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Some new weakly singular integral inequalities and their applications to fractional differential equations

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Abstract

Some new inequalities with weakly singular integral kernel are developed, which generalize some known inequalities and can be used in the qualitative and quantitative analysis of the solutions to certain fractional differential equations.

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Keywords: inequality; weakly singular integral kernel; fractional differential and integral equation; quantitative analysis

1 Introduction

The well-known inequalities, such as the Gronwall type inequality, the Gronwall-Bellman type inequality, the Henry-Gronwall type inequality, the Henry-Bihari type inequality and their variants in retarded form played important roles in the research of quantitative analysis of the solutions to differential and integral equations, as well as in the modeling of engineering and science problems. Recently, with the development of fractional differential equations, integral inequalities with weakly singular kernels have drawn more attention [1–11]. In 1981, Henry [12] proposed a method to find solutions and proved some results concerning linear integral inequalities with a weakly singular kernel. In 1997, Medveď [3] presented a new method to solve integral inequalities of Henry-Gronwall type and their Bihari version, then he got the explicit bounds with a quite simple formula, similar to the classic Gronwall-Bellman inequalities. Furthermore, he also obtained global solutions of the semilinear evolutions in [4].

In 2011, Ye and Gao [5] presented the integral inequalities of Henry-Gronwall type,

$$\begin{cases} u(t) \le a(t) + \int_{t_0}^{t} (t-s)^{\beta-1}[b(s)u(s) + c(s)u(s-r)]\,ds, & t \in [t_0, T), \\ u(t) \le \varphi(t), & t \in [t_0 - r, t_0). \end{cases}$$

In 2013, Shao and Meng [6] established Gronwall-Bellman type inequalities with a weakly singular integral

$$\begin{cases} u(t) \le a(t) + \int_{t_0}^{t} (t-s)^{\beta-1} b(s)u(s)\,ds + \int_{t_0}^{t} (t-s)^{\beta-1} p(s)u^{\gamma}(s-r)\,ds, & t \in [t_0, T), \\ u(t) \le \varphi(t), & t \in [t_0 - r, t_0). \end{cases}$$

In [7], Feng and Meng studied the following Gronwall-Bellman type inequalities:

$$u^p(x) \leq a(x) + \int_0^x b(t)u^q(t)\,dt + \frac{1}{\Gamma(\alpha)}h(x)\int_0^x (x-t)^{\alpha-1}L\bigl(t,u(t)\bigr)\,dt.$$

In 2008, by using the modified Medveď method, Ma and Pečarić [8] studied the inequality

$$u^p(t) \leq a(t) + b(t)\int_0^t \bigl(t^\alpha - s^\alpha\bigr)^{\beta-1}s^{\gamma-1}f(s)u^q(s)\,ds, \quad t \in R_+.$$

The aim of this paper was to give explicit bounds to some new nonlinear Henry-Gronwall type retarded integral inequalities with weakly singular integral kernel of the form

$$\begin{cases} u^p(t) \leq a(t) + \int_{t_0}^t (t-s)^{\beta-1}b(s)u^q(s)\,ds + \int_{t_0}^t (t-s)^{\beta-1}c(s)u^l(s-r)\,ds, & t \in I, \\ u(t) \leq \varphi(t), \quad t \in [t_0 - r, t_0), \end{cases}$$

and Gronwall-Bellman type integral inequalities with nonlinear weakly singular integral kernel of the form

$$u^p(t) \leq a(t) + b(t)\int_0^t (t-s)^{\beta-1}c(s)u^m(s)\,ds + d(t)\int_0^t \bigl(t^\alpha - s^\alpha\bigr)^{\beta-1}s^{\gamma-1}f(s)u^q(s)\,ds,$$

which can be used as handy and effective tools in the study of the delay fractional differential equations. We also give some examples to illustrate applications of our results.

2 Preliminary knowledge

In the following, R denotes the set of real numbers, N denotes the set of integer numbers, $R_+ = [0, +\infty)$, $I = [t_0, T) \subset R_+$, $T < \infty$. For convenience, we give some lemmas which will be used in the proof of the main results.

Lemma 2.1 (Jensen's inequality) *Let $n \in N$, $a_1, a_2, \ldots, a_n$ be nonnegative real numbers. Then, for $r > 1$,*

$$\left(\sum_{i=1}^n a_i\right)^r \leq n^{r-1}\sum_{i=1}^n a_i^r.$$

Lemma 2.2 (see [13]) *Let $a \geq 0$, $p \geq q \geq 0$, $p \neq 0$, then*

$$a^{\frac{q}{p}} \leq \frac{q}{p}K^{\frac{q-p}{p}}a + \frac{p-q}{p}K^{\frac{q}{p}} \quad \text{for any } K > 0.$$

Lemma 2.3 (see [14]) *Let α, β, γ, and p be positive constants, then*

$$\int_0^t \bigl(t^\alpha - s^\alpha\bigr)^{p(\beta-1)}s^{p(\gamma-1)}\,ds = \frac{t^\theta}{\alpha}B\left[\frac{p(\gamma-1)+1}{\alpha}, p(\beta-1)+1\right], \quad t \in R_+.$$

Here $B(m,n) = \int_0^1 s^{m-1}(1-s)^{n-1}\,ds$ $(m > 0, n > 0)$ is the B-function, $\theta = p[\alpha(\beta-1)+\gamma-1]+1$.

Definition 2.1 (see [15]) Let $[x,y,z]$ be an ordered parameter group of nonnegative real numbers. The group is said to *belong to the first class distribution* and denoted by $[x,y,z] \in I$ if the conditions $x \in (0,1]$, $y \in (\frac{1}{2},1)$, and $z \geq \frac{3}{2} - y$ are satisfied; the group is said to *belong to the second class distribution* and denoted by $[x,y,z] \in II$ if the conditions $x \in (0,1]$, $y \in (0,\frac{1}{2}]$, and $z > \frac{1-2y^2}{1-y^2}$ are satisfied.

Lemma 2.4 (see [15]) *Suppose that the positive constants* α, β, γ, p_1, *and* p_2 *satisfy the conditions*

(a) *if* $[\alpha,\beta,\gamma] \in I$, $p_1 = \frac{1}{\beta}$;
(b) *if* $[\alpha,\beta,\gamma] \in II$, $p_2 = \frac{1+4\beta}{1+3\beta}$, *then*

$$B\left[\frac{p_i(\gamma-1)+1}{\alpha}, p_i(\beta-1)+1\right] \in (0,\infty)$$

and

$$\theta_i = p_i\bigl[\alpha(\beta-1)+\gamma-1\bigr]+1 \geq 0$$

are valid for $i = 1,2$.

