Available online at www.sciencedirect.com

SCIENCE DIRECT

Applied Mathematics Letters 17 (2004) 375–380

Applied Mathematics Letters

www.elsevier.com/locate/aml

# Remarks on Some Relationships Between the Bernoulli and Euler Polynomials

H. M. SRIVASTAVA
Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3P4, Canada
harimsri@math.uvic.ca

Á. PINTÉR
Mathematical Institute, Kossuth Lajos University
H-4010 Debrecen, Hungary
apinter@zuse.math.klte.hu

*(Received and accepted July 2003)*

**Abstract**—In a recent paper which appeared in this journal, Cheon [1] rederived several known properties and relationships involving the classical Bernoulli and Euler polynomials. The object of the present sequel to Cheon's work [1] is to show (among other things) that the *main* relationship (proven in [1]) can easily be put in a much more general setting. Some analogous relationships between the Bernoulli and Euler polynomials are also considered. 

**Keywords**—Bernoulli polynomials, Euler polynomials, Generating functions, Bernoulli numbers, Euler numbers, Addition theorem, Multiplication theorem, Generalized Bernoulli polynomials and numbers, Generalized Euler polynomials and numbers.

## 1. INTRODUCTION

The classical *Bernoulli polynomials* $B_n(x)$ and the classical *Euler polynomials* $E_n(x)$ are usually defined by means of the following generating functions:

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}\qquad(|t|<2\pi) \tag{1}$$

and

$$\frac{2e^{xt}}{e^t+1}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}\qquad(|t|<\pi), \tag{2}$$

The present investigation was completed during the second-named author's visit to the Mathematical Institute of Kossuth Lajos University at Debrecen in July 2003. This work was supported, in part, by the *Natural Sciences and Engineering Research Council of Canada* under Grant OGP0007353 and, in part, by the *Netherlands Organization for Scientific Research* (NWO) and the *Hungarian Academy of Sciences*, by the HNFSR Grants N34001, F34981, and T42985, and by the FKFP Grant 3272-13/066/2001.

0893-9659/04/$ - see front matter  Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX
doi: 10.1016/S0893-9659(04)00041-2

respectively. The corresponding *Bernoulli numbers* $B_n$ and *Euler numbers* $E_n$ are given by

$$B_n := B_n(0) = (-1)^n B_n(1) = \left(2^{1-n} - 1\right)^{-1} B_n\left(\frac{1}{2}\right) \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\};\ \mathbb{N} := \{1, 2, 3, \dots\}) \tag{3}$$

and

$$E_n := 2^n E_n\left(\frac{1}{2}\right) \qquad (n \in \mathbb{N}_0), \tag{4}$$

respectively.

Numerous interesting (and useful) properties and relationships involving each of these polynomials and numbers can be found in many books and tables on this subject (see, for example, [2–6]). Recently, by making use of some fairly standard techniques based upon series rearrangement, Cheon [1] rederived each of the following results (cf. [1, p. 366, Theorem 1; p. 368, Theorem 3]):

$$B_n(x+1) = \sum_{k=0}^{n} \binom{n}{k} B_k(x) \qquad (n \in \mathbb{N}_0), \tag{5}$$

$$E_n(x+1) = \sum_{k=0}^{n} \binom{n}{k} E_k(x) \qquad (n \in \mathbb{N}_0), \tag{6}$$

and

$$B_n(x) = \sum_{\substack{k=0 \\ (k \neq 1)}}^{n} \binom{n}{k} B_k E_{n-k}(x) \qquad (n \in \mathbb{N}_0). \tag{7}$$

Both (5) and (6) are *well-known* (rather *classical*) results and are *obvious* special cases of the following familiar addition theorems:

$$B_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} B_k(x)\, y^{n-k} \qquad (n \in \mathbb{N}_0) \tag{8}$$

and

$$E_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} E_k(x)\, y^{n-k} \qquad (n \in \mathbb{N}_0), \tag{9}$$

when $y = 1$. Furthermore, Cheon's *main* result (7) is essentially the same as the following known relationship (cf., e.g., [3, p. 806, Entry (23.1.29)], [4, p. 29], and [6, p. 66, equation 1.6 (63)]):

$$B_n(x) = 2^{-n} \sum_{k=0}^{n} \binom{n}{k} B_{n-k} E_k(2x) \qquad (n \in \mathbb{N}_0) \tag{10}$$

or, equivalently,

$$2^n B_n\left(\frac{x}{2}\right) = \sum_{k=0}^{n} \binom{n}{k} B_k E_{n-k}(x) \qquad (n \in \mathbb{N}_0). \tag{11}$$

In Section 2 of the present sequel to Cheon's work [1], we propose to verify the equivalence of the relationships (7) and (11). And, in Section 3, we shall consider some interesting generalizations and analogues of the equivalent relationships (7) and (11).

