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On the analytical solution of some families of transcendental equations

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Abstract

The problem of finding the exact analytical closed-form solution of some families of transcendental equations is studied, in some detail, by the Special Trans Function Theory (STFT). The mathematical genesis of the analytical closed-form solution is presented, and the structure of the theoretical derivation, proofs and numerical results confirm the validity and base principle of the STFT. Undoubtedly, the proposed analytical approach implies the qualitative improvement of the conventional analytical and numerical methods.

Keywords: STFT; Analytical closed-form solution; Families of transcendental equations; EQID

1. Introduction

The subject of the theoretical analysis presented here is a broad family of transcendental equations of the form:

$$\Psi(\zeta) = U(\zeta)e^{-\Psi(\zeta)}, \quad U(\zeta) \in R^+ \tag{1}$$

where we shall restrict ourselves to the one-dimensional case, when $U(\zeta)$ are functions of the arbitrary variable ζ in real domain. We remark that the STFT can easily be extended on the higher dimensionality of formulae (1); namely, we can define a higher-dimensional equation (1) treating the new variables in like manner: $\Psi(\zeta_1, \zeta_{2,\ldots,\zeta_n}) = U(\zeta_1, \zeta_{2,\ldots,\zeta_n})\exp(-\Psi(\zeta_1, \zeta_{2,\ldots,\zeta_n}))$.

Note that the need for families of the transcendental equations' (1) analytical analysis arises in different disciplines. However, the general theoretical approach by the STFT [1–9], presented here, means that the mathematical analysis of Eq. (1) is invariant of any real problems in the applied physics or engineering domains. We have interest, of course, that the STFT used here is novel and rigorous in the mathematical sense and yields an exact analytical closed-form solution to the classical problem on the solvability of some families of transcendental equations (1). It goes without saying, since the essential idea of this presentation is the STFT as a general mathematical method to find the exact analytical closed-form solutions to a broad family of transcendental equations (1). In other words, we can restrict

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ourselves to a theoretical research of the STFT in the mathematical domain. It is a unique manner that the STFT becomes, in its broadest sense, mathematically usable.

Let us note that an equation proposed by Lambert [10] and studied by Euler is solved by W(x) Lambert's W-Function [10,11], but restricted to the $U(\zeta) < \exp(-1)$.

2. The Special Trans Function Theory for some families of transcendental equations (1)

In this section we attempt to find the exact analytical closed-form solution of transcendental equation (1), for an arbitrary real value of $U(\zeta)$.

Theorem 1. *Eq. (1) has the analytical closed-form solution*

$$\Psi(\zeta) = \mathrm{trans}_{+}(U(\zeta)) \tag{2}$$

where $\mathrm{trans}_{+}(U(\zeta))$ *is a new special function defined as*

$$\mathrm{trans}_{+}(U(\zeta)) = \lim_{x\to\infty}\left[\ln\left(\frac{\Phi(x+1,U(\zeta))}{\Phi(x,U(\zeta))}\right)\right] \tag{3}$$

and $\phi(x, U(\zeta))$ *is defined as*

$$\Phi(x,U(\zeta)) = \sum_{n=0}^{[x]} (U(\zeta))^n \,\frac{(x-n)^n}{n!} \tag{4}$$

where $[x]$ *denotes the greatest integer less than or equal to* x.

Proof. The transcendental equation (1) can be identified with a partial differential equation (EQID) of type

$$\frac{\partial\Phi(x,U(\zeta))}{\partial x} = U(\zeta)\Phi(x-1,U(\zeta)). \tag{5}$$

We will solve partial differential equation (5) in the set of originals of the Laplace transform. Thus, after taking the Laplace Transform, Eq. (5) takes the following form:

$$sF(s,U(\zeta)) - U(\zeta)F(s,U(\zeta))\mathrm{e}^{-s} = \Phi(0) \tag{6}$$

where $F(s, U(\zeta)) = L\{\Phi(x, U(\zeta))\}$. Therefore

$$F(s,U(\zeta)) = \frac{\Phi(0)}{s-U(\zeta)\mathrm{e}^{-s}} = \frac{\Phi(0)}{s\left(1-U(\zeta)\frac{\mathrm{e}^{-s}}{s}\right)}. \tag{7}$$

