized as follows. We start with notations and give some preliminaries in Section 2. In Section 3 we recall some results on associate spaces of local Morrey-type spaces and complementary local Morrey-type spaces. New characterization of the Morrey space was given in Section 4. In Section 5 we investigate some properties of an intersection and a union of complementary local Morrey-type spaces. New characterization of the pre-dual space of the Morrey space was given in Section 6. Finally, in Section 7 we recall known pre-dual spaces of the Morrey space and compare new one with known spaces.

## 2. Notations and preliminaries

Now we make some conventions. Throughout the paper, we always denote by $c$ and $C$ a positive constant which is independent of main parameters, but it may vary from line to line. By $A \lesssim B$ we mean that $A \le cB$ with some positive constant $c$ independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that $A$ and $B$ are equivalent. A constant, with subscript such as $c_1$, does not change in different occurrences. For a measurable set $E$, $\chi_E$ denotes the characteristic function of $E$.

Given a function $w$ defined on $(0, \infty)$, we say that $w$ satisfies the doubling condition if there exists a constant $D > 0$ such that for any $t > 0$, we have $w(2t) \le Dw(t)$. When $w$ satisfies this condition, we denote $w \in \triangle_2$, for short.

Unless a special remark is made, the differential element $dx$ is omitted when the integrals under consideration are the Lebesgue integrals.

Let $Y$ be a Banach space and $X$ its subspace. The closure of $X$ in $Y$ we will denote by $[X]_Y$.

Let $X$ and $Y$ be two Banach spaces. The symbol $X \hookrightarrow Y$ means that $X \subset Y$ and the natural embedding of $X$ in $Y$ is continuous. We say that $X$ *coincides* with $Y$ (and write $X = Y$) if $X$ and $Y$ are equivalent in the algebraic and topological sense (their norms are equivalent).

**Definition 2.1.** Banach spaces $X_\alpha, \alpha \in A$, form a Banach family if there exists a Banach space $W$ such that

$$X_\alpha \hookrightarrow W, \quad \alpha \in A.$$

If $(X_\alpha)_{\alpha\in A}$ is a Banach family, the concepts of its *sum* $\Sigma(X_\alpha)_{\alpha\in A}$ and *intersection* $\triangle(X_\alpha)_{\alpha\in A}$ will be introduced as follows.

**Definition 2.2** (*[6], Definition 2.1.35*)**.** The sum of a family $(X_\alpha)_{\alpha\in A}$ is a Banach space $X$ such that

(a) $X_\alpha \hookrightarrow X, \alpha \in A$;
(b) if for certain Banach space $Y$ we have

$$X_\alpha \hookrightarrow Y, \quad \alpha \in A, \quad \text{then } X \hookrightarrow Y.$$

Changing the direction of embeddings, we obtain from here the definition of the *intersection* of the family $(X_\alpha)_{\alpha\in A}$.

Note that the sum and intersection of a Banach family exist ([6], Proposition 2.1.36).

For a fixed $p$ with $p \in (0, +\infty]$, $p'$ denotes the conjugate exponent of $p$, namely,

$$p' := \begin{cases} \dfrac{p}{1-p} & \text{if} \quad 0 < p < 1, \\ +\infty & \text{if} \quad p = 1, \\ \dfrac{p}{p-1} & \text{if} \quad 1 < p < +\infty, \\ 1 & \text{if} \quad p = +\infty, \end{cases}$$

and $1/(+\infty) = 0, 0/0 = 0, 0 \cdot (\pm\infty) = 0$.

If $E$ is a nonempty measurable subset on $\mathbb{R}^n$ and $f$ is a measurable function on $E$, then we put

$$\|f\|_{L_p(E)} := \left(\int_E |f(y)|^p dy\right)^{\frac{1}{p}}, \quad 0 < p < +\infty,$$
$$\|f\|_{L_\infty(E)} := \sup\{\alpha : |\{y \in E : |f(y)| \ge \alpha\}| > 0\}.$$

If $I$ is a nonempty measurable subset on $(0, +\infty)$ and $g$ is a measurable function on $I$, then we define $\|g\|_{L_p(I)}$ and $\|g\|_{L_\infty(I)}$, correspondingly.

For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x, r)$ be the open ball centered at $x$ of radius $r$ and ${}^{\complement}B(x, r) := \mathbb{R}^n \setminus B(x, r)$.

Morrey spaces $\mathcal{M}_{p,\lambda}$ were introduced by Morrey in 1938 [1] and defined as follows: *for* $0 \le \lambda \le n$, $1 \le p \le \infty$, $f \in \mathcal{M}_{p,\lambda}$ *if* $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ *and*

$$\|f\|_{\mathcal{M}_{p,\lambda}} \equiv \|f\|_{\mathcal{M}_{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\ r>0} r^{\frac{\lambda-n}{p}} \|f\|_{L_p(B(x,r))} < \infty.$$

Note that $\mathcal{M}_{p,0} = L_\infty(\mathbb{R}^n)$ and $\mathcal{M}_{p,n} = L_p(\mathbb{R}^n)$. If $\lambda < 0$ or $\lambda > n$, then $\mathcal{M}_{p,\lambda} = \Theta$, where $\Theta$ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

In [7] Adams introduced a variant of Morrey-type spaces as follows: for $0 \le \lambda \le n$, $1 \le p, \theta \le \infty$, $f \in \mathcal{M}_{p\theta,\lambda}$ if $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ and

$$\|f\|_{\mathcal{M}_{p\theta,\lambda}} \equiv \|f\|_{\mathcal{M}_{p\theta,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n} \|r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))}\|_{L_\theta(0,\infty)} < \infty.$$

(If $\theta = \infty$, then $\mathcal{M}_{p\theta,\lambda} = \mathcal{M}_{p,\lambda}$.)

Let us recall definitions of local Morrey-type spaces and complementary local Morrey-type spaces.

**Definition 2.3** ([2]). Let $0 < p, \theta \le \infty$ and let $w$ be a non-negative measurable function on $(0,\infty)$. We denote by $LM_{p\theta,\omega}$ the local Morrey-type space, the space of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{LM_{p\theta,\omega}} \equiv \|f\|_{LM_{p\theta,\omega}(\mathbb{R}^n)} = \left\| w(r)\|f\|_{L_p(B(0,r))}\right\|_{L_\theta(0,\infty)}.$$

**Definition 2.4** ([4]). Let $0 < p, \theta \le \infty$ and let $w$ be a non-negative measurable function on $(0,\infty)$. We denote by ${}^{\complement}LM_{p\theta,\omega}$ the complementary local Morrey-type space, the space of all functions $f \in L_p({}^{\complement}B(0,t))$ for all $t > 0$ with finite quasinorm

$$\|f\|_{{}^{\complement}LM_{p\theta,\omega}} \equiv \|f\|_{{}^{\complement}LM_{p\theta,\omega}(\mathbb{R}^n)} = \left\| w(r)\|f\|_{L_p({}^{\complement}B(0,r))}\right\|_{L_\theta(0,\infty)}.$$

**Definition 2.5.** Let $0 < \theta \le \infty$. We denote by $\Omega_\theta$ the set of all non-negative measurable functions $\omega$ on $(0,\infty)$ such that

$$0 < \|\omega\|_{L_\theta(t,\infty)} < \infty, \quad t > 0,$$

and by ${}^{\complement}\Omega_\theta$ the set of all non-negative measurable functions $\omega$ on $(0,\infty)$ such that

$$0 < \|\omega\|_{L_\theta(0,t)} < \infty, \quad t > 0.$$

It is convenient to define local Morrey-type spaces and complementary local Morrey-type spaces at any fixed point $x \in \mathbb{R}^n$.

**Definition 2.6.** Let $0 < p, \theta \le \infty$ and let $w$ be a non-negative measurable function on $(0,\infty)$. For any fixed $x \in \mathbb{R}^n$ we denote by $LM_{p\theta,\omega}^{\{x\}}$, the local Morrey-type space: the space of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{LM_{p\theta,\omega}^{\{x\}}} := \left\| w(r)\|f\|_{L_p(B(x,r))}\right\|_{L_\theta(0,\infty)} = \|f(x+\cdot)\|_{LM_{p\theta,\omega}}.$$

**Definition 2.7.** Let $0 < p, \theta \le \infty$ and let $w$ be a non-negative measurable function on $(0,\infty)$. For any fixed $x \in \mathbb{R}^n$ we denote by ${}^{\complement}LM_{p\theta,w}^{\{x\}}$ the complementary local Morrey-type space, the space of all functions $f \in L_p({}^{\complement}B(x,t))$ for all $t > 0$ with finite quasinorm

$$\|f\|_{{}^{\complement}LM_{p\theta,w}^{\{x\}}} := \left\| w(r)\|f\|_{L_p({}^{\complement}B(x,r))}\right\|_{L_\theta(0,\infty)} = \|f(x+\cdot)\|_{{}^{\complement}LM_{p\theta,\omega}}.$$

Note that $LM_{p\theta,\omega} = LM_{p\theta,\omega}^{\{0\}}$ and ${}^{\complement}LM_{p\theta,\omega} = {}^{\complement}LM_{p\theta,w}^{\{0\}}$.

