

## RESEARCH ARTICLE

### A $K$ -fold Averaging Cross-validation Procedure

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Cross-validation type of methods have been widely used to facilitate model estimation and variable selection. In this work, we suggest a new  $K$ -fold cross validation procedure to select a candidate ‘optimal’ model from each hold-out fold and average the  $K$  candidate ‘optimal’ models to obtain the ultimate model. Due to the averaging effect, the variance of the proposed estimates can be significantly reduced. This new procedure results in more stable and efficient parameter estimation than the classical  $K$ -fold cross validation procedure. In addition, we show the asymptotic equivalence between the proposed and classical cross validation procedures in the linear regression setting. We also demonstrate the broad applicability of the proposed procedure via two examples of parameter sparsity regularization and quantile smoothing splines modeling. We illustrate the promise of the proposed method through simulations and a real data example.

*Key words:* Cross-validation; Model Averaging; Model Selection

#### 1. Introduction

Cross-validation ( $CV$ ) has been widely used for model selection in regression problems. Its theoretical properties and empirical performance have been extensively discussed in the literature including the pioneering work of Stone (1974, 1977) and Geisser (1975). Burman (1989) studied the properties of leave-one-out  $CV$  and  $K$ -fold  $CV$  procedures. Shao (1993) and Rao and Wu (2005) provided theoretical insights and asymptotic theories of model selection with a fixed number of variables for linear models. Wong (1983) examined consistency of  $CV$  in kernel nonparametric regression. Zhang (1993) showed superiority of leave- $K$ -out  $CV$  over leave-one-out  $CV$  in linear regression. Kohavi (1995)

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assessed the performance in terms of model estimation and variable selection at various values of  $K_1$  and  $K_2$  respectively in  $K_1$ -fold *CV* and leave- $K_2$ -out *CV*. For robust model selection, Ronchetti et al. (1997) suggested a robust loss function to measure the prediction error. Substantial practical and theoretical results of traditional and newer *CV* approaches were summarized in a review paper by Arlot and Celisse (2010).

In a typical  $K$ -fold *CV* procedure for a linear model, the data set is randomly and evenly split into  $K$  parts (if possible). A candidate model is built based on  $K - 1$  parts of the data set, called a training set. Prediction accuracy of this candidate model is then evaluated on a test set containing the data in the hold-out part. By respectively using each of the  $K$  parts as the test set and repeating the model building and evaluation procedure, we choose the model with the smallest cross-validation score (typically, the mean squared prediction error *MSPE*) as the ‘optimal’ model. Given  $p$  independent variables, there are in total of  $2^p - 1$  possible models. In the  $K$ -fold *CV* procedure, each model is in fact evaluated  $K$  times. Therefore, a single ‘optimal’ model is selected via  $K(2^p - 1)$  times of model evaluation.

In this work, we propose a new cross-validation procedure with the core idea of first choosing  $K$  candidate ‘optimal’ models to build the ultimate model. On each training set, the  $2^p - 1$  models are fitted and the candidate ‘optimal’ model is selected with the smallest cross-validation score obtained from the corresponding test set. This procedure is repeated on  $K$  training and test sets to obtain  $K$  candidate ‘optimal’ models. At last, the ultimate model is obtained with its parameter estimates as the average values across  $K$  candidate ‘optimal’ models. Therefore, we call the proposed method  $K$ -fold averaging cross-validation (*ACV*). Due to the averaging effect, efficiency of the final parameter estimates obtained by *ACV* improves over that of the traditional  $K$ -fold *CV*. We note that parameter estimates of *CV* and *ACV* are identical when all the candidate ‘optimal’ models from *ACV* are identical to the model selected by the traditional *CV*.

The proposed *ACV* procedure is also applicable to high dimensional data to determine the amount of penalty imposed by a regularization method, e.g., LASSO (Tibshirani 1996). In fact, the *ACV* procedure has broad applications as *CV* to problems wherein regularization parameters associated with a penalization method need to be determined. This will be demonstrated via an application to smoothing spline based nonparametric modeling (Nychka et al. 1995).

The paper is organized as follows. We describe the explicit expression of the *ACV* estimator and its asymptotic property in linear regression models in Section 2. We discuss the connection and distinction between *ACV* and *CV* in the applications of LASSO and smoothing spline based modeling in Section 3. We illustrate the proposed method through simulation studies in Section 4 and through application to a real data set in Section 5. And we provide some concluding remarks in Section 6.