Lemma 2.5 ([13]) *Let* $a(t), b(t) \in C(I,R_+)$, $p \geq q > 0$. *If* $u(t) \in C(I,R_+)$, *and*

$$u^p(t) \leq a(t) + \int_{t_0}^t b(s)u^q(s)\,ds,$$

then

$$u(t) \leq \left\{a(t) + \int_{t_0}^t A_1(s)e^{\int_s^t B_1(\tau)\,d\tau}\,ds\right\}^{\frac{1}{p}}, \quad t \in I. \tag{2.1}$$

Here

$$A_1(t) = b(t)\left[\frac{q}{p}K^{\frac{q-p}{p}}a(t) + \frac{p-q}{p}K^{\frac{q}{p}}\right], \qquad B_1(t) = \frac{q}{p}K^{\frac{q-p}{p}}b(t).$$

Lemma 2.6 *Let* $a(t), b(t), c(t) \in C(I,R_+)$, $p \geq q > 0$, $p \geq l > 0$, $\varphi(t) \in C([t_0-r,t_0],R_+)$, $a(t_0) = \varphi(t_0)$. *If* $u(t) \in C(I,R_+)$, *and*

$$\begin{cases} u^p(t) \leq a(t) + \int_{t_0}^t b(s)u^q(s)\,ds + \int_{t_0}^t c(s)u^l(s-r)\,ds, & t \in I, \\ u(t) \leq \varphi(t), & t \in [t_0-r,t_0). \end{cases} \tag{2.2}$$

Then, for $t \in [t_0, t_0+r)$,

$$u(t) \leq \left\{a(t) + \int_{t_0}^t c(s)\varphi^l(s-r)\,ds + \int_{t_0}^t A_2(s)e^{\int_s^t B_2(\tau)\,d\tau}\,ds\right\}^{\frac{1}{p}}. \tag{2.3}$$

Here $r \in R_+$,

$$A_2(t) = \frac{q}{p}K^{\frac{q-p}{p}}b(t)\left(a(t) + \int_{t_0}^t c(s)\varphi^l(s-r)\,ds\right) + \frac{p-q}{p}K^{\frac{q}{p}},$$
$$B_2(t) = B_1(t) = \frac{q}{p}K^{\frac{q-p}{p}}b(t).$$

For $t \in [t_0 + r, T)$,

$$u(t) \leq \left\{ a(t) + \int_{t_0}^{t} A_3(s) e^{\int_s^t B_3(\tau)\,d\tau}\,ds \right\}^{\frac{1}{p}}, \tag{2.4}$$

where

$$A_3(t) = \frac{q}{p} K^{\frac{q-p}{p}} a(t)b(t) + \frac{p-q}{p} K^{\frac{q}{p}} b(t) + \frac{l}{p} K^{\frac{l-p}{p}} a(t-r)c(t) + \frac{p-l}{p} K^{\frac{l}{p}} c(t),$$
$$B_3(t) = \frac{q}{p} K^{\frac{q-p}{p}} b(t) + \frac{l}{p} K^{\frac{l-p}{p}} c(t).$$

Proof For $t \in [t_0, t_0 + r)$, we get

$$u^p(t) \leq a(t) + \int_{t_0}^{t} c(s)\varphi^l(s-r)\,ds + \int_{t_0}^{t} b(s)u^q(s)\,ds,$$

then from Lemma 2.5, the inequality (2.3) holds.

For $t \in [t_0 + r, T)$, let $z(t) = \int_{t_0}^t b(s)u^q(s)\,ds + \int_{t_0}^t c(s)u^l(s-r)\,ds$, then $z(t_0) = 0$, $z(t)$ is non-decreasing, and

$$u^p(t) \leq a(t) + z(t).$$

Then from Lemma 2.2 we obtain, for any $K > 0$,

$$\begin{aligned} z'(t) &= b(t)u^q(t) + c(t)u^l(t-r) \\ &\leq b(t)\left[\frac{q}{p} K^{\frac{q-p}{p}} \big(a(t) + z(t)\big) + \frac{p-q}{p} K^{\frac{q}{p}}\right] \\ &\quad + c(t)\left[\frac{l}{p} K^{\frac{l-p}{p}} \big(a(t-r) + z(t-r)\big) + \frac{p-l}{p} K^{\frac{l}{p}}\right] \\ &\leq A_3(t) + B_3(t)z(t), \end{aligned}$$

and we have $z(t) \leq \int_{t_0}^t A_3(s) e^{\int_s^t B_3(\tau)\,d\tau}\,ds$. Then the inequality (2.4) holds. □

Lemma 2.7 ([16]) *Let* $a(t)$, $b(t,s)$, $b'(t,s)$ *be continuous and nonnegative. If* $u(t) \in C(I, R_+)$ *and*

$$u(t) \leq a(t) + \int_{t_0}^{t} b(t,s)u(s)\,ds,$$

then

$$u(t) \leq a(t) + e^{\int_{t_0}^t b(t,s)\,ds} \int_{t_0}^{t} e^{-\int_{t_0}^{\tau} b(\tau,s)\,ds} \left[a(\tau)b(\tau,\tau) + \int_{t_0}^{\tau} b'_{\tau}(\tau,s)a(s)\,ds \right] d\tau. \tag{2.5}$$

3 Main results

We are now to deal with a certain class of Henry-Gronwall type retarded inequalities with weak integral kernels.

Theorem 3.1 *Let* $a(t), b(t), c(t) \in C(I, R_+)$, $p \geq q > 0$, $p \geq l > 0$, $\varphi(t) \in C([t_0 - r, t_0], R_+)$, $a(t_0) = \varphi(t_0)$, $u(t) \in C(I, R_+)$. *Suppose*

$$\begin{cases} u^p(t) \leq a(t) + \int_{t_0}^t (t-s)^{\beta-1} b(s) u^q(s)\,ds + \int_{t_0}^t (t-s)^{\beta-1} c(s) u^l(s-r)\,ds, & t \in I, \\ u(t) \leq \varphi(t), & t \in [t_0 - r, t_0), \end{cases} \tag{3.1}$$

where $\beta > 0$.

(i) *Suppose that* $\beta > \frac{1}{2}$, *then*

$$\begin{cases} u(t) \leq \{a_1(t) + \int_{t_0}^t c_1(s)\varphi^l(s-r)\,ds + \int_{t_0}^t A_4(s) e^{\int_s^t B_4(\tau)\,d\tau}\,ds\}^{\frac{1}{p}}, & t \in [t_0, t_0 + r), \\ u(t) \leq \{a_1(t) + \int_{t_0}^t A_5(s) e^{\int_s^t B_5(\tau)\,d\tau}\,ds\}^{\frac{1}{p}}, & t \in [t_0 + r, T). \end{cases} \tag{3.2}$$

Here

$$\begin{aligned} a_1(t) &= 3a^2(t)e^{-2t}, \qquad b_1(t) = \frac{6\Gamma(2\beta-1)}{4^\beta} b^2(t) e^{2t(\frac{q}{p}-1)}, \\ c_1(t) &= \frac{6\Gamma(2\beta-1)}{4^\beta} c^2(s) e^{2t(\frac{l}{p}-1)}, \\ A_4(t) &= \frac{q}{p} K^{\frac{q-p}{p}} b_1(t) \left[a_1(t) + \int_{t_0}^t c_1(s)\varphi^l(s-r)\,ds\right] + \frac{p-q}{p} K^{\frac{q}{p}}, \\ B_4(t) &= \frac{q}{p} K^{\frac{q-p}{p}} b_1(t), \\ A_5(t) &= \left[\frac{q}{p} K^{\frac{q-p}{p}} a_1(t) + \frac{p-q}{p} K^{\frac{q}{p}}\right] b_1(t) + \left[\frac{l}{p} K^{\frac{l-p}{p}} a_1(t-r) + \frac{p-l}{p} K^{\frac{l}{p}}\right] c_1(t), \\ B_5(t) &= \frac{q}{p} K^{\frac{q-p}{p}} b_1(t) + \frac{l}{p} K^{\frac{l-p}{p}} c_1(t). \end{aligned}$$

(ii) *If* $0 < \beta \leq \frac{1}{2}$, *let* $q_1 = \frac{1+\beta}{\beta}$, $p_1 = 1 + \beta$, *then*

$$\begin{cases} u(t) \leq \{a_2(t) + \int_{t_0}^t c_2(s)\varphi^l(s-r)\,ds + \int_{t_0}^t A_6(s) e^{\int_s^t B_6(\tau)\,d\tau}\,ds\}^{\frac{1}{p}}, & t \in [t_0, t_0 + r), \\ u(t) \leq \{a_2(t) + \int_{t_0}^t A_7(s) e^{\int_s^t B_7(\tau)\,d\tau}\,ds\}^{\frac{1}{p}}, & t \in [t_0 + r, T). \end{cases} \tag{3.3}$$