By expanding we get

$$F(s,U(\zeta)) = \frac{\Phi(0)}{s}\sum_{n=0}^{\infty}(U(\zeta))^n\frac{\mathrm{e}^{-ns}}{s^n}. \tag{8}$$

The series (8) converges for $|U(\zeta)\frac{\mathrm{e}^{-s}}{s}| \ll 1$. Now, we can invert term by term to obtain the original

$$\Phi(x,U(\zeta)) = \sum_{n=0}^{\infty}(U(\zeta))^n\frac{(x-n)^n}{n!}H(x-n) \tag{9}$$

where $H(x-n)$ is Heaviside's unit function. The latter function series is identical to Eq. (4), for $x > n$. Finally, applying the Laplace Transform, the analytical solution to the partial differential equation (5) can be written in the closed form representation

$$\Phi(x,U(\zeta)) = \sum_{n=0}^{[x]} (U(\zeta))^n \,\frac{(x-n)^n}{n!}. \tag{10}$$

2.1. On the nature of the analytical solution (10)

The functional series (10) is the unique analytical closed-form solution to Eq. (5) according to Lerch's theorem. Consequently, we have the following statement:

Lemma 2. *For any $x > 1$ the functional series (10) satisfies Eq. (5).*

Proof. When we substitute Eq. (10) into (5), we obtain:

$$\begin{aligned}\frac{\partial \Phi(x,U(\zeta))}{\partial x} - U(\zeta)\Phi(x-1,U(\zeta)) &= \frac{\partial}{\partial x}\left(\sum_{n=0}^{[x]}(U(\zeta))^n \frac{(x-n)^n}{n!}\right) - U(\zeta)\sum_{n=0}^{[x-1]}(U(\zeta))^n \frac{(x-1-n)^n}{n!} \\ &= \sum_{n=1}^{[x]}(U(\zeta))^n \frac{(x-n)^{n-1}}{(n-1)!} - \sum_{n=0}^{[x-1]}(U(\zeta))^{n+1}\frac{(x-(n+1))^n}{n!} \\ &= \sum_{n=0}^{[x-1]}(U(\zeta))^{n+1}\frac{(x-(n+1))^n}{n!} \\ &\quad - \sum_{n=0}^{[x-1]}(U(\zeta))^{n+1}\frac{(x-(n+1))^n}{n!} = 0 \end{aligned} \tag{11}$$

for any $x \notin N$. For $x \in N$ and $x \neq 1$ $\Phi(x, U(\zeta))$ is differentiable and the proof follows as well. □

Remark 1. It can be easily shown that function (10) is not differentiable at $x = 1$, due to the term of first order. Namely, from Eq. (9) we can write $\varphi_n(x) = (x-n)^n H(x-n)$ and

$$\varphi_n(x) = \begin{Bmatrix} 0 & \text{for } x < n \\ (x-n)^n & \text{for } x > n \end{Bmatrix}. \tag{12}$$

Now, from Eq. (12) we have

$$\varphi_n'(x) = \begin{Bmatrix} 0 & \text{for } x < n \\ n(x-n)^{n-1} & \text{for } x \geq n \end{Bmatrix}. \tag{13}$$

Thus, $\varphi_n(x)$ for $n \neq 1$ is differentiable for all $x > 0$. $\varphi'_1(x)$ for $n = 1$ is differentiable for all $x > 0$ and $x \neq 1$.