In [2] global Morrey-type spaces $GM_{p\theta,w}$ were defined.

**Definition 2.8** ([2]). Let $0 < p, \theta \le \infty$ and let $w$ be a non-negative measurable function on $(0,\infty)$. We denote by $GM_{p\theta,w}$, the global Morrey-type space, the space of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ with finite quasinorms

$$\|f\|_{GM_{p\theta,w}} = \|f\|_{GM_{p\theta,w}(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n} \|f(x+\cdot)\|_{LM_{p\theta,\omega}} = \sup_{x\in\mathbb{R}^n} \|f\|_{LM_{p\theta,\omega}^{\{x\}}}.$$

Note that the space $GM_{p\theta,w}$ is the *intersection* space of the Banach family $(LM_{p\theta,\omega}^{\{x\}})_{x\in\mathbb{R}^n}$, that is, $GM_{p\theta,w} = \triangle(LM_{p\theta,\omega}^{\{x\}})$.

Note that

$$\|f\|_{LM_{p\infty,1}} = \|f\|_{GM_{p\infty,1}} = \|f\|_{L_p}.$$

Furthermore, $GM_{p\infty,r^{-\lambda/p}} \equiv \mathcal{M}_{p,\lambda}$, $0 < \lambda < n$. The interpolation properties of the spaces $GM_{p\infty,w}$ were studied by Spanne in [8]. The spaces $GM_{p\theta,r^{-\lambda}}$ were used by Lu [9] for studying the embedding theorems for vector fields of Hörmander type.

As mentioned in [4], the *intersection* $\triangle({}^{\complement}LM^{\{x\}}_{p\theta,w})$ of the Banach family $({}^{\complement}LM^{\{x\}}_{p\theta,w})_{x\in\mathbb{R}^n}$, defined by the finiteness of the quasi-norm

$$\|f\|_{\triangle({}^{\complement}LM^{\{x\}}_{p\theta,w})} = \sup_{x\in\mathbb{R}^n} \Big\|\omega(r)\|f\|_{L_p({}^{\complement}B(x,r))}\Big\|_{L_\theta(0,\infty)} = \sup_{x\in\mathbb{R}^n} \|f\|_{{}^{\complement}LM^{\{x\}}_{p\theta,w}}$$

is of no particular interest because this expression is equal to the product $\|f\|_{L_p(\mathbb{R}^n)} \times \|\omega\|_{L_\theta(0,\infty)}$. It means that

$$\triangle({}^{\complement}LM^{\{x\}}_{p\theta,w}) = \begin{cases} L_p(\mathbb{R}^n), & \text{if } \|w\|_{L_\theta(0,\infty)} < \infty \\ \Theta, & \text{if } \|w\|_{L_\theta(0,\infty)} = \infty. \end{cases}$$

It is natural to define global complementary Morrey-type space as a *sum* of a Banach family in the following way.

**Definition 2.9.** Let $0 < q, \theta \le \infty$ and let $w$ be a non-negative measurable function on $(0, \infty)$. We denote by ${}^{\complement}GM_{q\theta,w} := \Sigma_{x\in\mathbb{R}^n}({}^{\complement}LM^{\{x\}}_{q\theta,w})$, the complementary global Morrey space, the set of all functions $f$ such that $f = \sum_k f_k$ in the sense of distributions, where $f_k \in {}^{\complement}LM^{\{x_k\}}_{q\theta,w}$, $x_k \in \mathbb{R}^n$, and $\sum_k \|f_k\|_{{}^{\complement}LM^{\{x_k\}}_{q\theta,w}} < \infty$.

We define a quasi-norm in ${}^{\complement}GM_{q\theta,w}$

$$\|f\|_{{}^{\complement}GM_{q\theta,w}} := \inf_{f=\sum_k f_k} \sum_k \|f_k\|_{{}^{\complement}LM^{\{x_k\}}_{q\theta,w}},$$

where the infimum is taken over all representation of $f$ of the form $\sum_k f_k$, $f_k \in {}^{\complement}LM^{\{x_k\}}_{q\theta,w}$, $\sum_k \|f_k\|_{{}^{\complement}LM^{\{x_k\}}_{q\theta,w}} < \infty$ and $x_k \in \mathbb{R}^n$.

**Remark 2.10.** Note that in view of Lemma 7.1 this definition is correct (see [6, p. 110] and [10]).

## 3. Associate and dual spaces of local Morrey-type and complementary local Morrey-type spaces

Let $(\mathcal{R}, \mu)$ be a totally $\sigma$-finite non-atomic measure space. Let $\mathfrak{M}(\mathcal{R}, \mu)$ be the set of all $\mu$-measurable a.e. finite real functions on $\mathcal{R}$.

**Definition 3.1.** Let $X$ be a set of functions from $\mathfrak{M}(\mathcal{R}, \mu)$, endowed with a positively homogeneous functional $\|\cdot\|_X$, defined for every $f \in \mathfrak{M}(\mathcal{R}, \mu)$ and such that $f \in X$ if and only if $\|f\|_X < \infty$. We define the associate space $X'$ of $X$ as the set of all functions $f \in \mathfrak{M}(\mathcal{R}, \mu)$ such that $\|f\|_{X'} < \infty$, where

$$\|f\|_{X'} = \sup\left\{\int_{\mathcal{R}} |fg|d\mu : \|g\|_X \le 1\right\}.$$

In what follows we assume $\mathcal{R} = \mathbb{R}^n$ and $d\mu = dx$.

In [5] the associate spaces of local Morrey-type and complementary local Morrey-type spaces were calculated. Our method of construction of the pre-dual space of the Morrey space mainly based on these results. For the sake of completeness we recall some statements from [5].

**Theorem 3.2** (*[5], Theorem 4.5*). *Assume $1 \le p < \infty$, $0 < \theta \le \infty$. Let $\omega \in {}^{\complement}\Omega_\theta$. Set $X = {}^{\complement}LM_{p\theta,\omega}$.*

(i) *Let $0 < \theta \le 1$. Then*

$$\|f\|_{X'} \approx \sup_{t\in(0,\infty)} \|f\|_{L_{p'}(B(0,t))} \|\omega\|^{-1}_{L_\theta(0,t)},$$

*with the positive constants in equivalency independent of $f$.*

(ii) *Let $1 < \theta \le \infty$. Then*

$$\|f\|_{X'} \approx \left(\int_{(0,\infty)} \|f\|^{\theta'}_{L_{p'}(B(0,t))} d\left(-\|\omega\|^{-\theta'}_{L_\theta(0,t+)}\right)\right)^{\frac{1}{\theta'}} + \frac{\|f\|_{L_{p'}(\mathbb{R}^n)}}{\|\omega\|_{L_\theta(0,\infty)}},$$

*with the positive constants in equivalency independent of $f$.*

**Theorem 3.3** (*[5], Theorem 4.6*). *Assume $1 \le p < \infty$, $0 < \theta \le \infty$. Let $\omega \in \Omega_\theta$. Set $X = LM_{p\theta,\omega}$.*

(i) *Let $0 < \theta \le 1$. Then*

$$\|f\|_{X'} \approx \sup_{t\in(0,\infty)} \|f\|_{L_{p'}({}^{\complement}B(0,t))} \|\omega\|^{-1}_{L_\theta(t,\infty)},$$

*with the positive constants in equivalency independent of $f$.*

(ii) *Let* $1 < \theta \le \infty$. *Then*

$$\|f\|_{X'} \approx \left(\int_{(0,\infty)} \|f\|^{\theta'}_{L_{p'}(^{\complement}B(0,t))} d\|\omega\|^{-\theta'}_{L_\theta(t-,\infty)}\right)^{\frac{1}{\theta'}} + \frac{\|f\|_{L_{p'}(\mathbb{R}^n)}}{\|\omega\|_{L_\theta(0,\infty)}},$$

*with the positive constants in equivalency independent of* $f$.