Here

$$\begin{aligned} a_2(t) &= 3^{q_1-1} a^{q_1}(t) e^{-q_1 t}, \qquad b_2(t) = 3^{q_1-1} \left(\frac{\Gamma(1-(1-\beta)p_1)}{p_1^{1-(1-\beta)p_1}}\right)^{\frac{q_1}{p_1}} b^{q_1}(t) e^{(-q_1 + \frac{q_1 q}{p})t}, \\ c_2(t) &= 3^{q_1-1} \left(\frac{\Gamma(1-(1-\beta)p_1)}{p_1^{1-(1-\beta)p_1}}\right)^{\frac{q_1}{p_1}} c^{q_1}(t) e^{(-q_1 + \frac{q_1 l}{p})t}, \\ A_6(t) &= \frac{q}{p} K^{\frac{q-p}{p}} b_2(t) \left(a_2(t) + \int_{t_0}^t c_2(s)\varphi^l(s-r)\,ds\right) + \frac{p-q}{p} K^{\frac{q}{p}}, \\ B_6(t) &= \frac{q}{p} K^{\frac{q-p}{p}} b_2(t), \\ A_7(t) &= \left[\frac{q}{p} K^{\frac{q-p}{p}} a_2(t) + \frac{p-q}{p} K^{\frac{q}{p}}\right] b_2(t) + \left[\frac{l}{p} K^{\frac{l-p}{p}} a_2(t-r) + \frac{p-l}{p} K^{\frac{l}{p}}\right] c_2(t), \\ B_7(t) &= \frac{q}{p} K^{\frac{q-p}{p}} b_2(t) + \frac{l}{p} K^{\frac{l-p}{p}} c_2(t). \end{aligned}$$

Proof (i) Using the Cauchy-Schwarz inequality by (3.1), we get

$$\begin{aligned} u^p(t) &\le a(t) + \int_{t_0}^t (t-s)^{\beta-1} e^s e^{-s} b(s) u^q(s)\,ds + \int_{t_0}^t (t-s)^{\beta-1} e^s e^{-s} c(s) u^l(s-r)\,ds \\ &\le a(t) + \left(\int_{t_0}^t (t-s)^{2\beta-2} e^{2s}\,ds\right)^{\frac{1}{2}} \left(\int_{t_0}^t b^2(s) e^{-2s} u^{2q}(s)\,ds\right)^{\frac{1}{2}} \\ &\quad + \left(\int_{t_0}^t (t-s)^{2\beta-2} e^{2s}\,ds\right)^{\frac{1}{2}} \left(\int_{t_0}^t c^2(s) e^{-2s} u^{2l}(s-r)\,ds\right)^{\frac{1}{2}} \\ &\le a(t) + \left(\frac{2e^{2t}\Gamma(2\beta-1)}{4^\beta}\right)^{\frac{1}{2}} \\ &\quad \cdot \left[\left(\int_{t_0}^t b^2(s) e^{-2s} u^{2q}(s)\,ds\right)^{\frac{1}{2}} + \left(\int_{t_0}^t c^2(s) e^{-2s} u^{2l}(s-r)\,ds\right)^{\frac{1}{2}}\right]. \end{aligned}$$

Using Lemma 2.1 for $n = 3$, $r = 2$, we obtain

$$u^{2p}(t) \le 3a^2(t) + 3\frac{2e^{2t}\Gamma(2\beta-1)}{4^\beta}\left(\int_{t_0}^t b^2(s) e^{-2s} u^{2q}(s)\,ds + \int_{t_0}^t c^2(s) e^{-2s} u^{2l}(s-r)\,ds\right).$$

Let $v(t) = e^{-\frac{2}{p}t} u^2(t)$, then we get

$$\begin{aligned} v^p(t) &\le 3a^2(t) e^{-2t} + \frac{6\Gamma(2\beta-1)}{4^\beta} \\ &\quad \cdot \left(\int_{t_0}^t b^2(s) e^{2s(\frac{q}{p}-1)} v^q(s)\,ds + \int_{t_0}^t c^2(s) e^{2s(\frac{l}{p}-1)} v^l(s-r)\,ds\right). \end{aligned}$$

Using Lemma 2.6, we get (3.2).

(ii) By the hypothesis, we get $\frac{1}{p_1} + \frac{1}{q_1} = 1$, using the Hölder inequality by (3.1), we obtain

$$\begin{aligned} u^p(t) &\le a(t) + \left(\int_{t_0}^t (t-s)^{p_1(\beta-1)} e^{p_1 s}\,ds\right)^{\frac{1}{p_1}} \left(\int_{t_0}^t b^{q_1}(s) e^{-q_1 s} u^{q_1 q}(s)\,ds\right)^{\frac{1}{q_1}} \\ &\quad + \left(\int_{t_0}^t (t-s)^{p_1(\beta-1)} e^{p_1 s}\,ds\right)^{\frac{1}{p_1}} \left(\int_{t_0}^t e^{-q_1 s} c^{q_1}(s) u^{q_1 l}(s-r)\,ds\right)^{\frac{1}{q_1}} \\ &= a(t) + \left(\frac{e^{p_1 t}\Gamma(1-(1-\beta)p_1)}{p_1^{1-(1-\beta)p_1}}\right)^{\frac{1}{p_1}} \\ &\quad \cdot \left[\left(\int_{t_0}^t b^{q_1}(s) e^{-q_1 s} u^{q_1 q}(s)\,ds\right)^{\frac{1}{q_1}} + \left(\int_{t_0}^t c^{q_1}(s) e^{-q_1 s} u^{q_1 l}(s-r)\,ds\right)^{\frac{1}{q_1}}\right]. \end{aligned}$$

Using Lemma 2.1 for $n = 3$, $r = q_1$, we have

$$\begin{aligned} u^{pq_1}(t) &\le 3^{q_1-1} a^{q_1}(t) + 3^{q_1-1}\left(\frac{e^{p_1 t}\Gamma(1-(1-\beta)p_1)}{p_1^{1-(1-\beta)p_1}}\right)^{\frac{q_1}{p_1}} \\ &\quad \cdot \left[\int_{t_0}^t b^{q_1}(s) e^{-q_1 s} u^{q_1 q}(s)\,ds + \int_{t_0}^t c^{q_1}(s) e^{-q_1 s} u^{q_1 l}(s-r)\,ds\right]. \end{aligned}$$

Let $v(t) = e^{-\frac{q_1}{p}t}u^{q_1}(t)$, then we get

$$v^p(t) \leq 3^{q_1-1}a^{q_1}(t)e^{-q_1 t} + 3^{q_1-1}\left(\frac{\Gamma(1-(1-\beta)p_1)}{p_1^{1-(1-\beta)p_1}}\right)^{\frac{q_1}{p_1}} \cdot \left[\int_{t_0}^{t} b^{q_1}(s)e^{(-q_1+\frac{q_1 q}{p})s}v^q(s)\,ds + \int_{t_0}^{t} c^{q_1}(s)e^{(-q_1+\frac{q_1 l}{p})s}v^l(s-r)\,ds\right].$$

By Lemma 2.6 we get (3.3). This completes the proof of Theorem 3.1. □

Remark 3.1 If $p = q = 1$, then Theorem 3.1 becomes Theorem 5 in [5]. If $p = q = l = 1$, then Theorem 3.1 becomes Theorem 2.5 in [3]. If $p = l = 1$, $b(t) = 0$, then Theorem 3.1 becomes Theorem 2.4 in [3].