2.2. On the particular solution to the differential equation (5)

For Eq. (5), by using the method of separation of variables we can write

$$\Phi(x, U(\zeta)) = \Theta(U(\zeta))e^{\omega(U(\zeta))x} \tag{14}$$

and from Eqs. (5) and (14) we get

$$\omega(U(\zeta)) = U(\zeta)e^{-\omega(U(\zeta))}. \tag{15}$$

From Eqs. (1) and (15) it is not difficult to see that $\omega(U(\zeta)) = \Psi(\zeta) = \Psi(\zeta; U(\zeta))$. Namely, the particular solution of the form

$$\Phi_p(x, U(\zeta)) = \Theta(U(\zeta))e^{\omega(U(\zeta))x} = \Theta(U(\zeta))\exp(\Psi(\zeta; U(\zeta))x) = \Theta(U(\zeta))\exp(\Psi(\zeta)x) \tag{16}$$

satisfies the differential equation (5) under the condition that $\Psi(\zeta)$ satisfies Eq. (1).

Lemma 3. *The function (16) is an asymptotic function to Eq. (5) and, consequently,* $\lim_{x\to\infty}\left(\frac{\Phi(x,U(\zeta))}{\Phi_p(x,U(\zeta))}\right) = 1$.

Proof is given in [2] for $Y = Be^Y$ where $B \in R^+$; $Y \in R^+$. Under assumptions $B = -U(\zeta)$; $Y = -\Psi(\zeta)$ we have from $Y = B\exp(Y) \to -\Psi(\zeta) = -U(\zeta)\exp(-\Psi(\zeta))$ or $\Psi(\zeta) = U(\zeta)\exp(-\Psi(\zeta))$. Analogically, we have our proof finished. On the other hand, intuitively it is clear that the particular exponential solution is unique.

3. The unique solution principle

It is easily verified that from Eqs. (10) and (16), we can establish the following asymptotic expression:

$$\lim_{x\to\infty}\left(\frac{\Phi\,(x,U(\zeta))}{\Phi_p\,(x,U(\zeta))}\right)=1 \tag{17}$$

based on the functional theory. According to the unique solution principle and functional theory, we have

$$\lim_{x\to\infty}\left(\frac{\Phi(x+1,U(\zeta))}{\Phi(x,U(\zeta))}\right)=\frac{\Phi_p(x+1,U(\zeta))}{\Phi_p(x,U(\zeta))}=\frac{\Theta(U(\zeta))\mathrm{e}^{\omega(U(\zeta))(x+1)}}{\Theta(U(\zeta))\mathrm{e}^{\omega(U(\zeta))x}}=\mathrm{e}^{\omega(U(\zeta))}=\mathrm{e}^{\Psi(\zeta)} \tag{18}$$

and $\Psi(\zeta)=\mathrm{trans}_+(U(\zeta))=\lim_{x\to\infty}\left[\ln\left(\frac{\Phi(x+1,U(\zeta))}{\Phi(x,U(\zeta))}\right)\right]=\lim_{x\to\infty}\left(U(\zeta)\frac{\Phi(x,U(\zeta))}{\Phi(x+1,U(\zeta))}\right)$, since $\Psi(\zeta)=U(\zeta)\mathrm{e}^{-\Psi(\zeta)}$. □

More explicitly we have

$$\Psi(\zeta)=\mathrm{trans}_+(U(\zeta))=\lim_{x\to\infty}\left(U(\zeta)\left(\frac{\sum_{n=0}^{[x]}\frac{(U(\zeta))^n(x-n)^n}{n!}}{\sum_{n=0}^{[x+1]}\frac{(U(\zeta))^n(x+1-n)^n}{n!}}\right)\right). \tag{19}$$

The latter formulae is the definition of a new special function, $\mathrm{trans}_+(U(\zeta))$.

Note that an essential part of the Special Trans Function Theory is the existence of equality (17). From the previous section it is clear that applying the unique solution principle [1–9] we have (17). Now, we have completed our proof.

4. Numerical results analysis

For practical analysis and numerical calculation the formula (17) takes the form

$$\langle\Phi(x,U(\zeta))\rangle_{[P_+]}=\left\langle\Phi_p(x,U(\zeta))\right\rangle_{[P_+]} \tag{20}$$

where $\langle\Phi(x,U(\zeta))\rangle_{[P_+]}$ denotes the numerical value of the function $\Phi(x,U(\zeta))$ given with $\left[P_+\right]$ accurate digits. $\left[P_+\right]$ is defined as $\left[P_+\right]=[\ln(abs(G))]$ where the Error Function G is defined as: $G(\zeta)=(\Psi(\zeta)-U(\zeta)\exp(-\Psi(\zeta)))$.