## 2. $K$ -fold averaging cross-validation in a linear model

### 2.1. Basic setting

Consider a linear model

$$\mathbf{Y} = \mathbf{X}\beta + \epsilon,$$

where  $\mathbf{Y} = (y_1, \dots, y_n)'$  is the response vector,  $\mathbf{X} = \{x_{ij}\}$  is an  $n \times p$  design matrix for the full model, and  $\epsilon = (\epsilon_1, \dots, \epsilon_n)'$  is a vector of *iid* random variables with mean 0 and variance  $\sigma^2$ . We assume that  $p$  is fixed. All the observations are divided into  $K$  parts such that every part is mutually exclusive. Throughout this paper, we assume that  $K$  is finite and fixed. In this paper, we consider deterministic predictors, but when  $\mathbf{X}$  is random, the results are valid almost surely if the two assumptions in Section 2.3 and the assumption on  $\epsilon$  are satisfied almost surely for given  $\mathbf{X}$ .

We use  $n_k$  to denote the sample size in the  $k^{\text{th}}$  fold, where  $\sum_{k=1}^K n_k = n$ . We define  $M_k$  as the selected model based on the  $k^{\text{th}}$  hold-out fold (test set), and let its corresponding design matrix be  $\mathbf{X}_k$  with all observations. We further assume that  $\mathbf{X}_k$  is full rank for any  $k$ .

When the  $k^{\text{th}}$  fold is a test set, a design matrix in the  $k^{\text{th}}$  fold is defined as  $\mathbf{X}_{n_k}$ , and the design matrix of the corresponding training set is denoted as  $\mathbf{X}_{-n_k}$ .  $\mathbf{Y}_k$  and  $\mathbf{Y}_{-k}$  are the response observations corresponding to  $\mathbf{X}_{n_k}$  and  $\mathbf{X}_{-n_k}$ , respectively.

The choice of a candidate 'optimal' model is made based on the mean squared prediction error in the hold-out fold. The  $M_k$  can be represented as a subset of  $\{1, \dots, p\}$ . Let  $|M_k|$  be the number of elements (or, equivalently the number of independent variables) in  $M_k$ . Further, let  $\mathbf{X}_{M_k, -n_k}$  be the design matrix composed of variables of  $M_k$  in the training set which has a size of  $n - n_k$  by  $|M_k|$ . And we denote  $\tilde{\beta}_k$  as an estimator of  $\beta$  with  $M_k$ . For example, the  $\tilde{\beta}_k$  obtained from the least squares method can be expressed as  $(\mathbf{X}'_k \mathbf{X}_k)^{-1} \mathbf{X}'_k \mathbf{Y}$ , which is a length- $|M_k|$  vector.

Since the  $\tilde{\beta}_k$ s are averaged over  $k$  to obtain the ultimate parameter estimates, we need to ensure that each of them is of length  $p$ . For this purpose, we introduce the transformation matrix  $\mathbf{T}_k$ , which is of size  $p \times |M_k|$ . Each column vector of  $\mathbf{T}_k$  is composed of a 1 and  $(p - 1)$  0s. For example, if  $\mathbf{x}_2$  and  $\mathbf{x}_4$  are selected and  $p = 6$ , then  $M_k = \{2, 4\}$ , and  $\mathbf{T}_k$  contains two column vectors of  $(0, 1, 0, 0, 0, 0)'$  and  $(0, 0, 0, 1, 0, 0)'$ , where the position of 1 in the  $i^{\text{th}}$  column is specified in the  $i^{\text{th}}$  element of  $M_k$ , and the variables not contained in  $M_k$  are indexed as 0. As a result, the size of  $\tilde{\beta}_k$  is changed to  $p$  via multiplying by  $\mathbf{T}_k$ . Going back to the example,  $\tilde{\beta}_k = (1, 2)'$  yields  $\mathbf{T}_k \tilde{\beta}_k = (0, 1, 0, 2, 0, 0)'$ . And  $\mathbf{T}_k$  also indicates the selected model with  $\mathbf{X} \mathbf{T}_k = \mathbf{X}_k$ .