In fact more general results, which are important for our applications, are true.

**Theorem 3.4.** *Assume* $1 \le p < \infty$, $0 < \theta \le \infty$. *Let* $\omega \in {}^{\complement}\Omega_\theta$. *For any fixed* $x \in \mathbb{R}^n$ *set* $X = {}^{\complement}LM^{\{x\}}_{p\theta,\omega}$.

(i) *Let* $0 < \theta \le 1$. *Then*

$$\|f\|_{X'} \approx \sup_{t\in(0,\infty)} \|f\|_{L_{p'}(B(x,t))}\|\omega\|^{-1}_{L_\theta(0,t)},$$

*with the positive constants in equivalency independent of* $f$ *and* $x$.

(ii) *Let* $1 < \theta \le \infty$. *Then*

$$\|f\|_{X'} \approx \left(\int_{(0,\infty)} \|f\|^{\theta'}_{L_{p'}(B(x,t))} d\left(-\|\omega\|^{-\theta'}_{L_\theta(0,t+)}\right)\right)^{\frac{1}{\theta'}} + \frac{\|f\|_{L_{p'}(\mathbb{R}^n)}}{\|\omega\|_{L_\theta(0,\infty)}},$$

*with the positive constants in equivalency independent of* $f$ *and* $x$.

**Proof.** Let $x$ be any fixed point in $\mathbb{R}^n$. Then

$$\begin{aligned}\|f\|_{X'} &= \|f\|_{\left({}^{\complement}LM^{\{x\}}_{p\theta,\omega}\right)'}\\ &= \sup\left\{\int_{\mathbb{R}^n} |f(y)g(y)|dy : \|g\|_{{}^{\complement}LM^{\{x\}}_{p\theta,\omega}} \le 1\right\}\\ &= \sup\left\{\int_{\mathbb{R}^n} |f(x+y)g(x+y)|dy : \|g(x+\cdot)\|_{{}^{\complement}LM_{p\theta,\omega}} \le 1\right\}\\ &= \sup\left\{\int_{\mathbb{R}^n} |f(x+y)g(y)|dy : \|g\|_{{}^{\complement}LM_{p\theta,\omega}} \le 1\right\}\\ &= \|f(x+\cdot)\|_{\left({}^{\complement}LM_{p\theta,\omega}\right)'}.\end{aligned}$$

It remains to apply Theorem 3.2. □

**Theorem 3.5.** *Assume* $1 \le p < \infty$, $0 < \theta \le \infty$. *Let* $\omega \in \Omega_\theta$. *For any fixed* $x \in \mathbb{R}^n$ *set* $X = LM^{\{x\}}_{p\theta,\omega}$.

(i) *Let* $0 < \theta \le 1$. *Then*

$$\|f\|_{X'} \approx \sup_{t\in(0,\infty)} \|f\|_{L_{p'}(^{\complement}B(x,t))}\|\omega\|^{-1}_{L_\theta(t,\infty)},$$

*with the positive constants in equivalency independent of* $f$ *and* $x$.

(ii) *Let* $1 < \theta \le \infty$. *Then*

$$\|f\|_{X'} \approx \left(\int_{(0,\infty)} \|f\|^{\theta'}_{L_{p'}(^{\complement}B(x,t))} d\|\omega\|^{-\theta'}_{L_\theta(t-,\infty)}\right)^{\frac{1}{\theta'}} + \frac{\|f\|_{L_{p'}(\mathbb{R}^n)}}{\|\omega\|_{L_\theta(0,\infty)}},$$

*with the positive constants in equivalency independent of* $f$ *and* $x$.

The proof of Theorem 3.5 is similar to that of Theorem 3.4 (we only need to apply Theorem 3.3 instead of Theorem 3.2) and we omit it.

It was shown in [5] that for some values of parameters the dual spaces coincide with the associated spaces. Namely, the following theorems were proved.

**Theorem 3.6** (*[5], Theorem 5.1*). *Assume* $1 \le p < \infty$, $1 \le \theta < \infty$. *Let* $\omega \in \Omega_\theta$ *and* $\|\omega\|_{L_\theta(0,\infty)} = \infty$. *Then*

$$\left(LM_{p\theta,\omega}\right)^* = {}^{\complement}LM_{p'\theta',\widetilde{\omega}}, \tag{3.1}$$

*where* $\widetilde{\omega}(t) = \omega^{\theta-1}(t)\left(\int_t^\infty \omega^\theta(s)ds\right)^{-1}$, *under the following pairing:*

$$\langle f, g\rangle = \int_{\mathbb{R}^n} fg.$$

*Moreover* $\|f\|_{{}^{\complement}LM_{p'\theta',\widetilde{\omega}}} = \sup_g \left|\int_{\mathbb{R}^n} fg\right|$, *where the supremum is taken over all functions* $g \in LM_{p\theta,\omega}$ *with* $\|g\|_{LM_{p\theta,\omega}} \le 1$.

**Theorem 3.7** *([5], Theorem 5.2)***.** *Assume $1 \le p < \infty$, $1 \le \theta < \infty$. Let $\omega \in {}^{\complement}\Omega_\theta$ and $\|\omega\|_{L_\theta(0,\infty)} = \infty$. Then*

$$\left({}^{\complement}LM_{p\theta,\omega}\right)^* = LM_{p'\theta',\overline{\omega}}, \tag{3.2}$$

*where $\overline{\omega}(t) = \omega^{\theta-1}(t)\left(\int_0^t \omega^\theta(s)ds\right)^{-1}$, under the following pairing:*

$$\langle f, g\rangle = \int_{\mathbb{R}^n} fg.$$

*Moreover $\|f\|_{LM_{p'\theta',\overline{\omega}}} = \sup_g \left|\int_{\mathbb{R}^n} fg\right|$, where the supremum is taken over all functions $g \in {}^{\complement}LM_{p\theta,\omega} : \|g\|_{{}^{\complement}LM_{p\theta,\omega}} \le 1$.*

In fact more general results hold true.

**Theorem 3.8.** *Assume $1 \le p < \infty$, $1 \le \theta < \infty$. Let $\omega \in \Omega_\theta$ such that $\|\omega\|_{L_\theta(0,\infty)} = \infty$. Then for any $x \in \mathbb{R}^n$*

$$\left(LM^{\{x\}}_{p\theta,\omega}\right)^* = {}^{\complement}LM^{\{x\}}_{p'\theta',\widetilde{\omega}}, \tag{3.3}$$

*where $\widetilde{\omega}(t) = \omega^{\theta-1}(t)\left(\int_t^\infty \omega^\theta(s)ds\right)^{-1}$, under the following pairing:*

$$\langle f, g\rangle = \int_{\mathbb{R}^n} fg.$$

*Moreover $\|f\|_{{}^{\complement}LM^{\{x\}}_{p'\theta',\widetilde{\omega}}} = \sup_g \left|\int_{\mathbb{R}^n} fg\right|$, where the supremum is taken over all functions $g \in LM^{\{x\}}_{p\theta,\omega}$ with $\|g\|_{LM^{\{x\}}_{p\theta,\omega}} \le 1$.*

**Theorem 3.9.** *Assume $1 \le p < \infty$, $1 \le \theta < \infty$. Let $\omega \in {}^{\complement}\Omega_\theta$ such that $\|\omega\|_{L_\theta(0,\infty)} = \infty$. Then for any $x \in \mathbb{R}^n$*

$$\left({}^{\complement}LM^{\{x\}}_{p\theta,\omega}\right)^* = LM^{\{x\}}_{p'\theta',\overline{\omega}}, \tag{3.4}$$

*where $\overline{\omega}(t) = \omega^{\theta-1}(t)\left(\int_0^t \omega^\theta(s)ds\right)^{-1}$, under the following pairing:*

$$\langle f, g\rangle = \int_{\mathbb{R}^n} fg.$$

*Moreover $\|f\|_{LM^{\{x\}}_{p'\theta',\overline{\omega}}} = \sup_g \left|\int_{\mathbb{R}^n} fg\right|$, where the supremum is taken over all functions $g \in {}^{\complement}LM^{\{x\}}_{p\theta,\omega} : \|g\|_{{}^{\complement}LM^{\{x\}}_{p\theta,\omega}} \le 1$.*

Proofs of Theorems 3.8 and 3.9 are analogous to proofs of Theorems 3.6 and 3.7, respectively and we omit them.