Theorem 3.2 *Suppose that $u(t)$, $a(t)$, $b(t)$, $c(t)$, $d(t)$, $f(t)$ are nonnegative continuous functions for $t \in R_+$. Let p, m, q be constants with $p \geq m > 0$, $p \geq q > 0$. If*

$$u^p(t) \leq a(t) + b(t)\int_0^t (t-s)^{\beta-1}c(s)u^m(s)\,ds + d(t)\int_0^t \left(t^\alpha - s^\alpha\right)^{\beta-1}s^{\gamma-1}f(s)u^q(s)\,ds, \quad (3.4)$$

then for $[\alpha, \beta, \gamma] \in I$,

$$u(t) \leq \left\{a(t) + \left[\bar{a}_1(t) + e^{\int_0^t \bar{b}_1(t,s)\,ds}\int_0^t e^{-\int_0^\tau \bar{b}_1(\tau,s)\,ds} \cdot \left(\bar{a}_1(s)\bar{b}_1(\tau,\tau) + \int_0^\tau \bar{b}'_{1\tau}(\tau,s)\bar{a}_1(s)\,ds\right)d\tau\right]^{1-\beta}\right\}^{\frac{1}{p}}, \quad (3.5)$$

where

$$\bar{a}_1(t) = 3^{\frac{\beta}{1-\beta}}\left[b(t)e^t M_1^\beta A_1^{1-\beta}(t) + d(t)\left(N_1 t^{\theta_1}\right)^\beta B_1^{1-\beta}(t)\right]^{\frac{1}{1-\beta}},$$

$$\bar{b}_1(t,s) = 3^{\frac{\beta}{1-\beta}}\left[\left(\frac{m}{p}K^{\frac{m-p}{p}}b(t)e^t M_1^\beta\right)^{\frac{1}{1-\beta}}e^{-\frac{1}{1-\beta}s}c^{\frac{1}{1-\beta}}(s) + \left(\frac{q}{p}K^{\frac{q-p}{p}}d(t)\left(N_1 t^{\theta_1}\right)^\beta\right)^{\frac{1}{1-\beta}}f^{\frac{1}{1-\beta}}(s)\right],$$

$$M_1 = \frac{\Gamma(1-(1-\beta)\frac{1}{\beta})}{(\frac{1}{\beta})^{1-(1-\beta)\frac{1}{\beta}}} = \frac{\Gamma(2-\frac{1}{\beta})}{\beta^{\frac{1}{\beta}-2}},$$

$$N_1 = \frac{1}{\alpha}\cdot B\left(\frac{\frac{\gamma-1}{\beta}+1}{\alpha}, \frac{\beta-1}{\beta}+1\right) = \frac{1}{\alpha}\cdot B\left(\frac{\beta+\gamma-1}{\alpha}, \frac{2\beta-1}{\beta}\right),$$

$$\theta_1 = \frac{1}{\beta}\left[\alpha(\beta-1)+\gamma-1\right]+1,$$

$$A_1(t) = \int_0^t e^{-\frac{1}{1-\beta}s}c^{\frac{1}{1-\beta}}(s)\left(\frac{m}{p}K^{\frac{m-p}{p}}a(s) + \frac{p-m}{p}K^{\frac{m}{p}}\right)^{\frac{1}{1-\beta}}ds,$$

$$B_1(t) = \int_0^t f^{\frac{1}{1-\beta}}(s)\left(\frac{q}{p}K^{\frac{q-p}{p}}a(s) + \frac{p-q}{p}K^{\frac{q}{p}}\right)^{\frac{1}{1-\beta}}ds;$$

for $[\alpha,\beta,\gamma]\in II$,

$$u(t)\le\Big\{a(t)+\Big[\bar a_2(t)+e^{\int_0^t\bar b_2(t,s)\,ds}\int_0^t e^{-\int_0^\tau\bar b_2(\tau,s)\,ds}\cdot\Big(\bar a_2(s)\bar b_2(\tau,\tau)+\int_0^\tau\bar b'_{2\tau}(\tau,s)\bar a_2(s)\,ds\Big)\,d\tau\Big]^{\frac{\beta}{1+4\beta}}\Big\}^{\frac{1}{p}},\tag{3.6}$$

where

$$\bar a_2(t)=3^{\frac{1+3\beta}{\beta}}\big[b(t)e^tM_2^{\frac{1+3\beta}{1+4\beta}}A_2^{\frac{\beta}{1+4\beta}}(t)+d(t)\big(N_2t^{\theta_2}\big)^{\frac{1+3\beta}{1+4\beta}}B_2^{\frac{\beta}{1+4\beta}}(t)\big]^{\frac{1+4\beta}{\beta}},$$

$$\bar b_2(t,s)=3^{\frac{1+3\beta}{\beta}}\bigg[\Big(\frac{m}{p}K^{\frac{m-p}{p}}b(t)e^tM_2^{\frac{1+3\beta}{1+4\beta}}\Big)^{\frac{1+4\beta}{\beta}}e^{-\frac{1+4\beta}{\beta}s}c^{\frac{1+4\beta}{\beta}}(s)+\Big(\frac{q}{p}K^{\frac{q-p}{p}}d(t)\big(N_2t^{\theta_2}\big)^{\frac{1+3\beta}{1+4\beta}}\Big)^{\frac{1+4\beta}{\beta}}f^{\frac{1+4\beta}{\beta}}(s)\bigg],$$

$$M_2=\frac{\Gamma(1-(1-\beta)\frac{1+4\beta}{1+3\beta})}{(\frac{1+4\beta}{1+3\beta})^{1-(1-\beta)\frac{1+4\beta}{1+3\beta}}}=\frac{\Gamma(\frac{4\beta^2}{1+3\beta})}{(\frac{1+4\beta}{1+3\beta})^{\frac{4\beta^2}{1+3\beta}}},$$

$$N_2=\frac{1}{\alpha}\cdot B\bigg(\frac{\frac{1+4\beta}{1+3\beta}(\gamma-1)+1}{\alpha},\frac{1+4\beta}{1+3\beta}(\beta-1)+1\bigg)=\frac{1}{\alpha}\cdot B\bigg(\frac{\gamma-\beta+4\beta\gamma}{\alpha(1+3\beta)},\frac{4\beta^2}{1+3\beta}\bigg),$$

$$\theta_2=\frac{1+4\beta}{1+3\beta}\big[\alpha(\beta-1)+\gamma-1\big]+1,$$

$$A_2(t)=\int_0^t e^{-\frac{1+4\beta}{\beta}s}c^{\frac{1+4\beta}{\beta}}(s)\Big(\frac{m}{p}K^{\frac{m-p}{p}}a(s)+\frac{p-m}{p}K^{\frac{m}{p}}\Big)^{\frac{1+4\beta}{\beta}}\,ds,$$

$$B_2(t)=\int_0^t f^{\frac{1+4\beta}{\beta}}(s)\Big(\frac{q}{p}K^{\frac{q-p}{p}}a(s)+\frac{p-q}{p}K^{\frac{q}{p}}\Big)^{\frac{1+4\beta}{\beta}}\,ds.$$

Proof Define

$$v(t)=b(t)\int_0^t(t-s)^{\beta-1}c(s)u^m(s)\,ds+d(t)\int_0^t\big(t^\alpha-s^\alpha\big)^{\beta-1}s^{\gamma-1}f(s)u^q(s)\,ds,\tag{3.7}$$

then

$$u^p(t)\le a(t)+v(t)\quad\text{or}\quad u(t)\le\big\{a(t)+v(t)\big\}^{\frac{1}{p}}.\tag{3.8}$$