To measure the discrepancy between the observations and the predicted values, we use a squared error loss. The *MSPE* evaluated with the data  $\mathbf{Y}_k$  can be written as

$$MSPE_k = \frac{1}{n_k} (\mathbf{Y}_k - \mathbf{X}_{n_k} \hat{\beta}_{-n_k})' (\mathbf{Y}_k - \mathbf{X}_{n_k} \hat{\beta}_{-n_k}), \quad (1)$$

where the least squares estimate  $\hat{\beta}_{-n_k} = \mathbf{T}_k (\mathbf{X}'_{M_k, -n_k} \mathbf{X}_{M_k, -n_k})^{-1} \mathbf{X}'_{M_k, -n_k} \mathbf{Y}_{-k}$ . Note that  $\hat{\beta}_{-n_k}$  depends on the variables contained in  $M_k$ . Because  $\hat{\beta}_{-n_k}$  is not our final estimator, convergence of (1) to 0 does not necessarily deduce consistency of the estimator defined in (2); whereas it does in the method of leave- $n_k$ -out cross-validation described in Zhang (1993) and Shao (1993). Rather, we choose  $M_k$  or equivalently  $\mathbf{X}_k$  by minimizing (1), and use it to construct the ultimate estimator. The explicit form of the proposed parameter estimator based on the least squares method is provided below.

## 2.2. A new estimator

In the traditional  $K$ -fold *CV*, a selected model is what produces smallest prediction error where the number of predicted values is the same as sample size. That is,  $K$ -fold *CV*

selects a model which minimizes  $\sum_{k=1}^K MSPE_k/K$ .

The fundamental distinction of our method is to choose a candidate ‘optimal’ model  $\mathbf{X}_k$  based on the hold-out fold, and then iterate this procedure for  $k = 1, \dots, K$ . Note that the model selection at each iteration provides an opportunity to choose different (up to  $K$ ) plausible models, which is not the case in  $K$ -fold  $CV$ . At last, the  $K$  candidate ‘optimal’ models are averaged to yield an ultimate model.

The regression coefficient estimator of the model optimally chosen by the  $k^{th}$  fold is

$$\hat{\beta}_k = \mathbf{T}_k(\mathbf{X}'_k \mathbf{X}_k)^{-1} \mathbf{X}'_k \mathbf{Y},$$

where  $\mathbf{X}_k$  is an  $n \times |M_k|$  design matrix containing all the  $n$  values of the explanatory variables in  $M_k$  selected by the  $k^{th}$  fold according to (1). Note again that  $\mathbf{T}_k$  is the transformation matrix to ensure the length of  $\hat{\beta}_k$  to be  $p$ . Then, the ultimate estimator of the regression coefficients in the  $K$ -fold  $ACV$  procedure is defined as

$$\hat{\beta}_{ACV} = \frac{1}{K} \sum_{k=1}^K \hat{\beta}_k = \frac{1}{K} \sum_{k=1}^K \mathbf{T}_k(\mathbf{X}'_k \mathbf{X}_k)^{-1} \mathbf{X}'_k \mathbf{Y}, \quad (2)$$

Accordingly, the fitted value of  $\mathbf{Y}$  by  $ACV$  is

$$\hat{\mathbf{Y}}_{ACV} = \mathbf{X} \hat{\beta}_{ACV} = \frac{1}{K} \sum_{k=1}^K \mathbf{X}_k(\mathbf{X}'_k \mathbf{X}_k)^{-1} \mathbf{X}'_k \mathbf{Y} = \mathbf{H}_{ACV} \mathbf{Y}.$$

Let  $\mathbf{P}_k = \mathbf{X}_k(\mathbf{X}'_k \mathbf{X}_k)^{-1} \mathbf{X}'_k$ , then  $\mathbf{H}_{ACV} = \sum_{k=1}^K \mathbf{P}_k/K$ . Since  $\mathbf{X}$  is deterministic predictors, we have,

$$\begin{aligned} E(\mathbf{X} \hat{\beta}_{ACV}) &= \mathbf{H}_{ACV} E(\mathbf{Y}) = \mathbf{H}_{ACV} \mathbf{X} \beta, \\ Var(\mathbf{X} \hat{\beta}_{ACV}) &= Var(\mathbf{H}_{ACV} \mathbf{Y}) = \mathbf{H}'_{ACV} \mathbf{H}_{ACV} \sigma^2. \end{aligned} \quad (3)$$

Although each  $\mathbf{P}_k$  is a projection matrix,  $\mathbf{H}_{ACV}$  is not a projection matrix in general. This implies that  $\hat{\beta}_{ACV}$  is likely a biased estimator (Hoaglin and Welsch 1978). However, efficiency of the estimator can be substantially gained from averaging. To account for variability induced by randomly splitting the data, we repeat the splitting step several times in both  $CV$  and  $ACV$  procedures. Alternatively, we can use a balanced split. Note that there are  $n!/(n_1!n_2! \cdots n_K!)$  different ways of splitting data. Then, all observations have equal probability of being included in the training set and in the test set, thus there is no random variation arises from the splitting. In practice, the random variation from the split is ignorable, and only a few times of splitting observed to be enough in the simulation studies.