Now, from Eqs. (17) and (20) it is possible to establish the equality

$$\left\langle\frac{\Phi(x+1,U(\zeta))}{\Phi(x,U(\zeta))}\right\rangle_{[P_+]}=\left\langle\frac{\Phi_p(x+1,U(\zeta))}{\Phi_p(x,U(\zeta))}\right\rangle_{[P_+]}=\langle\exp(\Psi(\zeta))\rangle_{[P_+]} \tag{21}$$

and

$$\langle\Psi(\zeta)\rangle_{[P_+]}=\ln\left\langle\left(\frac{\Phi(x+1,U(\zeta))}{\Phi(x,U(\zeta))}\right)\right\rangle_{[P_+]}. \tag{22}$$

More explicitly, for fixed variable x, the $\Psi(\zeta)$ take the forms:

$$\langle\Psi(\zeta)\rangle_{[P_+]}=\left\langle\ln\left(\frac{\sum_{n=0}^{[x+1]}\frac{(U(\zeta))^n(x+1-n)^n}{n!}}{\sum_{n=0}^{[x]}\frac{(U(\zeta))^n(x-n)^n}{n!}}\right)\right\rangle_{[P_+]}=\left\langle U(\zeta)\left(\frac{\sum_{n=0}^{[x]}\frac{(U(\zeta))^n(x-n)^n}{n!}}{\sum_{n=0}^{[x+1]}\frac{(U(\zeta))^n(x+1-n)^n}{n!}}\right)\right\rangle_{[P_+]}. \tag{23}$$

From a theoretical point of view solutions (23) for $\langle\Psi(\zeta)\rangle_{[P_+]}$ can be found with an arbitrary order of accuracy by taking an appropriate value of x ([2,9]). The numerical results obtained by using formulae (23) are given in Table 1. It is not difficult to see that analytical solution (23) gives impressive results which suggest that STFT works. Namely, when we consider the number of accurate digits P_+ we have to deal with a sumlimit $[x]$. It may be convenient to

Table 1
The numerical comparison analysis between STFT and classical approach

$U(\zeta)$	$\underset{M}{[x]}$	$\Psi(\zeta)$ (Eq. (23)), $([P_+])$, $W(U(\zeta))$ — Lambert, $([P_+]_W)$
10^{-3}	3	0.0009990014973385, $([P_+] = 17)$, 0.00099900149 $([P_+]_W = 12)$
10^{-3}	4	0.00099900149733853088, (21), 0.00099900149733, (15)
10^{-3}	5	0.00099900149733853088995, (24), 0.0009990014973385, (17)
10^{-3}	6	0.0009990014973385308899578 27, (28), 0.000999001497338530 8, (20)
10^{-3}	7	0.000999001497338530889957827 87,(30),0.00099900149733853088, (21)
0.1	10	0.0912765271608622 64, $([P_+] = 19)$, 0.09127652, $([P_+]_W = 09)$
0.1	12	0.09127652716086226429 98, (23), 0.091276527, (10)
0.1	14	0.0912765271608622642998 9572, (27), 0.0912765271, (11)
0.1	16	0.0912765271608622642998 95721423, (31), 0.09127652716, (12)
0.1	18	0.091276527160862264299895 72142318, (33), 0.0912765271608 (14)
10^3	60	5.249602, $([P_+] = 7)$, –, $([P_+]_W = 0)$
10^3	80	5.24960285240, (12), –, (0)
10^3	100	5.249602852401, (13), –, (0)
10^3	120	5.2496028524015962, (17), –, (0)
10^3	140	5.2496028524015962271, (20), –, (0)
10^3	150	5.24960285240159622 7126, (22), –, (0)
10^9	600	17.841725967421, $([P_+] = 14)$, –, $([P_+]_W = 0)$
10^9	800	17.841725967421469, (17), –, (0)
10^9	900	17.84172596742146918, (19), –, (0)
10^9	10^3	17.8417259674214691825 40600, $([P_+] = 26)$, –, $([P_+]_W = 0)$