By Lemma 2.2, from (3.7), for any $K>0$, we have

$$\begin{aligned}v(t)&\le b(t)\int_0^t(t-s)^{\beta-1}c(s)\Big[\frac{m}{p}K^{\frac{m-p}{p}}\big(a(s)+v(s)\big)+\frac{p-m}{p}K^{\frac{m}{p}}\Big]\,ds\\&\quad+d(t)\int_0^t\big(t^\alpha-s^\alpha\big)^{\beta-1}s^{\gamma-1}f(s)\Big[\frac{q}{p}K^{\frac{q-p}{p}}\big(a(s)+v(s)\big)+\frac{p-q}{p}K^{\frac{q}{p}}\Big]\,ds\\&=A(t)+\frac{m}{p}K^{\frac{m-p}{p}}b(t)\int_0^t(t-s)^{\beta-1}c(s)v(s)\,ds\end{aligned}$$

$$\begin{aligned} &+ \frac{q}{p}K^{\frac{q-p}{p}} d(t) \int_0^t \left(t^\alpha - s^\alpha\right)^{\beta-1} s^{\gamma-1} f(s) v(s)\, ds \\ &= A(t) + \frac{m}{p}K^{\frac{m-p}{p}} b(t) \cdot B(t) + \frac{q}{p}K^{\frac{q-p}{p}} d(t) \cdot C(t). \end{aligned} \tag{3.9}$$

Here

$$\begin{aligned} A(t) &= b(t) \int_0^t (t-s)^{\beta-1} c(s) \left[\frac{m}{p}K^{\frac{m-p}{p}} a(s) + \frac{p-m}{p} K^{\frac{m}{p}} \right] ds \\ &\quad + d(t) \int_0^t \left(t^\alpha - s^\alpha\right)^{\beta-1} f(s) s^{\gamma-1} \left[\frac{q}{p}K^{\frac{q-p}{p}} a(s) + \frac{p-q}{p} K^{\frac{q}{p}} \right] ds, \\ B(t) &= \int_0^t (t-s)^{\beta-1} c(s) v(s)\, ds, \\ C(t) &= \int_0^t \left(t^\alpha - s^\alpha\right)^{\beta-1} s^{\gamma-1} f(s) v(s)\, ds. \end{aligned}$$

Now we try to estimate $A(t)$, $B(t)$, and $C(t)$.

If $[\alpha,\beta,\gamma] \in I$, let $p_1 = \frac{1}{\beta}$, $q_1 = \frac{1}{1-\beta}$; if $[\alpha,\beta,\gamma] \in II$, let $p_2 = \frac{1+4\beta}{1+3\beta}$, $q_2 = \frac{1+4\beta}{\beta}$. Then $\frac{1}{p_i} + \frac{1}{q_i} = 1$ for $i = 1, 2$. Using Hölder's inequality, Lemma 2.3, and Lemma 2.4 we get

$$\begin{aligned} A(t) &\le b(t) \left[\int_0^t (t-s)^{p_i(\beta-1)} e^{p_i s}\, ds \right]^{\frac{1}{p_i}} \\ &\quad \cdot \left[\int_0^t e^{-q_i s} c^{q_i}(s) \left(\frac{m}{p}K^{\frac{m-p}{p}} a(s) + \frac{p-m}{p} K^{\frac{m}{p}} \right)^{q_i} ds \right]^{\frac{1}{q_i}} \\ &\quad + d(t) \left[\int_0^t \left(t^\alpha - s^\alpha\right)^{p_i(\beta-1)} s^{p_i(\gamma-1)}\, ds \right]^{\frac{1}{p_i}} \\ &\quad \cdot \left[\int_0^t f^{q_i}(s) \left(\frac{q}{p}K^{\frac{q-p}{p}} a(s) + \frac{p-q}{p} K^{\frac{q}{p}} \right)^{q_i} ds \right]^{\frac{1}{q_i}} \\ &= b(t) \left(\frac{e^{p_i t} \Gamma(1-(1-\beta)p_i)}{p_i^{1-(1-\beta)p_i}} \right)^{\frac{1}{p_i}} \\ &\quad \cdot \left[\int_0^t e^{-q_i s} c^{q_i}(s) \left(\frac{m}{p}K^{\frac{m-p}{p}} a(s) + \frac{p-m}{p} K^{\frac{m}{p}} \right)^{q_i} ds \right]^{\frac{1}{q_i}} \\ &\quad + d(t) \left[\frac{t^{\theta_i}}{\alpha} B\left(\frac{p_i(\gamma-1)+1}{\alpha}, p_i(\beta-1)+1 \right) \right]^{\frac{1}{p_i}} \\ &\quad \cdot \left[\int_0^t f^{q_i}(s) \left(\frac{q}{p}K^{\frac{q-p}{p}} a(s) + \frac{p-q}{p} K^{\frac{q}{p}} \right)^{q_i} ds \right]^{\frac{1}{q_i}} \\ &= b(t) e^t M_i^{\frac{1}{p_i}} A_i^{\frac{1}{q_i}}(t) + d(t) \left(N_i t^{\theta_i}\right)^{\frac{1}{p_i}} B_i^{\frac{1}{q_i}}(t). \end{aligned} \tag{3.10}$$

Here

$$\begin{aligned} &\theta_i = p_i\left[\alpha(\beta-1) + \gamma - 1\right] + 1, \\ &M_i = \frac{\Gamma(1-(1-\beta)p_i)}{p_i^{1-(1-\beta)p_i}}, \qquad N_i = \frac{1}{\alpha} B\left(\frac{p_i(\gamma-1)+1}{\alpha}, p_i(\beta-1)+1 \right), \end{aligned}$$

$$A_i(t) = \int_0^t e^{-q_i s} c^{q_i}(s)\left(\frac{m}{p}K^{\frac{m-p}{p}} a(s) + \frac{p-m}{p}K^{\frac{m}{p}}\right)^{q_i} ds,$$
$$B_i(t) = \int_0^t f^{q_i}(s)\left(\frac{q}{p}K^{\frac{q-p}{p}} a(s) + \frac{p-q}{p}K^{\frac{q}{p}}\right)^{q_i} ds, \quad i = 1,2.$$

Similarly, we can get

$$\begin{aligned} B(t) &\le \left[\int_0^t (t-s)^{p_i(\beta-1)} e^{p_i s}\, ds\right]^{\frac{1}{p_i}} \left[\int_0^t e^{-q_i s} c^{q_i}(s) v^{q_i}(s)\, ds\right]^{\frac{1}{q_i}} \\ &= e^t M_i^{\frac{1}{p_i}} \left[\int_0^t e^{-q_i s} c^{q_i}(s) v^{q_i}(s)\, ds\right]^{\frac{1}{q_i}} \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} C(t) &\le \left[\int_0^t \left(t^\alpha - s^\alpha\right)^{p_i(\beta-1)} s^{p_i(\gamma-1)}\, ds\right]^{\frac{1}{p_i}} \left[\int_0^t f^{q_i}(s) v^{q_i}(s)\, ds\right]^{\frac{1}{q_i}} \\ &= \left(N_i t^{\theta_i}\right)^{\frac{1}{p_i}} \left[\int_0^t f^{q_i}(s) v^{q_i}(s)\, ds\right]^{\frac{1}{q_i}}. \end{aligned} \tag{3.12}$$