**Connection with Bagging:** The bootstrap aggregating approach (*Bagging*) suggested by Breiman (1996) is similar to our proposed method in terms of the repeated parameter estimation and variable selection steps. The *Bagging* procedure first generates a sample of size  $n$ . Based on the generated sample, it performs model selection according to a criterion such as  $K$ -fold  $CV$  and obtains the least squares estimator,  $\hat{\beta}_1^{Bagging}$ . After repeating this process  $B$  times,  $(\hat{\beta}_1^{Bagging}, \dots, \hat{\beta}_B^{Bagging})$  are obtained and averaged to form a final estimate. The distinction is that Bagging procedure requires generating bootstrap samples from the original data set, while the proposed  $ACV$  always use original data

set and average only ‘optimal’ models obtained from each hold-out fold to form a final model.

**Connection with BMA:** The *ACV* method also shares some similarity with Bayesian model averaging (*BMA*) (Raftery et al. 1997) where the posterior distribution of  $\beta$  given data  $D$  is

$$Pr(\beta|D) = \sum_{v=1}^V Pr(\beta|M_v, D)Pr(M_v|D),$$

where  $V = 2^p - 1$ . This is a weighted average of the posterior distribution of  $\beta$ , wherein the weight corresponds to the posterior model probabilities. Based on this formulation, we can view *ACV* as a special case of *BMA* by setting  $Pr(M_v|D) = 1/K$  if  $M_v \in \{M_1, \dots, M_K\}$  and  $Pr(M_v|D) = 0$  otherwise. Thus, we average those ‘optimally’ selected models based on the  $MSP E_k$  defined in (1) with the equal weight, instead of averaging across all  $2^p - 1$  possible models as in *BMA*. In fact, *ACV* can be modified to obtain the similar form of *BMA*. We define  $\hat{\beta}_k^{FMA} = \sum_{v=1}^V w_{k,v} \hat{\beta}_{k,v}$ , where  $\hat{\beta}_{k,v}$  is the estimate of the  $v^{th}$  model and its corresponding weight is  $w_{k,v}$ . Because we concern finding the smallest  $MSP E_k$  among  $2^p - 1$  possible ones using data in the  $k^{th}$  hold-out fold, we use the  $MSP E_k$  values from the  $v^{th}$  model,  $MSP E_{k,v}$ , to estimate  $w_{k,v}$ . For example, we can use  $w_{k,v} = 1/\sqrt{MSP E_{k,v}}$  that is normalized to sum up to 1. Hence, we can build a new version of model averaging as  $1/K \sum_{k=1}^K \hat{\beta}_k^{FMA}$ . The properties of this model averaging method will not be pursued in this work.

Remark: The *ACV* procedure can be integrated with robust model estimation. For example, instead of the mean one may consider using a component-wise median estimation among the candidate models. M-estimator (Huber and Ronchetti 2009) or other types of robust estimators can also be used.

### 2.3. Theoretical Properties

We denote the true model as  $M_*$ . A model  $M$  can be classified into two categories (Shao 1993):

- Category I:  $M \supseteq M_*$
- Category II: At least one component of  $M_*$  is not contained in  $M$ .

Now, we introduce the following assumptions :

Assumption A.  $n/n_k \rightarrow K$  for all  $k = 1, \dots, K$ .

Assumption B. For  $M_k$  in Category II,

$$\liminf_{n \rightarrow \infty} n^{-1} \beta' \mathbf{X}' (\mathbf{I}_n - \mathbf{P}_k) \mathbf{X} \beta = b_k > 0, \text{ and } n^{-1} \beta' \mathbf{X}' \mathbf{P}_k \mathbf{X} \beta \rightarrow 0.$$

Assumption B is adopted from Zhang (1993). Notice that  $b_k$  is non-zero when  $M_k$  is in Category II, but becomes zero when  $M_k$  is in category I.