## 4. New characterization of Morrey space

In this section, we give new characterization of classical Morrey space.

Note that

$$g \mapsto \inf_{x\in\mathbb{R}^n} \int_0^\infty r^{\frac{n-\lambda}{p}-1} \|g\|_{L_{p'}({}^{\complement}B(x,r))} dr$$

is the positively homogeneous functional on $\bigcup_{x\in\mathbb{R}^n} {}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1}$.

Denote by

$$\widetilde{\mathcal{M}}_{p,\lambda} := \left\{f \in \mathfrak{M}(\mathbb{R}^n, dx) : \|f\|_{\widetilde{\mathcal{M}}_{p,\lambda}} < \infty\right\} \tag{4.1}$$

the associate space of the set of functions $\bigcup_{x\in\mathbb{R}^n} {}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1}$, where

$$\|f\|_{\widetilde{\mathcal{M}}_{p,\lambda}} := \sup\left\{\int_{\mathbb{R}^n} |fg| : \inf_{x\in\mathbb{R}^n} \int_0^\infty r^{\frac{n-\lambda}{p}-1} \|g\|_{L_{p'}({}^{\complement}B(x,r))} dr \le 1\right\}. \tag{4.2}$$

To study properties of the space $\widetilde{\mathcal{M}}_{p,\lambda}$, the following results are useful.

**Lemma 4.1.** *Let $1 \le p < \infty$ and $0 < \lambda < n$. Then the inequality*

$$\int_{\mathbb{R}^n} |fg| \le C \|f\|_{\mathcal{M}_{p,\lambda}} \inf_{x\in\mathbb{R}^n} \int_0^\infty r^{\frac{n-\lambda}{p}-1} \|g\|_{L_{p'}(^{\complement}B(x,r))} dr, \tag{4.3}$$

*holds with positive constant $C$ independent of functions $f$ and $g$.*

**Proof.** Let $x$ be any fixed point in $\mathbb{R}^n$. For $\theta = \infty$ and $w(t) = t^{\frac{\lambda-n}{n}}$ Theorem 3.5 (part (ii)) implies the following inequality

$$\int_{\mathbb{R}^n} |fg| \le C \sup_{t>0} t^{\frac{\lambda-n}{p}} \|f\|_{L_p(B(x,t))} \int_0^\infty r^{\frac{n-\lambda}{p}-1} \|g\|_{L_{p'}(^{\complement}B(x,r))} dr, \tag{4.4}$$

with constant $C$ independent of $f$, $g$ and $x$. Therefore

$$\begin{aligned}\int_{\mathbb{R}^n} |fg| &\le C \sup_{x\in\mathbb{R}^n,\, t>0} t^{\frac{\lambda-n}{p}} \|f\|_{L_p(B(x,t))} \int_0^\infty r^{\frac{n-\lambda}{p}-1} \|g\|_{L_{p'}(^{\complement}B(x,r))} dr \\ &= C\|f\|_{\mathcal{M}_{p,\lambda}} \int_0^\infty r^{\frac{n-\lambda}{p}-1} \|g\|_{L_{p'}(^{\complement}B(x,r))} dr. \end{aligned} \tag{4.5}$$

In view of arbitrariness of $x$ we arrive at (4.3). □

**Lemma 4.2.** *Let $1 \le p < \infty$ and $0 < \lambda < n$. Then*

$$\inf_{x\in\mathbb{R}^n} \int_0^\infty r^{\frac{n-\lambda}{p}-1} \|g\|_{L_{p'}(^{\complement}B(x,r))} dr = 0 \tag{4.6}$$

*if and only if $g = 0$ on $\mathbb{R}^n$.*

**Proof.** Obviously,

$$\inf_{x\in\mathbb{R}^n} \int_0^\infty r^{\frac{n-\lambda}{p}-1} \|g\|_{L_{p'}(^{\complement}B(x,r))} dr = 0,$$

when $g = 0$ a.e. on $\mathbb{R}^n$.

Now assume that $\inf_{x\in\mathbb{R}^n} \int_0^\infty r^{\frac{n-\lambda}{p}-1} \|g\|_{L_{p'}(^{\complement}B(x,r))} dr = 0$. For any fixed $R > 0$ consider the function $f = \chi_{B(0,R)}$. Obviously, $f \in \mathcal{M}_{p,\lambda}$, since $\|\chi_{B(0,R)}\|_{\mathcal{M}_{p,\lambda}} \approx R^{\lambda/p}$. Then by the inequality (4.3), we have $\int_{B(0,R)} |f| = 0$, therefore, $f = 0$ a.e. on $B(0,R)$. From arbitrariness of $R$, we get that $f = 0$ a.e. on $\mathbb{R}^n$. □

**Lemma 4.3.** *Let $1 \le p < \infty$ and $0 < \lambda < n$. Then*

$$\bigcup_{x\in\mathbb{R}^n} {}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1} \subset L_1^{\mathrm{loc}}(\mathbb{R}^n).$$

**Proof.** Let $g$ be any function from $\bigcup_{x\in\mathbb{R}^n} {}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1}$. Then there exists $x \in \mathbb{R}^n$ such that $g \in {}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1}$. Let $R$ be any fixed positive number. Since the function $f = \chi_{B(x,R)} \in \mathcal{M}_{p,\lambda}$ and $\|f\|_{\mathcal{M}_{p,\lambda}} \approx R^{\lambda/p}$, by the inequality (4.5) we get

$$\int_{B(x,R)} |g(y)|dy \le CR^{\frac{\lambda}{p}} \|g\|_{^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1}} < \infty.$$

In view of arbitrariness of $R$ we get that $g \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$. □

**Lemma 4.4.** *Assume $1 \le p < \infty$ and $0 < \lambda < n$. Moreover, let $f \in L_{p'}^{\mathrm{loc}}(\mathbb{R}^n)$. Then for any fixed $x \in \mathbb{R}^n$ and $R > 0$*

$$f\chi_{B(x,R)} \in {}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1}.$$

**Proof.** Indeed, for any fixed $x \in \mathbb{R}^n$ and $R:\ 0 < R < \infty$, we get

$$\begin{aligned}\|f\chi_{B(x,R)}\|_{^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1}} &= \int_0^\infty r^{\frac{n-\lambda}{p}-1} \|f\chi_{B(x,R)}\|_{L_{p'}(^{\complement}B(x,r))} dr \\ &= \int_0^\infty r^{\frac{n-\lambda}{p}-1} \left( \int_{^{\complement}B(x,r)\cap B(x,R)} |f|^{p'} \right)^{\frac{1}{p'}} dr\end{aligned}$$

$$= \int_0^R r^{\frac{n-\lambda}{p}-1} \left( \int_{\complement_{B(x,r)} \cap B(x,R)} |f|^{p'} \right)^{\frac{1}{p'}} dr$$

$$\le \left( \int_{B(x,R)} |f|^{p'} \right)^{\frac{1}{p'}} \int_0^R r^{\frac{n-\lambda}{p}-1} dr$$

$$= c_1 R^{\frac{n-\lambda}{p}} \left( \int_{B(x,R)} |f|^{p'} \right)^{\frac{1}{p'}} < \infty. \quad \square$$

Our main result in this section reads as follows.

**Theorem 4.5.** *Assume* $1 \le p < \infty$ *and* $0 < \lambda < n$. *Then*

$$\|f\|_{\mathcal{M}_{p,\lambda}} \approx \|f\|_{\widetilde{\mathcal{M}}_{p,\lambda}}. \tag{4.7}$$

**Proof.** By Lemma 4.1, it is easy to see that

$$\|f\|_{\widetilde{\mathcal{M}}_{p,\lambda}} \lesssim \|f\|_{\mathcal{M}_{p,\lambda}}.$$

Let us prove the opposite estimate, that is, $\|f\|_{\mathcal{M}_{p,\lambda}} \lesssim \|f\|_{\widetilde{\mathcal{M}}_{p,\lambda}}$. If $\|f\|_{\widetilde{\mathcal{M}}_{p,\lambda}} = \infty$, then there is nothing to prove. Assume that $\|f\|_{\widetilde{\mathcal{M}}_{p,\lambda}} < \infty$.