From (3.9), (3.11), and (3.12) we get

$$\begin{aligned} v(t) \le{}& A(t) + \frac{m}{p}K^{\frac{m-p}{p}} b(t) e^t M_i^{\frac{1}{p_i}} \left[\int_0^t e^{-q_i s} c^{q_i}(s) v^{q_i}(s)\, ds\right]^{\frac{1}{q_i}} \\ &+ \frac{q}{p}K^{\frac{q-p}{p}} d(t)\left(N_i t^{\theta_i}\right)^{\frac{1}{p_i}} \left[\int_0^t f^{q_i}(s) v^{q_i}(s)\, ds\right]^{\frac{1}{q_i}}. \end{aligned}$$

Using Lemma 2.1 and (3.10) we can get

$$\begin{aligned} v^{q_i}(t) \le{}& 3^{q_i-1} A^{q_i}(t) + 3^{q_i-1}\left(\frac{m}{p}K^{\frac{m-p}{p}} b(t) e^t M_i^{\frac{1}{p_i}}\right)^{q_i} \int_0^t e^{-q_i s} c^{q_i}(s) v^{q_i}(s)\, ds \\ &+ 3^{q_i-1}\left(\frac{q}{p}K^{\frac{q-p}{p}} d(t)\left(N_i t^{\theta_i}\right)^{\frac{1}{p_i}}\right)^{q_i} \int_0^t f^{q_i}(s) v^{q_i}(s)\, ds \\ ={}& 3^{q_i-1} A^{q_i}(t) + e_i(t)\int_0^t e^{-q_i s} c^{q_i}(s) v^{q_i}(s)\, ds + g_i(t)\int_0^t f^{q_i}(s) v^{q_i}(s)\, ds \\ \le{}& 3^{q_i-1}\left[b(t) e^t M_i^{\frac{1}{p_i}} A_i^{\frac{1}{q_i}}(t) + d(t)\left(N_i t^{\theta_i}\right)^{\frac{1}{p_i}} B_i^{\frac{1}{q_i}}(t)\right]^{q_i} \\ &+ \int_0^t \left[e_i(t) e^{-q_i s} c^{q_i}(s) + g_i(t) f^{q_i}(s)\right] v^{q_i}(s)\, ds. \end{aligned} \tag{3.13}$$

Here

$$e_i(t) = 3^{q_i-1}\left(\frac{m}{p}K^{\frac{m-p}{p}} b(t) e^t M_i^{\frac{1}{p_i}}\right)^{q_i}, \qquad g_i(t) = 3^{q_i-1}\left(\frac{q}{p}K^{\frac{q-p}{p}} d(t)\left(N_i t^{\theta_i}\right)^{\frac{1}{p_i}}\right)^{q_i}.$$

By Lemma 2.7 we have

$$v^{q_i}(t) \le \bar{a}(t) + e^{\int_0^t \bar{b}(t,s)\, ds} \int_0^t e^{-\int_0^\tau \bar{b}(\tau,s)\, ds}\left[\bar{a}(s)\bar{b}(\tau,\tau) + \int_0^\tau \bar{b}'_\tau(\tau,s)\bar{a}(s)\, ds\right] d\tau. \tag{3.14}$$

Here

$$\bar{a}_i(t) = 3^{q_i-1}\big[b(t)e^t M_i^{\frac{1}{p_i}} A_i^{\frac{1}{q_i}}(t) + d(t)\big(N_i t^{\theta_i}\big)^{\frac{1}{p_i}} B_i^{\frac{1}{q_i}}(t)\big]^{q_i},$$
$$\bar{b}_i(t,s) = e_i(t)e^{-q_i s}c^{q_i}(s) + g_i(t)f^{q_i}(s).$$

Combining (3.8) and (3.14) we get

$$u(t) \le \left\{a(t) + \left[\bar{a}_i(t) + e^{\int_0^t \bar{b}_i(t,s)\,ds}\int_0^t e^{-\int_0^\tau \bar{b}_i(\tau,s)\,ds} \cdot \left(\bar{a}_i(s)\bar{b}_i(\tau,\tau) + \int_0^\tau \bar{b}'_{i\tau}(\tau,s)\bar{a}_i(s)\,ds\right)d\tau\right]^{\frac{1}{q_i}}\right\}^{\frac{1}{p}}. \tag{3.15}$$

Considering the two situations for $i = 1, 2$ and using the parameters α, β, and γ to p_i, q_i and θ_i in (3.15), we can get (3.4) and (3.5), respectively. This completes the proof of Theorem 3.2. □

Corollary 3.1 *Suppose that the conditions of Theorem* 3.2 *hold, if* $b(t) = 0$, *then*

$$u^p(t) \le a(t) + d(t)\int_0^t \big(t^\alpha - s^\alpha\big)^{\beta-1}s^{\gamma-1}f(s)u^q(s)\,ds$$

and we deduce the following results.

For $[\alpha,\beta,\gamma] \in I$,

$$u(t) \le \left\{a(t) + \left[\bar{a}_3(t) + e^{\int_0^t \bar{b}_3(t,s)\,ds}\int_0^t e^{\int_0^\tau \bar{b}_3(\tau,s)\,ds} \cdot \left(\bar{a}_3(s)\bar{b}_3(\tau,\tau) + \int_0^\tau \bar{b}'_{3\tau}(\tau,s)\bar{a}_3(s)\,ds\right)d\tau\right]^{1-\beta}\right\}^{\frac{1}{p}}, \tag{3.16}$$

where

$$\bar{a}_3(t) = \big(3^\beta d(t)N_1^\beta t^{(\alpha+1)(\beta-1)+\gamma}\big)^{\frac{1}{1-\beta}}B_1(t),$$
$$\bar{b}_3(t,s) = 3^{\frac{\beta}{1-\beta}}\left(\frac{q}{p}K^{\frac{q-p}{p}}d(t)N_1^\beta t^{(\alpha+1)(\beta-1)+\gamma}f(s)\right)^{\frac{1}{1-\beta}},$$

N_1, $B_1(t)$ *are the same as in Theorem* 3.2.

For $[\alpha,\beta,\gamma] \in II$,

$$u(t) \le \left\{a(t) + \left[\bar{a}_4(t) + e^{\int_0^t \bar{b}_4(t,s)\,ds}\int_0^t e^{\int_0^\tau \bar{b}_4(\tau,s)\,ds} \cdot \left(\bar{a}_4(s)\bar{b}_4(\tau,\tau) + \int_0^\tau \bar{b}'_{4\tau}(\tau,s)\bar{a}_4(s)\,ds\right)d\tau\right]^{\frac{\beta}{1+4\beta}}\right\}^{\frac{1}{p}}, \tag{3.17}$$

where

$$\bar{a}_4(t) = 3^{\frac{1+3\beta}{\beta}}\left[d(t)N_2^{\frac{1+3\beta}{1+4\beta}}t^{\frac{[\alpha(\beta-1)+\gamma](1+3\beta)+\beta}{1+3\beta}}\right]^{\frac{1+4\beta}{\beta}}B_2(t),$$

$$\bar{b}_4(t,s) = 3^{\frac{1+3\beta}{\beta}} \left(\frac{q}{p} K^{\frac{q-p}{p}} d(t) N_2^{\frac{1+3\beta}{1+4\beta}} t^{\frac{[\alpha(\beta-1)+\gamma](1+3\beta)+\beta}{1+3\beta}} f(s) \right)^{\frac{1+4\beta}{\beta}},$$

N_2, $B_2(t)$ are the same as in Theorem 3.2.