We define the mean squared error (*MSE*) of *ACV* as

$$MSE_{ACV} = \frac{1}{n} E((\mathbf{X}\beta - \mathbf{X}\hat{\beta}_{ACV})'(\mathbf{X}\beta - \mathbf{X}\hat{\beta}_{ACV})). \tag{4}$$

Note that  $MSE_{ACV}$  is defined based on the ultimate parameter estimates and uses all the explanatory variables, which is different from  $MSPE_k$  used to select the candidate ‘optimal’ model in the hold-out fold  $k$ . Using (3), (4) is expressed as

$$\begin{aligned}
 MSE_{ACV} &= \frac{1}{n}tr(Var(\mathbf{X}\hat{\beta}_{ACV})) + \frac{1}{n}\|E(\mathbf{X}\hat{\beta}_{ACV} - \mathbf{X}\beta)\|^2 \\
 &= \frac{1}{n}tr(\mathbf{H}'_{ACV}\mathbf{H}_{ACV})\sigma^2 + \frac{1}{n}\beta'\mathbf{X}'(\mathbf{I}_n - \mathbf{H}_{ACV})'(\mathbf{I}_n - \mathbf{H}_{ACV})\mathbf{X}\beta,
 \end{aligned}
 \tag{5}$$

where  $\mathbf{I}_n$  is an identity matrix of size  $n$ , and  $tr(\cdot)$  is a trace.

Now, let the model chosen by the traditional  $K$ -fold  $CV$  as  $\mathbf{X}_o$ . Then, the  $MSE$  of the traditional  $K$ -fold  $CV$  ( $MSE_{CV}$ ) is obtained by replacing  $\mathbf{H}_{ACV}$  with the projection matrix of  $\mathbf{X}_o$ .

$$MSE_{CV} = \frac{d}{n}\sigma^2 + \frac{1}{n}\beta'\mathbf{X}'(\mathbf{I}_n - \mathbf{P}_{CV})\mathbf{X}\beta,
 \tag{6}$$

where  $d = tr(\mathbf{P}_{CV}) \leq p$  and  $\mathbf{P}_{CV} = \mathbf{X}_o(\mathbf{X}'_o\mathbf{X}_o)^{-1}\mathbf{X}'_o$ . Then, it is readily seen that  $MSE_{CV} = \frac{d}{n}\sigma^2 = O(n^{-1})$  when  $\mathbf{X}_o$  is in Category I, since the second term on the right-hand side (RHS) of equation (6) is 0. When  $\mathbf{X}_o$  is in Category II,  $MSE_{CV}$  is dominated by  $\frac{1}{n}\beta'\mathbf{X}'(\mathbf{I}_n - \mathbf{P}_{CV})\mathbf{X}\beta$ , which is greater than 0 by Assumption B.

In Theorem 1, we show that the two estimates chosen by  $CV$  and  $ACV$  are asymptotically equivalent as the sample size increases.

**Theorem 2.1:** *Let  $M_f = \cup_{k=1}^K M_k$  be a union of  $M_k$  for  $k = 1, \dots, K$ . Under Assumptions A and B,*

- (a)  $MSE_{ACV} = MSE_{CV} + O(n^{-1})$  if  $M_f$  and  $\mathbf{X}_o$  are in Category I.
- (b)  $MSE_{ACV} = MSE_{CV} + o(1)$  if  $M_f$  and  $\mathbf{X}_o$  are in Category II.

Note that the rate of decrease in both  $MSE_{ACV}$  and  $MSE_{CV}$  is subject to inclusion of the true model. This implies that inclusion of the true model will determine the asymptotic behavior of  $ACV$  and  $CV$ , even though the two sets of parameter estimates are asymptotically equivalent.

**Remark:** Some existing work use a slightly different but more realistic definition of MSE, since  $\beta$  is unknown in practice, which can be written as  $MSE_{ACV} = \frac{1}{n}E((\mathbf{Y} - \mathbf{X}\hat{\beta}_{ACV})'(\mathbf{Y} - \mathbf{X}\hat{\beta}_{ACV}))$ , and  $MSE_{CV} = \frac{1}{n}E((\mathbf{Y} - \mathbf{X}\hat{\beta}_{CV})'(\mathbf{Y} - \mathbf{X}\hat{\beta}_{CV}))$ . Using this definition, we can derive the same asymptotic equivalence of  $\hat{\beta}_{ACV}$  and  $\hat{\beta}_{CV}$ .