## 2. EQUIVALENCE OF RELATIONSHIPS (7) AND (11)

For both Bernoulli and Euler polynomials, the following multiplication theorems are well known (cf., e.g., [2, p. 37, equation 1.13 (11); p. 41, equations 1.14 (8) and 1.14 (9)]):

$$B_n(mx) = m^{n-1} \sum_{j=0}^{m-1} B_n\left(x + \frac{j}{m}\right) \qquad (n \in \mathbb{N}_0;\ m \in \mathbb{N}) \tag{12}$$

and

$$E_n(mx) = \begin{cases} m^n \sum_{j=0}^{m-1} (-1)^j E_n\left(x + \frac{j}{m}\right) & (n \in \mathbb{N}_0;\ m = 1, 3, 5, \dots), \\ -\frac{2}{n+1} m^n \sum_{j=0}^{m-1} (-1)^j B_{n+1}\left(x + \frac{j}{m}\right) & (n \in \mathbb{N}_0;\ m = 2, 4, 6, \dots), \end{cases} \tag{13}$$

which, together, would yield the following relationships between these two polynomials when $m = 2$ (*with*, of course, $n \longmapsto n-1$ and $x \longmapsto x/2$) [3, p. 806, Entry (23.1.27)]:

$$\begin{aligned} E_{n-1}(x) &= \frac{2^n}{n}\left[B_n\left(\frac{x+1}{2}\right) - B_n\left(\frac{x}{2}\right)\right] \\ &= \frac{2}{n}\left[B_n(x) - 2^n B_n\left(\frac{x}{2}\right)\right] \qquad (n \in \mathbb{N}). \end{aligned} \tag{14}$$

Since $B_1 = -1/2$, by separating the *second* ($k = 1$) term of the sum in (11), we readily find from (11) that

$$2^n B_n\left(\frac{x}{2}\right) = \sum_{\substack{k=0 \\ (k \neq 1)}}^{n} \binom{n}{k} B_k E_{n-k}(x) - \frac{n}{2} E_{n-1}(x) \qquad (n \in \mathbb{N}_0), \tag{15}$$

which, in light of the *second* relationship in (14), immediately yields (7). And, by simply reversing these steps, we can easily deduce (11) from (7).

## 3. GENERALIZATIONS AND ANALOGUES OF THE EQUIVALENT RELATIONSHIPS (7) AND (11)

For a real or complex parameter $\alpha$, the *generalized Bernoulli polynomials* $B_n^{(\alpha)}(x)$ and the *generalized Euler polynomials* $E_n^{(\alpha)}(x)$, *each* of degree $n$ in $x$ *as well as* in $\alpha$, are defined by means of the following generating functions (see, for details, [6, Section 1.6], [7, p. 253 *et seq.*], and [8, Section 2.8]):

$$\left(\frac{t}{e^t - 1}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x) \frac{t^n}{n!} \qquad (|t| < 2\pi;\ 1^\alpha := 1) \tag{16}$$

and

$$\left(\frac{2}{e^t + 1}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x) \frac{t^n}{n!} \qquad (|t| < \pi;\ 1^\alpha := 1), \tag{17}$$

respectively. Clearly, we have

$$B_n^{(1)}(x) = B_n(x) \quad \text{and} \quad E_n^{(1)}(x) = E_n(x) \qquad (n \in \mathbb{N}_0). \tag{18}$$

Moreover, it is easily observed from (16) and (17) that

$$B_n^{(\alpha+\beta)}(x+y)=\sum_{k=0}^{n}\binom{n}{k}B_k^{(\alpha)}(x)\,B_{n-k}^{(\beta)}(y)\qquad (n\in\mathbb{N}_0) \tag{19}$$

and

$$E_n^{(\alpha+\beta)}(x+y)=\sum_{k=0}^{n}\binom{n}{k}E_k^{(\alpha)}(x)\,E_{n-k}^{(\beta)}(y)\qquad (n\in\mathbb{N}_0), \tag{20}$$

respectively. In fact, several *further* addition theorems analogous to the *well-known* (rather *classical*) results (19) and (20) were considered, two decades ago, by Srivastava *et al.* [9] (see also [6, p. 62, equations 1.6 (26) and 1.6 (27); p. 66, equation 1.6 (68)]).