Corollary 3.2 *Suppose that the conditions of Theorem* 3.2 *hold*, $\alpha = \gamma = 1$. *If*

$$u^p(t) \le a(t) + b(t) \int_0^t (t-s)^{\beta-1} c(s) u^m(s)\,ds + d(t) \int_0^t (t-s)^{\beta-1} f(s) u^q(s)\,ds,$$

then for $[\alpha, \beta, \gamma] \in I$,

$$u(t) \le \left\{ a(t) + \left[\bar{a}_5(t) + e^{\int_0^t \bar{b}_1(t,s)\,ds} \int_0^t e^{\int_0^\tau \bar{b}_1(\tau,s)\,ds} \cdot \left(\bar{a}_5(s) \bar{b}_1(\tau,\tau) + \int_0^\tau \bar{b}'_{5\tau}(\tau,s) \bar{a}_5(s)\,ds \right) d\tau \right]^{1-\beta} \right\}^{\frac{1}{p}}, \tag{3.18}$$

where

$$\bar{a}_5(t) = 3^{\frac{\beta}{1-\beta}} \left[b(t) e^t M_1^\beta A_1^{1-\beta}(t) + d(t) \left(\bar{N}_1 t^{\bar{\theta}_1} \right)^\beta B_1^{1-\beta}(t) \right]^{\frac{1}{1-\beta}},$$

$$\bar{b}_5(t,s) = 3^{\frac{\beta}{1-\beta}} \left[\left(\frac{m}{p} K^{\frac{m-p}{p}} b(t) M_1^\beta \right)^{\frac{1}{1-\beta}} e^{\frac{1}{\beta-1}s} c^{\frac{1}{\beta-1}}(s) + \left(\frac{q}{p} K^{\frac{q-p}{p}} d(t) \left(N_1 t^{\bar{\theta}_1} \right)^\beta \right)^{\frac{1}{1-\beta}} f^{\frac{1}{1-\beta}}(s) \right],$$

$$\bar{N}_1 = B\left(1, \frac{2\beta-1}{\beta}\right), \qquad \bar{\theta}_1 = \frac{1}{\beta}(\beta-1) + 1 = \frac{2\beta-1}{\beta},$$

M_1, $A_1(t)$, $B_1(t)$ are the same as in Theorem 3.2.

For $[\alpha, \beta, \gamma] \in II$,

$$u(t) \le \left\{ a(t) + \left[\bar{a}_6(t) + e^{\int_0^t \bar{b}_6(t,s)\,ds} \int_0^t e^{\int_0^\tau \bar{b}_6(\tau,s)\,ds} \cdot \left(\bar{a}_6(s) \bar{b}_6(\tau,\tau) + \int_0^\tau \bar{b}'_{6\tau}(\tau,s) \bar{a}_6(s)\,ds \right) d\tau \right]^{\frac{\beta}{1+4\beta}} \right\}^{\frac{1}{p}}, \tag{3.19}$$

where

$$\bar{a}_6(t) = 3^{\frac{1+3\beta}{\beta}} \left[b(t) e^t M_2^{\frac{1+3\beta}{1+4\beta}} A_2^{\frac{\beta}{1+4\beta}}(t) + d(t) \left(\bar{N}_2 t^{\bar{\theta}_2} \right)^{\frac{1+3\beta}{1+4\beta}} B_2^{\frac{\beta}{1+4\beta}}(t) \right]^{\frac{1+4\beta}{\beta}},$$

$$\bar{b}_6(t,s) = 3^{\frac{1+3\beta}{\beta}} \left[\left(\frac{m}{p} K^{\frac{m-p}{p}} b(t) e^t M_2^{\frac{1+3\beta}{1+4\beta}} \right)^{\frac{1+4\beta}{\beta}} e^{-\frac{1+4\beta}{\beta}s} c^{\frac{1+4\beta}{\beta}}(s) + \left(\frac{q}{p} K^{\frac{q-p}{p}} d(t) \left(\bar{N}_2 t^{\bar{\theta}_2} \right)^{\frac{1+3\beta}{1+4\beta}} \right)^{\frac{1+4\beta}{\beta}} f^{\frac{1+4\beta}{\beta}}(s) \right],$$

$$\bar{N}_2 = B\left(1, \frac{4\beta^2}{1+3\beta}\right),$$

$$\bar{\theta}_2 = \frac{1+4\beta}{1+3\beta}(\beta-1) + 1 = \frac{4\beta^2}{1+3\beta},$$

M_2, $A_2(t)$, $B_2(t)$ are the same as in Theorem 3.2.

Remark 3.2 If $b(t) = 0$, then Theorem 3.2 becomes Theorem 2.6 in [2].

Remark 3.3 If $b(t) = d(t) = 1$, $p = m = \alpha = \gamma = 1$, then Theorem 3.2 becomes Theorem 4 in [4]. Due to the difference in methods, our results are not the same as in [2]. The results in Theorem 3.2 have the simple exponential function of Theorem 4 in [2].

4 Applications

(1) Consider the delay fractional differential equations with initial condition

$$\begin{cases} {}_{t_0}^{C}D_t^{\beta} x^p(t) = f(t, x(t), x(t-r)), \quad t \in I = [t_0, T), \\ {}_{t_0}^{C}D_t^{\beta} x(t_0) = b_k, \quad k = 0, 1, 2, \ldots, m-1, \\ x(t) = \varphi(t), \quad t \in I = [t_0 - r, t_0). \end{cases} \tag{4.1}$$

Here ${}_{t_0}^{C}D_t^{\beta}$ is the Caputo fractional derivative of order β $(m-1 \le \beta < m)$, $r \in R_+$ is a real constant, $f(t, y, z) \in C(I \times R^2, R)$, φ is a given continuously differentiable function on $[t_0 - r, t_0]$ up to order m $(m = -[-\beta])$. In this case, we denote $\varphi^k(t_0) = b_k$, $k = 0, 1, 2, \ldots, m-1$.

In [17], the initial value problem (4.1) is equivalent to the Volterra fractional integral equation

$$\begin{cases} x^p(t) = \sum_{k=0}^{m-1} \frac{b_k}{k!}(t-t_0)^k + \frac{1}{\Gamma(\beta)} \int_{t_0}^t (t-s)^{\beta-1} f(s, x(s), x(s-r))\, ds, \quad t \in I, \\ x(t) = \varphi(t), \quad t \in I = [t_0 - r, t_0). \end{cases} \tag{4.2}$$

The next theorem deals with the estimates of the solution of (4.2).

Theorem 4.1 *Suppose that* $|f(t, y, z)| \le b(t)|y|^q + c(t)|z|^l$, $t \in I$, $m, p, q, l \in R_+$, $p \ge q$, $p \ge l$, $b(t), c(t) \in C(I, R_+)$. *If* $x(t)$ *is the solution of initial problem* (4.2), *then the following estimations hold.*

(i) *Suppose that* $\beta > \frac{1}{2}$, *then*

$$\begin{cases} |x(t)| \le \{g_1(t) + \int_{t_0}^t k_1(s)\varphi^l(s-r)\, ds + \int_{t_0}^t F_1(s) e^{\int_s^t G_1(\tau)\, d\tau}\, ds\}^{\frac{1}{p}}, \quad t \in [t_0, t_0 + r), \\ |x(t)| \le \{g_1(t) + \int_{t_0}^t F_2(s) e^{\int_s^t G_2(\tau)\, d\tau}\, ds\}^{\frac{1}{p}} \quad t \in [t_0 + r, T). \end{cases} \tag{4.3}$$