Now, we show the reduced variance of  $\hat{\beta}_{ACV}$  under the finite sample.

**Lemma 2.2:** *Under the setting in Section 2.1, we have*

$$\det(var(\hat{\beta}_{ACV})) = \det(var(\frac{1}{K} \sum_{k=1}^K \hat{\beta}_k)) \leq \det(var(\hat{\beta}_{CV})).$$

*The inequality holds when  $\hat{\beta}_k$ s are all identical.*

This implies that the variance of  $\hat{\beta}_{ACV}$  is smaller than  $\hat{\beta}_{CV}$  (as  $\sqrt{\det(A_k)}$  measures the volume of  $A_k$ ) due to the averaging effect except for the case when  $\hat{\beta}_k$ s are all identical. Under our empirical experiments in Section 4, the magnitude of the reduced variance

tends to be larger than the allowed bias, thus results in more accurate model selection by *ACV*. This indicates the proposed method could take advantage of bias-variance tradeoff.

### 3. Integration of *ACV* with other methods

**Lasso penalization:** we introduce some applications of *ACV* beyond the traditional linear regression problem with a fairly small  $p$  as discussed in Section 2. When the number of variables  $p$  is large, it is impractical to fit all  $2^p - 1$  models. Rather, we use a penalization method such as LASSO (Tibshirani 1996), which is widely used for handling high-dimensional data. Such penalization methods convert a model selection problem to controlling the amount of regularization applied to the parameter estimates. Specifically, LASSO is defined as

$$\hat{\beta}^{LASSO}(\lambda) = \arg \min_{\beta \in \mathbb{R}^p} (||\mathbf{Y} - \mathbf{X}\beta||^2 + \lambda \sum_{j=1}^p |\beta_j|).$$

Because  $\lambda$  takes a real number in  $[0, \infty)$ , we cannot consider all the possible models. We examine some candidate values of  $\lambda$  in a reasonable range. With  $K$ -fold *CV*, a  $\hat{\lambda}_{CV}$  that produces the smallest cross validated score is claimed as ‘optimal.’

In this case, *ACV* intends to select  $K$  candidate ‘optimal’ values of  $\hat{\lambda}_k$  based on the cross-validated score from hold-out fold  $k = 1, \dots, K$ . Let  $\hat{\beta}_{-n_k}^{LASSO}(\lambda)$  be a LASSO estimate that is obtained based on all the samples excluding the ones in the  $k^{th}$  fold. The ultimate estimate of  $\lambda$  follows as  $\hat{\lambda}_{ACV} = \frac{1}{K} \sum_{k=1}^K \hat{\lambda}_k$ , where

$$\hat{\lambda}_k = \arg \min_{\lambda \in [0, \infty)} \left( \frac{1}{n_k} ||\mathbf{Y}_k - \mathbf{X}_{n_k} \hat{\beta}_{-n_k}^{LASSO}(\lambda)||^2 \right).$$

Such way of averaging enables obtaining a more stable estimate of the penalization parameter. Because the possible number of  $\lambda$  values is infinite, a manageable solution is to use several values of  $\lambda$  on a discrete grid for assessment. Note that  $\hat{\lambda}_{ACV}$  is likely to be more accurate than the estimate of *CV* using the same grid, due to the averaging effect of *ACV*.

Moreover, the computational efficiency of *ACV* is similar to *CV*. In both the methods, the major computation comes from estimating model with the  $(K - 1)$  folds of training data and from evaluating the performance with the hold-out fold. For *ACV*, the additional steps of choosing  $K$  candidate ‘optimal’ values and averaging could significantly improve the performance at trivial additional computation cost. We make comparisons between  $\hat{\beta}^{LASSO}(\hat{\lambda}_{CV})$  and  $\hat{\beta}^{LASSO}(\hat{\lambda}_{ACV})$  via simulation studies that will be described later.

**Quantile smoothing splines:** The cross-validation method is also frequently used in many different versions of smoothing splines to control the amount of roughness of the fitted function. Quantile smoothing splines are developed for targeting conditional quantiles in a nonparametric fashion. There are several different definitions of such smoothing splines; we use the method proposed by Bloomfield and Steiger (1983) and Nychka et al.