From the generating functions (16) and (17), it follows also that

$$B_n^{(\alpha)}(x+1)-B_n^{(\alpha)}(x)=nB_{n-1}^{(\alpha-1)}(x)\qquad (n\in\mathbb{N}_0) \tag{21}$$

and

$$E_n^{(\alpha)}(x+1)+E_n^{(\alpha)}(x)=2E_n^{(\alpha-1)}(x)\qquad (n\in\mathbb{N}_0), \tag{22}$$

respectively. Furthermore, since

$$B_n^{(0)}(x)=E_n^{(0)}(x)=x^n\qquad (n\in\mathbb{N}_0), \tag{23}$$

upon setting $\beta=0$ in the addition theorems (19) and (20) *and* interchanging $x$ and $y$, we obtain

$$B_n^{(\alpha)}(x+y)=\sum_{k=0}^{n}\binom{n}{k}B_k^{(\alpha)}(y)\,x^{n-k}\qquad (n\in\mathbb{N}_0) \tag{24}$$

and

$$E_n^{(\alpha)}(x+y)=\sum_{k=0}^{n}\binom{n}{k}E_k^{(\alpha)}(y)\,x^{n-k}\qquad (n\in\mathbb{N}_0), \tag{25}$$

respectively. Obviously, the familiar addition theorems (8) and (9) correspond to the special cases of (24) and (25), respectively, when $\alpha=1$.

Next, by combining (21) and (24) (*with* $x=1$ *and* $y\longmapsto x$), we find that

$$B_n^{(\alpha-1)}(x)=\frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}B_k^{(\alpha)}(x)\qquad (n\in\mathbb{N}_0), \tag{26}$$

which, in the special case when $\alpha=1$, immediately yields the following familiar expansion (cf., e.g., [4, p. 26]):

$$x^n=\frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}B_k(x)\qquad (n\in\mathbb{N}_0) \tag{27}$$

in series of the classical Bernoulli polynomials $\{B_n(x)\}_{n=0}^{\infty}$. In precisely the same manner, the addition theorem (25) in conjunction with (22) would lead us to

$$E_n^{(\alpha-1)}(x)=\frac{1}{2}\left[E_n^{(\alpha)}(x)+\sum_{k=0}^{n}\binom{n}{k}E_k^{(\alpha)}(x)\right]\qquad (n\in\mathbb{N}_0) \tag{28}$$

and

$$x^n=\frac{1}{2}\left[E_n(x)+\sum_{k=0}^{n}\binom{n}{k}E_k(x)\right]\qquad (n\in\mathbb{N}_0). \tag{29}$$

In view of (23), this last familiar expansion (29) (cf., e.g., [4, p. 30]) in series of the classical Euler polynomials $\{E_n(x)\}_{n=0}^{\infty}$ is indeed an immediate consequence of (28) when $\alpha=1$.

Making use of some of the above known formulas and identities, we now prove an interesting generalization of the equivalent relationships (7) and (11), which is given by Theorem 1 below.

THEOREM 1. *The following relationship:*

$$B_n^{(\alpha)}(x+y)=\sum_{k=0}^{n}\binom{n}{k}\left[B_k^{(\alpha)}(y)+\frac{k}{2}B_{k-1}^{(\alpha-1)}(y)\right]E_{n-k}(x) \qquad (\alpha\in\mathbb{C};\ n\in\mathbb{N}_0) \tag{30}$$

*holds true between the generalized Bernoulli polynomials and the classical Euler polynomials.*

PROOF. First of all, upon suitably substituting from (29) into the right-hand side of (24), we get

$$\begin{aligned} B_n^{(\alpha)}(x+y)&=\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}B_k^{(\alpha)}(y)\left[E_{n-k}(x)+\sum_{j=0}^{n-k}\binom{n-k}{j}E_j(x)\right]\\ &=\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}B_k^{(\alpha)}(y)\,E_{n-k}(x)\\ &\quad+\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}B_k^{(\alpha)}(y)\sum_{j=0}^{n-k}\binom{n-k}{j}E_j(x), \end{aligned} \tag{31}$$

which, by inverting the order of summation, yields

$$B_n^{(\alpha)}(x+y)=\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}B_k^{(\alpha)}(y)\,E_{n-k}(x)+\frac{1}{2}\sum_{j=0}^{n}\binom{n}{j}E_j(x)\sum_{k=0}^{n-j}\binom{n-j}{k}B_k^{(\alpha)}(y). \tag{32}$$

The innermost sum in (32) can be evaluated by means of (24) itself *with*, of course,

$$x=1 \quad\text{and}\quad n\longmapsto n-j \qquad (0\leqq j\leqq n;\ n,j\in\mathbb{N}_0).$$

We thus find from (32) that

$$B_n^{(\alpha)}(x+y)=\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}B_k^{(\alpha)}(y)\,E_{n-k}(x)+\frac{1}{2}\sum_{j=0}^{n}\binom{n}{j}B_{n-j}^{(\alpha)}(y+1)\,E_j(x) \tag{33}$$

or, equivalently, that

$$B_n^{(\alpha)}(x+y)=\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\left[B_k^{(\alpha)}(y)+B_k^{(\alpha)}(y+1)\right]E_{n-k}(x), \tag{34}$$

which, in light of the recurrence relation (21), leads us at once to relationship (30) asserted by Theorem 1.