Observe that for $g \in L_{p'}^{\mathrm{loc}}(\mathbb{R}^n)$ the inequality

$$\int_{B(x,R)} |fg| \le C R^{\frac{n-\lambda}{p}} \left( \int_{B(x,R)} |g|^{p'} \right)^{\frac{1}{p'}} \|f\|_{\widetilde{\mathcal{M}}_{p,\lambda}} \tag{4.8}$$

holds with constant $C > 0$ independent of $f$, $g$, $x$ and $R$. Indeed, let $x$ be any fixed point in $\mathbb{R}^n$ and $R > 0$. When $\int_{B(x,R)} |g|^{p'} = 0$ there is nothing to prove, since in this case $g = 0$ a.e. on $B(x, R)$. Assume that $\int_{B(x,R)} |g|^{p'} > 0$. Denote

$$h(y) = \frac{g(y)\chi_{B(x,R)}(y)}{c_1 R^{\frac{n-\lambda}{p}} \left( \int_{B(x,R)} |g|^{p'} \right)^{\frac{1}{p'}}}. \tag{4.9}$$

By Lemma 4.4

$$h \in {}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1},$$

and moreover, $\|h\|_{{}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1}} \le 1$. Consequently,

$$\inf_{x\in\mathbb{R}^n} \int_0^\infty r^{\frac{n-\lambda}{p}-1} \|h\|_{L_{p'}(\complement_{B(x,r)})} dr \le 1.$$

Therefore

$$\int_{\mathbb{R}^n} |hf| \le \|f\|_{\widetilde{\mathcal{M}}_{p,\lambda}}, \tag{4.10}$$

and from (4.9), we get (4.8).

The inequality (4.8) implies that $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$. By the Theorem of Resonance (see [11, Lemma 27, p. 283]) we get that $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$.

The function $g := |f|^{p-1}\chi_{B(x,R)} \in L_{p'}^{\mathrm{loc}}(\mathbb{R}^n)$, and if we put $g$ into the inequality (4.8), we obtain

$$\int_{B(x,R)} |f|^p \le c R^{\frac{n-\lambda}{p}} \left( \int_{B(x,R)} |f|^p \right)^{\frac{1}{p'}} \|f\|_{\widetilde{\mathcal{M}}_{p,\lambda}}.$$

Therefore,

$$R^{\frac{\lambda-n}{p}} \left( \int_{B(x,R)} |f|^p \right)^{\frac{1}{p}} \le c\|f\|_{\widetilde{\mathcal{M}}_{p,\lambda}}.$$

Since a constant $c$ is independent of $x$ and $R$, we get

$$\|f\|_{\mathcal{M}_{p,\lambda}} \le c\|f\|_{\widetilde{\mathcal{M}}_{p,\lambda}}. \quad \square$$

## 5. An intersection and a union of complementary local Morrey-type spaces

In this section, we investigate some properties of an intersection and a union of complementary local Morrey-type spaces. The following lemma is true.

**Theorem 5.1.** *Let $0 < p, \theta \le \infty$ and $w \in {}^{\complement}\Omega_\theta \bigcap \triangle_2$. Then for any $x_1,\ x_2 \in \mathbb{R}^n$, $x_1 \ne x_2$*

$$ {}^{\complement}LM^{\{x_1\}}_{p\theta,w}(\mathbb{R}^n) \bigcap {}^{\complement}LM^{\{x_2\}}_{p\theta,w}(\mathbb{R}^n) = {}^{\complement}LM_{p\theta,w}(\mathbb{R}^n) \bigcap L_p(\mathbb{R}^n). \tag{5.1} $$

**Proof.** In order to prove that

$$ {}^{\complement}LM^{\{x_1\}}_{p\theta,w}(\mathbb{R}^n) \bigcap {}^{\complement}LM^{\{x_2\}}_{p\theta,w}(\mathbb{R}^n) \subset {}^{\complement}LM_{p\theta,w}(\mathbb{R}^n) \bigcap L_p(\mathbb{R}^n) \tag{5.2} $$

observe that for $x_1 \ne x_2$

$$ {}^{\complement}LM^{\{x_1\}}_{p\theta,w}(\mathbb{R}^n) \bigcap {}^{\complement}LM^{\{x_2\}}_{p\theta,w}(\mathbb{R}^n) \subset L_p(\mathbb{R}^n). \tag{5.3} $$

Indeed, let $f \in {}^{\complement}LM^{\{x_1\}}_{p\theta,w}(\mathbb{R}^n) \bigcap {}^{\complement}LM^{\{x_2\}}_{p\theta,w}(\mathbb{R}^n)$. Since for any $R > 0$ and $x \in \mathbb{R}^n$

$$ \begin{aligned} \|f\|_{{}^{\complement}LM^{\{x\}}_{p\theta,\omega}} &= \left( \int_0^\infty w(r)^\theta \|f\|^\theta_{L_p({}^{\complement}B(x,r))} dr \right)^{\frac{1}{\theta}} \\ &\ge \left( \int_0^R w(r)^\theta \|f\|^\theta_{L_p({}^{\complement}B(x,r))} dr \right)^{\frac{1}{\theta}} \\ &\gtrsim \left( \int_0^R w(r)^\theta dr \right)^{\frac{1}{\theta}} \|f\|_{L_p({}^{\complement}B(x,R))} \end{aligned} \tag{5.4} $$

and $w \in {}^{\complement}\Omega_\theta$, then $f \in L_p({}^{\complement}B(x_i,R))$, $i = 1, 2$. Denote $R_0 = |x_1 - x_2|/2$. Then

$$ \|f\|_{L_p(\mathbb{R}^n)} \le \|f\|_{L_p({}^{\complement}B(x_1,R_0))} + \|f\|_{L_p({}^{\complement}B(x_2,R_0))} < \infty. $$

It proves (5.3). It remains to show that

$$ {}^{\complement}LM^{\{x_1\}}_{p\theta,w}(\mathbb{R}^n) \bigcap {}^{\complement}LM^{\{x_2\}}_{p\theta,w}(\mathbb{R}^n) \subset {}^{\complement}LM_{p\theta,w}(\mathbb{R}^n). \tag{5.5} $$

Let $f \in {}^{\complement}LM^{\{x\}}_{p\theta,w}(\mathbb{R}^n)$, where $x$ is a fixed point in $\mathbb{R}^n$. For any $r > 2|x|$ we have ${}^{\complement}B(0,r) \subset {}^{\complement}B(x, r/2)$. Indeed, for $y \in B(x, r/2)$ we get $|y| \le |y - x| + |x| \le r/2 + r/2 = r$, that is, $B(x, r/2) \subset B(0, r)$. Using $w \in \triangle_2$,

$$ \begin{aligned} \|f\|_{{}^{\complement}LM_{p\theta,\omega}} &= \left( \int_0^\infty w(r)^\theta \|f\|^\theta_{L_p({}^{\complement}B(0,r))} dr \right)^{\frac{1}{\theta}} \\ &= \left( \left\{ \int_0^{2|x|} + \int_{2|x|}^\infty \right\} w(r)^\theta \|f\|^\theta_{L_p({}^{\complement}B(0,r))} dr \right)^{\frac{1}{\theta}} \\ &\lesssim \left( \int_0^{2|x|} w(r)^\theta dr \right)^{\frac{1}{\theta}} \|f\|_{L_p(\mathbb{R}^n)} + \left( \int_{2|x|}^\infty w(r)^\theta \|f\|^\theta_{L_p({}^{\complement}B(x,r/2))} dr \right)^{\frac{1}{\theta}} \\ &\lesssim \left( \int_0^{2|x|} w(r)^\theta dr \right)^{\frac{1}{\theta}} \|f\|_{L_p(\mathbb{R}^n)} + \left( \int_0^\infty w(r)^\theta \|f\|^\theta_{L_p({}^{\complement}B(x,r))} dr \right)^{\frac{1}{\theta}} \\ &\approx \left( \int_0^{2|x|} w(r)^\theta dr \right)^{\frac{1}{\theta}} \|f\|_{L_p(\mathbb{R}^n)} + \|f\|_{{}^{\complement}LM^{\{x\}}_{p\theta,\omega}}. \end{aligned} \tag{5.6} $$

Let $f \in {}^{\complement}LM^{\{x_1\}}_{p\theta,w}(\mathbb{R}^n) \bigcap {}^{\complement}LM^{\{x_2\}}_{p\theta,w}(\mathbb{R}^n)$. By (5.3) $f \in L_p(\mathbb{R}^n)$. Then by (5.6), we get that $f \in {}^{\complement}LM_{p\theta,\omega}$, since $w \in {}^{\complement}\Omega_\theta$.