(1995). Basically, a quantile smoothing spline is taken as the minimizer of

$$\frac{1}{n} \sum_{i=1}^n \rho_q(y_i - g(x_i)) + \lambda \int \{g''(x)\}^2 dx,$$

where  $\rho_q(u) = u\{q - I(u < 0)\}$  is the check function of Koenker and Bassett (1978) and  $0 < q < 1$ . With a conditional probability density of  $f_{Y|X}(Y|X)$ , the  $q^{th}$  conditional quantile function  $g_q(x)$  is a function of  $x$  such that  $q = \int_{-\infty}^{g_q(x)} f_{Y|X}(y|x) dy$ . Here, the roughness of the fitted curves is determined by the value of  $\lambda$ . We define  $\hat{\lambda}_{CV}^q$  as the minimizer of the cross-validated score where the check loss function is used for validation. Similar to the procedure of conducting *ACV* under LASSO, we can obtain the *ACV* version of  $\hat{\lambda}_{ACV}^q$  for quantile smoothing splines. We use simulations to investigate the empirical properties of  $\hat{\lambda}_{CV}^q$  and  $\hat{\lambda}_{ACV}^q$ .

#### 4. Simulations

In empirical studies, it is desirable for  $K$ , the number of folds, to be reasonably large in order to receive the benefit of averaging. We consider  $K = 5$  and  $10$  throughout the simulation studies. We investigate the performance of *CV* and *ACV* in three scenarios: the traditional linear regression model, LASSO regularization method, and quantile smoothing splines. For these scenarios, the corresponding *ACV* methods described in Section 2 and Section 3 are used. The classical *CV* procedure is described as follows.

##### The *CV* Procedure

- (1) Randomly and evenly split the data set into  $K$  folds.
- (2) Use  $K - 1$  folds of data as a training data set to fit the linear model (or, LASSO, and quantile smoothing splines).
- (3) With the fitted models in (2), predict the value of the response variable in the hold-out fold.
- (4) From the response variable in hold-out fold (say,  $k^{th}$  fold), calculate mean squared prediction error by  $MSPE_k = \sum_{i \in \mathbb{N}_k} (y_i - \hat{y}_i^{pred})^2 / n_k$ , where  $\hat{y}_i^{pred}$  is the predicted value for  $y_i$  and  $\mathbb{N}_k$  is the data set in  $k^{th}$  fold.
- (5) Repeat (2) through (4) for  $K$  times so that each of  $K$  fold is used as a hold-out fold from which we obtain  $MSPE_1, \dots, MSPE_K$ .
- (6) Each candidate model obtains a prediction performance measure  $\sum_{k=1}^K MSPE_k / K$ . The ‘optimal’ model which minimizes  $\sum_{k=1}^K MSPE_k / K$  will be selected.

##### 4.1. Linear model

Assume a linear model  $\mathbf{Y} = \mathbf{X}\beta + \epsilon$  as described in Section 2.1 with  $\epsilon_i$ s following an *iid* standard normal distribution. We consider the number of samples  $n = 200$ . The  $X$  is generated from a multivariate normal distribution with mean 0 and the correlation between the  $i^{th}$  and  $j^{th}$  covariates  $\rho = 0.5^{|i-j|}$ . We consider three cases of the true values of  $\beta$ : (i)  $\beta = (3, 0, 0, 0, 0, 0, 0, 0)'$ ; (ii)  $\beta = (3, 1.5, 0, 0, 2, 0, 0, 0)'$ ; and (iii)  $\beta = (2, 2, 2, 2, 2, 2, 2, 2)'$  that represent sparse, intermediate, and dense parameter space, respectively.



Table 1. Estimate of  $MSE$  (multiplied by  $10^2$ ) and its standard error (in parentheses, multiplied by  $10^2$ ) based on 1000 MC samples for  $CV$ ,  $ACV$  and  $Bagging$ .

	$\beta=(3,0,0,0,0,0,0,0)'$	$\beta=(3,1.5,0,0,2,0,0,0)'$	$\beta=(2,2,2,2,2,2,2,2)'$
$CV, K = 5$	516.4 (13.6)	593.1 (13.3)	806.3 (13.0)
$ACV, K = 5$	405.8 (8.7)	521.1 (5.8)	806.3 (13.0)
$reduction(\%)$	21.43	12.15	0
$CV, K = 10$	509.3 (13.7)	590.9 (13.3)	806.3 (13.0)
$ACV, K = 10$	370.1 (7.6)	553.4 (5.8)	806.3 (13.0)
$reduction(\%)$	27.34	6.35	0
$Bagging$	811.4(13.1)	808.3(13.1)	810.6(13.0)