REMARK 1. In terms of the generalized Bernoulli numbers $\{B_n^{(\alpha)}\}_{n=0}^{\infty}$, by setting $y=0$ in Theorem 1, we obtain the following special case:

$$B_n^{(\alpha)}(x)=\sum_{k=0}^{n}\binom{n}{k}\left(B_k^{(\alpha)}+\frac{k}{2}B_{k-1}^{(\alpha-1)}\right)E_{n-k}(x) \qquad (\alpha\in\mathbb{C};\ n\in\mathbb{N}_0). \tag{35}$$

Since, by definition (16),

$$B_1^{(\alpha)}=-\frac{\alpha}{2} \quad\text{and}\quad B_n^{(0)}=\delta_{n,0} \qquad (n\in\mathbb{N}_0), \tag{36}$$

a *further* special case of (35) when $\alpha=1$ would yield the equivalent relationships (7) and (11), $\delta_{m,n}$ being the Kronecker delta.

REMARK 2. Alternatively, in view of (23), assertion (30) of Theorem 1 gives us the following (presumably new) relationship between the classical Bernoulli and the classical Euler polynomials when $\alpha=1$:

$$B_n(x+y)=\sum_{k=0}^{n}\binom{n}{k}\left[B_k(y)+\frac{k}{2}y^{k-1}\right]E_{n-k}(x), \tag{37}$$

which, by letting $y\to 0$, immediately yields the equivalent relationships (7) and (11) once again.

Finally, by appealing instead to (25) and (27), our demonstration of Theorem 1 can be applied *mutatis mutandis* in order to derive an interesting analogue of Theorem 1, which is given by Theorem 2 below.

THEOREM 2. *The following relationship:*

$$E_n^{(\alpha)}(x+y) = \sum_{k=0}^{n} \frac{2}{k+1}\binom{n}{k}\left[E_{k+1}^{(\alpha-1)}(y) - E_{k+1}^{(\alpha)}(y)\right] B_{n-k}(x) \qquad (\alpha \in \mathbb{C};\ n \in \mathbb{N}_0) \tag{38}$$

*holds true between the generalized Euler polynomials and the classical Bernoulli polynomials.*

REMARK 3. In light of (23), a special case of assertion (38) of Theorem 2 when $\alpha = 1$ gives us the following relationship:

$$E_n(x+y) = \sum_{k=0}^{n} \frac{2}{k+1}\binom{n}{k}\left[y^{k+1} - E_{k+1}(y)\right] B_{n-k}(x), \tag{39}$$

which, upon setting $y = 0$, immediately yields

$$E_n(x) = -\sum_{k=0}^{n} \frac{2}{k+1}\binom{n}{k} E_{k+1}(0)\, B_{n-k}(x) \qquad (n \in \mathbb{N}_0). \tag{40}$$

This last relationship (40) between the classical Euler and the classical Bernoulli polynomials is evidently analogous to the equivalent relationships (7) and (11). In fact, since [5, p. 29]

$$E_n(0) = (-1)^n E_n(1) = \frac{2\left(1-2^{n+1}\right)}{n+1} B_{n+1} \qquad (n \in \mathbb{N}), \tag{41}$$

relationship (40) can easily be rewritten in the following *equivalent* form:

$$E_{n-2}(x) = 2\binom{n}{2}^{-1} \sum_{k=0}^{n-2} \binom{n}{k}\left(2^{n-k}-1\right) B_{n-k} B_k(x) \qquad (n \in \mathbb{N} \setminus \{1\}), \tag{42}$$

which incidentally is a known result recorded by (for example) Abramowitz and Stegun [3, p. 806, Entry (23.1.28)] (see also [4, p. 29]).

## REFERENCES


1. G.-S. Cheon, A note on the Bernoulli and Euler polynomials, *Appl. Math. Lett.* **16** (3), 365–368, (2003).
2. A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Higher Transcendental Functions, Volume I*, McGraw-Hill, New York, (1953).
3. M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions with Formulas Graphs and Mathematical Tables*, National Bureau of Standards, Washington, DC, (1964).
4. W. Magnus, F. Oberhettinger and R.P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, Third Enlarged Edition, Springer-Verlag, New York, (1966).
5. E.R. Hansen, *A Table of Series and Products*, Prentice Hall, Englewood Cliffs, NJ, (1975).
6. H.M. Srivastava and J. Choi, *Series Associated with the Zeta and Related Functions*, Kluwer Academic, Dordrecht, (2001).
7. A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Higher Transcendental Functions, Volume III*, McGraw-Hill, New York, (1955).
8. Y.L. Luke, *The Special Functions and Their Approximations, Volume I*, Academic Press, New York, (1969).
9. H.M. Srivastava, J.-L. Lavoie and R. Tremblay, A class of addition theorems, *Canad. Math. Bull.* **26**, 438–445, (1983).