consider the P_+ as a function of $[x]$. The latter functional dependence $P_+ = P_+([x])$ sometimes behaves like the ordinary linear function. We remark that, of course, the linear functional form of $P_+ = P_+([x])$ is very usable in the problems in the applied physics or engineering domains. Let us consider this point in more detail. The number of accurate digits in the numerical structure of $\Psi(\zeta)$ depends upon the $[x]$. This implies that we have arbitrary degree of accuracy. Note that this may be a disadvantage in some cases (large $[x]$ for small P_+), for instance. An important study concerning the number of accurate digits in the numerical structure of transcendental numbers has been formulated in [1,2,9].

In an exact way the number of accurate digits in the practical applicable $\Psi(\zeta)$ is in accordance with physical (or engineering) requirements of exactness. Accordingly, the final form of solution (23) continues to stay in the domain of an analytical form, regardless of the number of accurate digits in the numerical structure of $\Psi(\zeta)$ which are obtained by computer calculation.

Note, the subject of the analysis presented in this section is also the numerical efficiency comparison between the famous Lambert $W(U(\zeta))$ function and the special $\text{trans}_+(U(\zeta))$ function (Perovich) (Table 1). It is not difficult to see that $\text{trans}_+(U(\zeta))$ has superior accuracy and computational efficiency for all values of $U(\zeta)$. We have used infinite precision arithmetic and validated the accuracy and computational complexity of the two formulas with Mathematica ver. 5.2. The reference value of the solution has been computed by the numerical equation solver FindRoot and by the built-in function ProductLog.

The Lambert $W(U(\zeta))$ and $\text{trans}_+(U(\zeta))$ functions have been timed by the Mathematica function Timing to compare their execution time. The Timing command includes only CPU time spent in the Mathematica kernel for computing an expression. All computations were carried out on a 2 GHz Pentium 4 platform with 512 MB RAM, under Windows XP Pro SP1 operating system, with Mathematica program, version 5.2 by Wolfram Research, Inc. In this case we have also that $\text{trans}_+(U(\zeta))$ has superior time computational efficiency for all values of $U(\zeta)$. It becomes clear, from the presented numerical values in Table 1, that the proposed formula for $\text{trans}_+(U(\zeta))$ gives superior results. Note, the Lambert W series for $U(\zeta) > \frac{1}{\exp(1)}$ becomes nonconvergent and cannot be used for practical numerical computation. Consequently, for $U(\zeta) > \frac{1}{\exp(1)}$ the formula (23) becomes a unique existing analytical expression for calculation of $\Psi(\zeta)$. Thus, it is a novel theoretical result.

Finally, we shall illustrate the STFT application by some examples from the applied physics and engineering domains. For convenience we restrict ourselves to the one dimensional transcendental equation given below:

Example 1. The transcendental equation: $Y(\zeta) = B(\zeta)\mathrm{e}^{Y(\zeta)}$, for $U(\zeta) < \frac{1}{\exp(1)}$ and for $Y(\zeta) < 1$, has the analytical solution: $Y(\zeta) = \mathrm{trans}_-(B(\zeta))$, where $\mathrm{trans}_-(B(\zeta)) = \lim_{x\to\infty}\left[\ln\left(\frac{\Phi_-(x+1,B(\zeta))}{\Phi_-(x,B(\zeta))}\right)\right]$ and $\Phi_-(x, B(\zeta)) = \sum_{n=0}^{[x]} \frac{(-1)^n(B(\zeta))^n(x-n)^n}{n!}$.