Here

$$g_1(t) = 3e^{-2t}\left[\sum_{k=0}^{m-1} \frac{|b_k|}{k!}(t-t_0)^k\right]^2, \qquad h_1(t) = \frac{6\Gamma(2\beta-1)}{4^{\beta}} e^{2t(\frac{q}{p}-1)}\left[\frac{b(t)}{\Gamma(\beta)}\right]^2,$$

$$k_1(t) = \frac{6\Gamma(2\beta-1)}{4^{\beta}}\left[\frac{c(t)}{\Gamma(\beta)}\right]^2 e^{2t(\frac{l}{p}-1)},$$

$$F_1(t) = \frac{q}{p}K^{\frac{q-p}{p}} h_1(t)\left[g_1(t) + \int_{t_0}^t k_1(s)\varphi^l(s-r)\, ds\right] + \frac{p-q}{p}K^{\frac{q}{p}}, \qquad G_1(t) = \frac{q}{p}K^{\frac{q-p}{p}} h_1(t),$$

$$F_2(t) = \left[\frac{q}{p}K^{\frac{q-p}{p}} g_1(t) + \frac{p-q}{p}K^{\frac{q}{p}}\right] h_1(t) + \left[\frac{l}{p}K^{\frac{l-p}{p}} g_1(t-r) + \frac{p-l}{p}K^{\frac{l}{p}}\right] k_1(t),$$

$$G_2(t) = \frac{q}{p}K^{\frac{q-p}{p}} h_1(t) + \frac{l}{p}K^{\frac{l-p}{p}} k_1(t).$$

(ii) *If* $0 < \beta \leq \frac{1}{2}$, *then*

$$\begin{cases} |x(t)| \leq \{g_2(t) + \int_{t_0}^t k_2(s)\varphi^l(s-r)\,ds + \int_{t_0}^t F_3(s)e^{\int_s^t G_3(\tau)\,d\tau}\,ds\}^{\frac{1}{p}}, & t \in [t_0, t_0 + r), \\ |x(t)| \leq \{g_2(t) + \int_{t_0}^t F_4(s)e^{\int_s^t G_4(\tau)\,d\tau}\,ds\}^{\frac{1}{p}}, & t \in [t_0 + r, T). \end{cases} \tag{4.4}$$

Here

$$\begin{aligned} g_2(t) &= 3^{\frac{1}{\beta}} e^{-\frac{1+\beta}{\beta}t}\left[\sum_{k=0}^{m-1} \frac{|b_k|}{k!}(t-t_0)^k\right]^{\frac{1+\beta}{\beta}}, \\ h_2(t) &= 3^{\frac{1}{\beta}}\left(\frac{\Gamma(\beta^2)}{(1+\beta)^{\beta^2}}\right)^{\frac{1}{\beta}} e^{[-\frac{1+\beta}{\beta}+\frac{q}{p}(1+\beta)]t}\left[\frac{b(t)}{\Gamma(\beta)}\right]^{\frac{1+\beta}{\beta}}, \\ k_2(t) &= 3^{\frac{1}{\beta}}\left(\frac{\Gamma(\beta^2)}{(1+\beta)^{\beta^2}}\right)^{\frac{1}{\beta}} e^{[-\frac{1+\beta}{\beta}+\frac{q}{p}(1+\beta)]t}\left[\frac{c(t)}{\Gamma(\beta)}\right]^{\frac{1+\beta}{\beta}}, \\ F_3(t) &= \frac{q}{p}K^{\frac{q-p}{p}}h_2(t)\left[g_2(t) + \int_{t_0}^t k_2(s)\varphi^l(s-r)\,ds\right] + \frac{p-q}{p}K^{\frac{q}{p}}, \qquad G_3(t) = \frac{q}{p}K^{\frac{q-p}{p}}h_2(t), \\ F_4(t) &= \left[\frac{q}{p}K^{\frac{q-p}{p}}g_2(t) + \frac{p-q}{p}K^{\frac{q}{p}}\right]h_2(t) + \left[\frac{l}{p}K^{\frac{l-p}{p}}g_2(t-r) + \frac{p-l}{p}K^{\frac{l}{p}}\right]k_2(t), \\ G_4(t) &= \frac{q}{p}K^{\frac{q-p}{p}}h_2(t) + \frac{l}{p}K^{\frac{l-p}{p}}k_2(t). \end{aligned}$$

Proof By (4.2), we derive that

$$\begin{cases} |x(t)|^p \leq \sum_{k=0}^{m-1}\frac{|b_k|}{k!}(t-t_0)^k + \frac{1}{\Gamma(\beta)}\int_{t_0}^t (t-s)^{\beta-1}b(s)|x(s)|^q\,ds \\ \qquad + \frac{1}{\Gamma(\beta)}\int_{t_0}^t (t-s)^{\beta-1}c(s)|x(s-r)|^l\,ds, \quad t \in I, \\ |x(t)| = |\varphi(t)|, \quad t \in [t_0 - r, t_0). \end{cases} \tag{4.5}$$

Using Theorem 3.1, we get the desired conclusion. This proves the results (4.3) and (4.4). □

(2) In this section, based on the definition of Riemann-Liouville (R-L) and Erdélyi-Kober (E-K) fractional integral, we will study the boundedness of a certain FDE with R-L fractional operator and E-K fractional operator. The definitions of two fractional operators are given below.

Definition 4.1 ([17]) The R-L fractional integral and fractional derivative of order α of the function $f(x) \in C(R_+, R)$ are given by

$$\begin{aligned} I^\alpha f(x) &= \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)\,dt, \quad \alpha > 0, \\ D^\alpha f(x) &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x (x-t)^{-\alpha}f(t)\,dt, \quad 0 < \alpha < 1, \end{aligned}$$

provided that the right side is point-wise defined on R_+.

Definition 4.2 ([18, 19]) The E-K fractional integral of continuous function $f(x) \in C(R_+, R)$ is defined by

$$I_\beta^{\gamma,\delta} f(x) = \frac{x^{-\beta(\gamma+\delta)}}{\Gamma(\delta)} \int_0^x \bigl(x^\beta - t^\beta\bigr)^{\delta-1} t^{\beta\gamma} f(t)\, d\bigl(t^\beta\bigr), \quad \delta, \gamma, \beta \in R_+,$$

provided that the right side is point-wise defined on R_+.

Consider the following Volterra type integral equations:

$$u^p(t) - \frac{b(t)}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} c(s) u^m(s)\, ds - \frac{d(t)}{\Gamma(\beta)} \int_0^t \bigl(t^\alpha - s^\alpha\bigr)^{\beta-1} s^{\gamma-1} f(s) u^q(s)\, ds = a(t). \quad (4.6)$$

Theorem 4.2 *Let $u(t), a(t), b(t), c(t), d(t), f(t) \in C[0,\infty)$, p, m, q, α, β, γ be the same as in Theorem* 3.2. *Then, for any $K > 0$, every solution of equation* (4.6) *has the bounds and the same modality as* (3.5), (3.6), *in which we have $|u(t)|$, $|a(t)|$, $\frac{|b(t)|}{\Gamma(\beta)}$, $|c(t)|$, $\frac{|d(t)|}{\Gamma(\beta)}$, $|f(t)|$ instead of $u(t)$, $a(t)$, $b(t)$, $c(t)$, $d(t)$, and $f(t)$.*

Proof From (4.6) we have

$$\begin{aligned} \bigl|u(t)\bigr|^p \le{}& \bigl|a(t)\bigr| + \frac{|b(t)|}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \bigl|c(s)\bigr| \bigl|u(s)\bigr|^m\, ds \\ &- \frac{|d(t)|}{\Gamma(\beta)} \int_0^t \bigl(t^\alpha - s^\alpha\bigr)^{\beta-1} s^{\gamma-1} \bigl|f(s)\bigr| \bigl|u(s)\bigr|^q\, ds. \end{aligned}$$

Then by the difference in the selection of $a(t)$, $b(t)$, $c(t)$, $d(t)$, $f(t)$, p, m, q, α, β, γ, we get the desired results. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
RX carried out the generalized weakly singular integral inequalities and completed the corresponding proof.
FM participated in Section 4 - Applications. All authors read and approved the final manuscript.

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