On the other hand, using the same argument as in the proof of estimate (5.6), one can prove that

$$ \|f\|_{{}^{\complement}LM^{\{x\}}_{p\theta,\omega}} \lesssim \left( \int_0^{2|x|} w(r)^\theta dr \right)^{\frac{1}{\theta}} \|f\|_{L_p(\mathbb{R}^n)} + \|f\|_{{}^{\complement}LM_{p\theta,\omega}} $$

holds for any $x \in \mathbb{R}^n$. This gives

$$ {}^{\complement}LM_{p\theta,w}(\mathbb{R}^n) \bigcap L_p(\mathbb{R}^n) \subset {}^{\complement}LM^{\{x\}}_{p\theta,w}(\mathbb{R}^n), \quad x \in \mathbb{R}^n. \quad \square $$

**Corollary 5.2.** *Let $0 < p, \theta \le \infty$ and $w \in {}^{\complement}\Omega_\theta \bigcap \triangle_2$. Then*

$$ \bigcap_{x \in \mathbb{R}^n} {}^{\complement}LM^{\{x\}}_{p\theta,w}(\mathbb{R}^n) = {}^{\complement}LM_{p\theta,w}(\mathbb{R}^n) \bigcap L_p(\mathbb{R}^n). \tag{5.7} $$

**Theorem 5.3.** *Let $0 < p, \theta \le \infty$ and $w \in {}^{\complement}\Omega_\theta \bigcap \triangle_2$. Then for any $x \in \mathbb{R}^n$*

$$ \left[ {}^{\complement}LM_{p\theta,w}(\mathbb{R}^n) \bigcap L_p(\mathbb{R}^n) \right]_{{}^{\complement}LM^{\{x\}}_{p\theta,w}(\mathbb{R}^n)} = {}^{\complement}LM^{\{x\}}_{p\theta,w}(\mathbb{R}^n), \tag{5.8} $$

*that is, the set ${}^{\complement}LM_{p\theta,w}(\mathbb{R}^n) \bigcap L_p(\mathbb{R}^n)$ is dense in ${}^{\complement}LM^{\{x\}}_{p\theta,w}(\mathbb{R}^n)$.*

**Proof.** Let $x$ be any fixed point $\mathbb{R}^n$. For $f \in {}^{\complement}LM^{\{x\}}_{p\theta,w}(\mathbb{R}^n)$ and any $k \in \mathbb{N}$, denote $f_k = f\chi_{B(x,k)\setminus B(x,\frac{1}{k})}$. It is evident that $f_k \to f$, $k \to \infty$ a.e in $\mathbb{R}^n$. By Lebesgue's Dominated Convergence Theorem, we get that

$$ \|f - f_k\|_{{}^{\complement}LM^{\{x\}}_{p\theta,w}(\mathbb{R}^n)} \to 0, \qquad k \to \infty. $$

On the other hand, it is evident that $f_k \in L_p(\mathbb{R}^n)$, $k \in \mathbb{N}$. Since $f_k \in {}^{\complement}LM^{\{x\}}_{p\theta,w}(\mathbb{R}^n)$, by (5.6), we get that $f_k \in {}^{\complement}LM_{p\theta,w}(\mathbb{R}^n)$. Finally, we arrive at $f_k \in {}^{\complement}LM_{p\theta,w}(\mathbb{R}^n) \bigcap L_p(\mathbb{R}^n)$. $\square$

**Theorem 5.4.** *Let $0 < p, \theta \le \infty$ and $w \in {}^{\complement}\Omega_\theta \bigcap \triangle_2$. Then for any $x_1, x_2 \in \mathbb{R}^n$ such that $x_1 \neq x_2$*

$$ \left[ {}^{\complement}LM^{\{x_1\}}_{p\theta,w}(\mathbb{R}^n) \bigcap {}^{\complement}LM^{\{x_2\}}_{p\theta,w}(\mathbb{R}^n) \right]_{{}^{\complement}LM^{\{x_i\}}_{p\theta,w}(\mathbb{R}^n)} = {}^{\complement}LM^{\{x_i\}}_{p\theta,w}(\mathbb{R}^n), \quad i = 1, 2. \tag{5.9} $$

**Proof.** The statement immediately follows from Theorems 5.1 and 5.3. $\square$

## 6. New characterization of the pre-dual space of the Morrey space

In this section, we prove that the space ${}^{\complement}GM_{p'1,\frac{n-\lambda}{p}-1}$ is the pre-dual space of the Morrey space $GM_{p\infty,\frac{\lambda-n}{p}}$.

**Theorem 6.1.** *Let $1 \le p < \infty$ and $0 < \lambda < n$. Then*

$$ \left( {}^{\complement}GM_{p'1,\frac{n-\lambda}{p}-1} \right)^* = GM_{p\infty,\frac{\lambda-n}{p}}. $$

**Proof.** Let $f \in GM_{p\infty,\frac{\lambda-n}{p}}$ and $g \in {}^{\complement}GM_{p'1,\frac{n-\lambda}{p}-1}$. For any representation of $g = \sum_k g_k$ such that $g_k \in {}^{\complement}LM^{\{x_k\}}_{p'1,\frac{n-\lambda}{p}-1}$ and $\sum_k \|g_k\|_{{}^{\complement}LM^{\{x_k\}}_{p'1,\frac{n-\lambda}{p}-1}} < \infty$, where $x_k \in \mathbb{R}^n$, we have

$$ \left| \int_{\mathbb{R}^n} f(x)g(x)dx \right| = \left| \int_{\mathbb{R}^n} f(x) \sum_k g_k(x)dx \right| \le \sum_k \int_{\mathbb{R}^n} |f(x)g_k(x)|dx. $$

Applying (4.4), we get

$$ \left| \int_{\mathbb{R}^n} f(x)g(x)dx \right| \lesssim \|f\|_{GM_{p\infty,\frac{\lambda-n}{p}}} \sum_k \|g_k\|_{{}^{\complement}LM^{\{x_k\}}_{p'1,\frac{n-\lambda}{p}-1}}. \tag{6.1} $$

Since (6.1) holds for any representation of $g$, then

$$ \left| \int_{\mathbb{R}^n} f(x)g(x)dx \right| \lesssim \|f\|_{GM_{p\infty,\frac{\lambda-n}{p}}} \|g\|_{{}^{\complement}GM_{p'1,\frac{n-\lambda}{p}-1}}. \tag{6.2} $$

It proves that $GM_{p\infty,\frac{\lambda-n}{p}} \subset \left( {}^{\complement}GM_{p'1,\frac{n-\lambda}{p}-1} \right)^*$.

Let us show that $\left({}^{\complement}GM_{p'1,\frac{n-\lambda}{p}-1}\right)^* \subset GM_{p\infty,\frac{\lambda-n}{p}}$. It follows from the definition of the space ${}^{\complement}GM_{p'1,\frac{n-\lambda}{p}-1}$ that for any fixed $x \in \mathbb{R}^n$

$$ {}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1} \subset {}^{\complement}GM_{p'1,\frac{n-\lambda}{p}-1}, $$

and

$$ \|f\|_{{}^{\complement}GM_{p'1,\frac{n-\lambda}{p}-1}} \le \|f\|_{{}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1}}. $$

If $L \in \left({}^{\complement}GM_{p'1,\frac{n-\lambda}{p}-1}\right)^*$, then for any $x \in \mathbb{R}^n$

$$ |L(f)| \le C\|f\|_{{}^{\complement}GM_{p'1,\frac{n-\lambda}{p}-1}} \le C\|f\|_{{}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1}} \tag{6.3} $$

for every $f \in {}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1}$. Thus $L \in \left({}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1}\right)^*$. By Theorem 3.7,

$$ \left({}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1}\right)^* = LM^{\{x\}}_{p\infty,\frac{\lambda-n}{p}}, $$

and there exists unique $g_x \in LM^{\{x\}}_{p\infty,\frac{\lambda-n}{p}}$ such that

$$ L(f) = \int_{\mathbb{R}^n} f(z)g_x(z)dz \tag{6.4} $$

for any $f \in {}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1}$. It is easy to see that if $x_1,\ x_2 \in \mathbb{R}^n$, $x_1 \ne x_2$, then $g_{x_1} = g_{x_2}$.