Example 2. The transcendental equation: $Y(\zeta) = B(\zeta)\mathrm{e}^{Y(\zeta)}, U(\zeta) < \frac{1}{\exp(1)}$; analytical solution, for $Y(\zeta) > 1$: $Y(\zeta) = \mathrm{trans}_>(B(\zeta))$ where the $\mathrm{trans}_>(B(\zeta))$ takes the form: $\mathrm{trans}_>(B(\zeta)) = 1 + \lim_{x\to\infty}\left[\ln\left(\frac{V(x+1,B(\zeta))}{V(x,B(\zeta))}\right)\right]$ where $V(x, B(\zeta)) = \sum_{n=0}^{[x-1]}\sum_{k=0}^{n}\sum_{m=0}^{k}\sum_{p=0}^{n-k} \frac{(-1)^{2n-k}(B(\zeta))^p(x-1-k)^{m+p}}{(k-m)!m!(n-k-p)!p!(m+p)!}$.

Example 3. The transcendental equation $\mathrm{e}^{-2Y(\zeta)} - \alpha(\zeta)Y(\zeta)\mathrm{e}^{-Y(\zeta)} - \beta(\zeta)$, for $0 < \beta < 1$; $\alpha > 0$, has the analytical solution $Y(\zeta) = \mathrm{trans}_{VD}(\alpha(\zeta), \beta(\zeta)) = \lim_{x\to\infty}\left[\ln\left(\frac{\varphi_{VD}(x+1,\alpha(\zeta),\beta(\zeta))}{\varphi_{VD}(x,\alpha(\zeta),\beta(\zeta))}\right)\right]$, where $\varphi_{VD}(x, \alpha(\zeta), \beta(\zeta)) = \sum_{n=0}^{\left[\frac{x-1}{2}\right]}\sum_{k=0}^{n}\sum_{m=0}^{k}\sum_{p=0}^{n-k} \frac{(-1)^{k+m+p}(\beta(\zeta))^p n!(x-m-k-1)^{m+1+p}}{(k-m)!m!(n-k-p)!p!(m+1+p)!(\alpha(\zeta))^{m+p}}$, etc.

We remark that the latter analytical results in Examples 2 and 3 are new, first presented in this paper.

5. Conclusions

From the previous sections it is obvious that the STFT is a consistent, general approach to solving transcendental equations and that it is independent of the type of transcendental equation. This means that in the same manner we can obtain arbitrary Special Trans Functions for very different transcendental equations (for instance, Examples 1–3).

The new formula (23) being derived in this paper, using the STFT, is valid (see Table 1) in the numerical sense. Namely, the obtained analytical solutions apart from the theoretical value have practical application. The form of the EQID can be routinely applied, according to the intuitive assumptions. The theoretical accuracy of the STFT is unlimited, and extreme precision is attainable with this approach [2,9].

Note that one of the advantages of the Special Trans Function Theory, as a mathematical method, is its applicability for arbitrary nonlinear forms. On the other hand in the STFT we have the possibility to obtain different gradients. It is not difficult to see that the latter statement implies the rigorous analytical analysis for any problem. Thus, the introduction of new analytical sensitivity parameters directly follows in the form $\frac{\partial\Psi}{\partial B}$, $\frac{\partial\Psi}{\partial\zeta}$. For example: $\frac{\partial\Psi}{\partial\zeta} = \frac{\partial U}{\partial\zeta}\frac{1}{U}\Psi - \frac{\partial\Psi}{\partial\zeta}\Psi$ or $\frac{\partial\Psi}{\partial\zeta}(1+\Psi) = \frac{\partial U}{\partial\zeta}\frac{\Psi}{U}$ and, consequently $\frac{\partial\Psi(\zeta)}{\partial\zeta} = \frac{\Psi(\zeta)}{U(\zeta)(1+\Psi(\zeta))}\frac{\partial U(\zeta)}{\partial\zeta}$ or $\frac{\partial\Psi(\zeta)}{\partial\zeta} = \frac{\mathrm{trans}_+(U(\zeta))}{U(\zeta)(1+\mathrm{trans}_+(U(\zeta)))}\frac{\partial U(\zeta)}{\partial\zeta}$. The latter expression is a significant and novel contribution of the STFT.

According to the authors' knowledge, this is the first direct application of the STFT to the mathematical genesis of the analytical closed form solution to some families of transcendental equations (1) independently of any physical process or phenomena.

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