Indeed, by (6.4) we get that

$$ \int_{\mathbb{R}^n} (g_{x_1} - g_{x_2})f = 0 \tag{6.5} $$

for any $f \in {}^{\complement}LM^{\{x_1\}}_{p'1,\frac{n-\lambda}{p}-1}(\mathbb{R}^n) \bigcap {}^{\complement}LM^{\{x_2\}}_{p'1,\frac{n-\lambda}{p}-1}(\mathbb{R}^n)$. By Theorem 5.4, we have

$$ \left[{}^{\complement}LM^{\{x_1\}}_{p'1,\frac{n-\lambda}{p}-1}(\mathbb{R}^n) \bigcap {}^{\complement}LM^{\{x_2\}}_{p'1,\frac{n-\lambda}{p}-1}(\mathbb{R}^n)\right]_{{}^{\complement}LM^{\{x_i\}}_{p'1,\frac{n-\lambda}{p}-1}(\mathbb{R}^n)} = {}^{\complement}LM^{\{x_i\}}_{p'1,\frac{n-\lambda}{p}-1}(\mathbb{R}^n), \quad i = 1, 2. $$

In view of the fact that ${}^{\complement}LM^{\{x_i\}}_{p'1,\frac{n-\lambda}{p}-1}(\mathbb{R}^n)$, $i = 1, 2$ are Banach spaces and the intersection

$$ {}^{\complement}LM^{\{x_1\}}_{p'1,\frac{n-\lambda}{p}-1}(\mathbb{R}^n) \bigcap {}^{\complement}LM^{\{x_2\}}_{p'1,\frac{n-\lambda}{p}-1}(\mathbb{R}^n) $$

is the subspace of both of them, we get that $g_{x_1} = g_{x_2}$. By (6.3)

$$ \left|\int_{\mathbb{R}^n} f(z)g(z)dz\right| \le C\|f\|_{{}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1}}. $$

Thus

$$ \|g\|_{LM^{\{x\}}_{p\infty,\frac{\lambda-n}{p}}} \le C. $$

Since constant $C$ does not depend on $x$, we get that

$$ \|g\|_{GM_{p\infty,\frac{\lambda-n}{p}}} < \infty. $$

Summarizing, we have already proved that there exists unique $g \in GM_{p\infty,\frac{\lambda-n}{p}}$

$$ L(f) = \int_{\mathbb{R}^n} fg \tag{6.6} $$

for every $f \in \bigcup_{x\in\mathbb{R}^n} {}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1}$. Let us prove that (6.6) holds for any $f \in {}^{\complement}GM_{p'1,\frac{n-\lambda}{p}-1}$. Let $f$ be any function from $GM_{p'1,\frac{n-\lambda}{p}-1}$ and $f = \sum_k f_k$ be any representation of $f$ with $f_k \in {}^{\complement}LM^{\{x_k\}}_{p'1,\frac{n-\lambda}{p}-1}$ and $\sum_k \|f_k\|_{{}^{\complement}LM^{\{x_k\}}_{p'1,\frac{n-\lambda}{p}-1}} < \infty$, where $x_k \in \mathbb{R}^n$. For finite representation there is nothing to prove. Assume that the representation is infinite, that is, $f = \sum_{k=1}^{\infty} f_k$. Since

$$\left|L(f) - L\left(\sum_{k=1}^m f_k\right)\right| = \left|L\left(\sum_{k=m+1}^{\infty} f_k\right)\right| \le \sum_{k=m+1}^{\infty} |L(f_k)| \lesssim \sum_{k=m+1}^{\infty} \|f_k\|_{{}^{\complement}LM^{\{x_k\}}_{p'1,\frac{n-\lambda}{p}-1}} \to 0, \quad m \to \infty,$$

and, using (4.4),

$$\left|\int_{\mathbb{R}^n} fg - \int_{\mathbb{R}^n}\left(\sum_{k=1}^m f_k\right)g\right| = \left|\int_{\mathbb{R}^n}\left(f - \sum_{k=1}^m f_k\right)g\right| = \left|\int_{\mathbb{R}^n}\left(\sum_{k=m+1}^{\infty} f_k\right)g\right| \le \sum_{k=m+1}^{\infty}\int_{\mathbb{R}^n}|f_k g| \lesssim \|g\|_{GM_{p\infty,\lambda}} \sum_{k=m+1}^{\infty} \|f_k\|_{{}^{\complement}LM^{\{x_k\}}_{p'1,\frac{n-\lambda}{p}-1}} \to 0, \quad m \to \infty,$$

we arrive at

$$L(f) = \int_{\mathbb{R}^n} fg$$

for any $f \in {}^{\complement}GM_{p'1,\frac{n-\lambda}{p}-1}$. □

## 7. Equivalent pre-dual spaces

For $p > 1$ there are already three characterizations of the pre-dual space of a Morrey space in the literature. First, in 1986, C.T. Zorko proved the following theorem.

**Theorem** ([12], Theorem 5). *Let $p \in (1,\infty)$ and $\lambda \in (0,n)$. Then a pre-dual space of $\mathcal{M}_{p,\lambda}$ is $\mathcal{Z}_{p',\lambda}$ in the following sense: if $g \in \mathcal{M}_{p,\lambda}$, then $\int_{\mathbb{R}^n} fg$ is an element of $(\mathcal{Z}_{p',\lambda})^*$. Moreover, for any $L \in (\mathcal{Z}_{p',\lambda})^*$, there exists $g \in \mathcal{M}_{p,\lambda}$ such that*

$$L(f) = \int_{\mathbb{R}^n} fg, \quad f \in \mathcal{Z}_{p',\lambda}.$$

The space $\mathcal{Z}_{p',\lambda}$ is defined by the set of all functions $f$ on $\mathbb{R}^n$ with the norm

$$\|f\|_{\mathcal{Z}_{p',\lambda}} = \inf\left\{\|\{c_k\}\|_{\ell^1} : f = \sum_k c_k a_k\right\} < \infty,$$

where $a_k$ is a $(p', n-\lambda)$-atom and $\|\{c_k\}\| = \sum_k |c_k| < \infty$, and the infimum is taken over all possible atomic decompositions of $f$. Additionally, we say that a function $a$ on $\mathbb{R}^n$ is an $(p', n-\lambda)$-atom provided that $a$ is supported on a ball $B \subset \mathbb{R}^n$ and satisfies

$$\|a\|_{p'} \le \frac{1}{|B|^{\frac{n-\lambda}{np}}}.$$

Second, in 1998, E.A. Kalita obtained another description of the pre-dual space of a Morrey space as follows.

**Theorem** ([13], Theorem 1). *Let $p \in (1,\infty)$ and $\lambda \in (0,n)$. Then a pre-dual space of $\mathcal{M}_{p,\lambda}$ is $\mathcal{K}_{p',\lambda}$ in the following sense: if $g \in \mathcal{M}_{p,\lambda}$, then $\int_{\mathbb{R}^n} fg$ is an element of $(\mathcal{K}_{p',\lambda})^*$. Moreover, for any $L \in (\mathcal{K}_{p',\lambda})^*$, there exists $g \in \mathcal{M}_{p,\lambda}$ such that*

$$L(f) = \int_{\mathbb{R}^n} fg, \quad f \in \mathcal{K}_{p',\lambda}.$$

The $\mathcal{K}_{p',\lambda}$ consists of all functions $f$ on $\mathbb{R}^n$ with the quasi-norm

$$\|f\|_{\mathcal{K}_{p',\lambda}} = \inf_{\sigma}\left(\int_{\mathbb{R}^n} |f|^{p'}\omega_\sigma^{1-p'}\right)^{\frac{1}{p'}},$$

where

$$\omega_\sigma(x) = \int_{\mathbb{R}^{n+1}_+} r^{-(n-\lambda)} 1_{\mathbb{R}^1_+}(r - |x-y|) d\sigma(y,r),$$

and where the infimum is taken over all $\sigma \in M^+(\mathbb{R}^{n+1}_+)$ (the class of all nonnegative Radon measures on the upper half space $\mathbb{R}^{n+1}_+ = \{(x,r) : x \in \mathbb{R}^n,\ r > 0\}$) with normalization $\sigma(\mathbb{R}^{n+1}_+) = 1$.

Third, in 2004, D.R. Adams and J. Xiao obtained another description of the pre-dual space of a Morrey space as follows.

**Theorem** ([14], Theorem 2.3)**.** *Let $p \in (1,\infty)$ and $\lambda \in (0,n)$. Then the pre-dual space of $\mathcal{M}_{p,\lambda}$ is $\mathcal{H}_{p',\lambda}$ under the following pairing:*

$$\langle f, g\rangle = \int_{\mathbb{R}^n} fg.$$

*Moreover,*

$$\|f\|_{\mathcal{M}_{p,\lambda}} = \sup_g \left| \int_{\mathbb{R}^n} fg \right|,$$

*where the supremum is taken over all functions $g \in \mathcal{H}_{p',\lambda}$ with $\|g\|_{\mathcal{H}_{p',\lambda}} \le 1$.*

We say that $g$ is in $\mathcal{H}_{p',\lambda}$ if

$$\|g\|_{\mathcal{H}_{p',\lambda}} = \inf_\omega \left( \int_{\mathbb{R}^n} |g|^{p'} \omega^{1-p'} \right)^{\frac{1}{p'}} < \infty, \tag{7.1}$$

where the infimum is over all nonnegative functions $\omega$ on $\mathbb{R}^n$ satisfying

$$\|\omega\|_{L^1(\Lambda^{(\infty)}_{n-\lambda})} \le 1.$$

Here $\Lambda^{(\infty)}_d$, $0 < d \le n$, denotes the $d$-dimensional Hausdorff capacity, that is,

$$\Lambda^{(\infty)}_d(E) = \inf \sum r_j^d,$$

where the infimum is taken over all countable coverings of $E \subset \mathbb{R}^n$ by open balls of radius $r_j$.

The following relationship is obtained in [14].

**Theorem** ([14], Theorem 3.3)**.** *Let $p \in (1,\infty)$ and $\lambda \in (0,n)$. Then $\mathcal{Z}_{p',\lambda} = \mathcal{K}_{p',\lambda} = \mathcal{H}_{p',\lambda}$ with*

$$\|\cdot\|_{\mathcal{Z}_{p',\lambda}} \approx \|\cdot\|_{\mathcal{K}_{p',\lambda}} \approx \|\cdot\|_{\mathcal{H}_{p',\lambda}}.$$

Let us compare ${}^{\complement}GM_{p'1,\frac{n-\lambda}{p}-1}$ with known pre-dual spaces. The following Lemma is true.

**Lemma 7.1.** *Let $1 \le p < \infty$ and $0 < \lambda < n$. Then*

$$\bigcup_{x\in\mathbb{R}^n} {}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1} \subset \mathcal{Z}_{p',\lambda}.$$

**Proof.** Let $x$ be any point in $\mathbb{R}^n$ and let $f$ be any function from ${}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1}$. It is possible to decompose $f$ in the following way:

$$f = \sum_k 2^{k\frac{n-\lambda}{p}} \|f\|_{L_{p'}(B(x,2^k)\setminus B(x,2^{k-1}))} \frac{f\chi_{B(x,2^k)\setminus B(x,2^{k-1})}}{2^{k\frac{n-\lambda}{p}} \|f\|_{L_{p'}(B(x,2^k)\setminus B(x,2^{k-1}))}}.$$

Denote

$$\lambda_k := 2^{k\frac{n-\lambda}{p}} \|f\|_{L_{p'}(B(x,2^k)\setminus B(x,2^{k-1}))} \quad \text{and} \quad a_k := \frac{f\chi_{B(x,2^k)\setminus B(x,2^{k-1})}}{2^{k\frac{n-\lambda}{p}} \|f\|_{L_{p'}(B(x,2^k)\setminus B(x,2^{k-1}))}}.$$

Note that $a_k$ is $(p', n-\lambda)$-atom. Indeed, it is obvious that $\operatorname{supp} a_k \subset B(x, 2^k)$. On the other hand,

$$\|a_k\|_{L_{p'}(\mathbb{R}^n)} = \frac{1}{2^{k\frac{n-\lambda}{p}}} \approx \frac{1}{|B(x,2^k)|^{\frac{n-\lambda}{np}}}.$$

Note that $\{\lambda_k\} \in \ell^1$. Indeed,

$$\sum_k |\lambda_k| = \sum_k 2^{k\frac{n-\lambda}{p}} \|f\|_{L_{p'}(B(x,2^k)\setminus B(x,2^{k-1}))}$$
$$\lesssim \sum_k \int_{2^{k-2}}^{2^{k-1}} r^{\frac{n-\lambda}{p}-1} dr \|f\|_{L_{p'}({}^{\complement}B(x,2^{k-1}))}$$
$$\lesssim \sum_k \int_{2^{k-2}}^{2^{k-1}} r^{\frac{n-\lambda}{p}-1} \|f\|_{L_{p'}({}^{\complement}B(x,r))} dr$$
$$\lesssim \int_0^{\infty} r^{\frac{n-\lambda}{p}-1} \|f\|_{L_{p'}({}^{\complement}B(x,r))} dr = \|f\|_{{}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1}}.$$

Recall the following fact: if $\psi$ is a testing function supported in $B(x_1, r_1)$ and $a$ is an atom, we have

$$\left|\int a(x)\psi(x)dx\right| \le r_1^{-\frac{\lambda}{p}} \|\psi\|_\infty$$

(see, for instance, [12]).

Therefore $f = \sum_k \lambda_k a_k$ in the sense of distributions. Thus $f \in \mathcal{Z}_{p',\lambda}$ and

$$\|f\|_{\mathcal{Z}_{p',\lambda}} \lesssim \|f\|_{{}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1}} < \infty. \quad \square$$

Finally, we prove that the space ${}^{\complement}GM_{p'1,\frac{n-\lambda}{p}-1}$ coincides with known pre-dual spaces, namely, the following theorem is true.

**Theorem 7.2.** *Let* $1 \le p < \infty$ *and* $0 < \lambda < n$. *Then*

$${}^{\complement}GM_{p'1,\frac{n-\lambda}{p}-1} = \mathcal{Z}_{p',\lambda}.$$

**Proof.** At first prove that

$$\mathcal{Z}_{p',\lambda} \subset {}^{\complement}GM_{p'1,\frac{n-\lambda}{p}-1}.$$

Let $f \in \mathcal{Z}_{p',\lambda}$. Suppose $f = \sum_k c_k a_k$, where each $a_k$ is $(p', n-\lambda)$-atom supported in some ball $B(x_k, r_k)$ and $\sum_k |\lambda_k| < \infty$. Observe that $a_k \in {}^{\complement}LM^{\{x_k\}}_{p'1,\frac{n-\lambda}{p}-1}$. Indeed,

$$\|a\|_{{}^{\complement}LM^{\{x_k\}}_{p'1,\frac{n-\lambda}{p}-1}} = \int_0^{\infty} r^{\frac{n-\lambda}{p}-1} \|a_k\|_{L_{p'}({}^{\complement}B(x_k,r))} dr$$
$$= \int_0^{r_k} r^{\frac{n-\lambda}{p}-1} \|a_k\|_{L_{p'}({}^{\complement}B(x_k,r))} dr$$
$$\lesssim r_k^{\frac{n-\lambda}{p}} \|a_k\|_{L_{p'}(\mathbb{R}^n)} \lesssim \frac{r_k^{\frac{n-\lambda}{p}}}{|B(x_k, r_k)|^{\frac{n-\lambda}{np}}} = c_1 < \infty.$$

Then

$$\sum_k \|c_k a_k\|_{{}^{\complement}LM^{\{x_k\}}_{p'1,\frac{n-\lambda}{p}-1}} \lesssim c_1 \sum_k |c_k| < \infty,$$

that is, $f \in {}^{\complement}GM_{p'1,\frac{n-\lambda}{p}-1}$.

Conversely, by Lemma 7.1

$$\bigcup_{x\in\mathbb{R}^n} {}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1} \subset \mathcal{Z}_{p',\lambda}.$$

Assume that $f \in {}^{\complement}GM_{p'1,\frac{n-\lambda}{p}-1}$ and let $f = \sum_k f_k$ be any representation of $f$ with $f_k \in {}^{\complement}LM^{\{x_k\}}_{p'1,\frac{n-\lambda}{p}-1}$ and

$$\sum_k \|f_k\|_{{}^{\complement}LM^{\{x_k\}}_{p'1,\frac{n-\lambda}{p}-1}} < \infty,$$

where $x_k \in \mathbb{R}^n$. Then

$$\|f\|_{\mathcal{Z}_{p',\lambda}} = \left\|\sum_k f_k\right\|_{\mathcal{Z}_{p',\lambda}} \le \sum_k \|f_k\|_{\mathcal{Z}_{p',\lambda}} \lesssim \sum_k \|f_k\|_{{}^{\complement}LM^{\{x_k\}}_{p'1,\frac{n-\lambda}{p}-1}} < \infty. \quad \square$$

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