To measure accuracy of the estimates, we use the mean squared error ( $MSE$ ) defined as

$$\begin{aligned} MSE &= E_{\hat{\beta}} \|\mathbf{X}\hat{\beta} - \mathbf{X}\beta\|^2 \\ &= E_{\hat{\beta}} \{(\hat{\beta} - \beta)' \mathbf{X}' \mathbf{X} (\hat{\beta} - \beta)\}. \end{aligned} \quad (7)$$

With  $L$  Monte Carlo (MC) samples ( $L = 1000$ ), the estimate  $\widehat{MSE}$  can be computed as  $1/L \sum_{l=1}^L \{(\hat{\beta}^l - \beta)' \mathbf{X}'_l \mathbf{X}_l (\hat{\beta}^l - \beta)\}$ , where  $\hat{\beta}^l$  is the estimate of  $\beta$  for the  $l^{th}$  MC sample.

In addition,  $Bagging$  procedure is applied to the same MC samples. Since  $CV$  and  $ACV$  choose a model from  $2^8 - 1 = 255$  possible models, we let  $Bagging$  procedure to select from 255 re-sampled (with replacement) data sets from one MC sample. The size of re-sampled data set is 200. The results of both  $CV$  and  $ACV$  are reported in Table 1. In Table 1, “ $reduction(\%)$ ” denotes the reduction of  $MSE$  achieved by  $ACV$  in comparison to  $CV$ . We see that the largest reduction is attained in case (i) which is the sparse scenario, while no reduction is made in the dense case (iii). In the dense case (iii),  $ACV$  is able to select the true model  $K$  times which leads to the accurate ultimate model via averaging, and thus results in the identical values of  $MSE$  as  $CV$ . Performance of  $Bagging$  is consistent over the three cases, and thus relatively works fine in the dense case, but is outperformed by  $CV$  and  $ACV$ .

#### 4.2. Lasso penalization

We also investigate the performance of  $CV$  and  $ACV$  in the case of large  $p$  with the usage of sparsity penalization. We use the similar simulation set-up with  $n = 200$  in the previous section, but increase  $p$  to 1000 and set  $\rho = 0$ . The  $\beta$  is set to have 50 non-zero values and 950 zeros. Two different sets of non-zero values are considered as follows: (i) 25 values are randomly generated from  $Unif(1, 2)$ , and the other 25 from  $Unif(-2, -1)$ ; and (ii) 25 values are taken to be 1 and the others are with values of  $-1$ . We interrogate 200 equally spaced values of  $\lambda$  in  $(0.001, 0.4)$  with  $K = 10$ .

We compare  $CV$  and  $ACV$  in terms of parameter estimation and identification of true non-zero parameters. We again use  $MSE$  in (7) to measure the estimation accuracy. The mean and its standard error of 1000  $MSE$  values, respectively, obtained from 1000 MC samples in each case are reported in Table 2. We see that  $ACV$  reduces  $MSE$  by about 30% comparing to  $CV$  in each case. To investigate accuracy of detecting the 50 true

Table 2. Estimates of  $MSE$  and its standard error (in parentheses) based on 1000 MC samples for  $CV$  and  $ACV$ .

	$CV$	$ACV$	$reduction(\%)$
case (i)	573.2 (17.8)	381.3 (6.3)	33.5
case (ii)	487.4 (13.3)	342.8 (5.6)	29.7

Table 3. The distribution of the number of detected true non-zero  $\beta$ s of  $CV$  subtracted from that of  $ACV$ , based on 1000 MC samples.

	-6 to -4	-3	-2	-1	0	1	2	3	4	5 to 7	8 to 15
case (i)	14	17	51	140	352	170	108	62	31	39	16
case (ii)	14	15	61	138	306	181	112	58	46	48	21

non-zero parameters, we focus on the 50 variables with the largest absolute values of the parameter estimates, and record the number of detected true non-zero ones among them. In Table 3, we report the distribution of the number of detected true non-zero  $\beta$ s of  $CV$  subtracted from that of  $ACV$ . In case (i),  $ACV$  and  $CV$  show the identical detection accuracy for 352 times out of 1000 runs. Note that  $ACV$  performs better than  $CV$  for 426 times, while  $CV$  outperforms  $ACV$  for 222 times. In case (ii),  $ACV$  detects more true non-zero  $\beta$ s than  $CV$  for 466 times, but fewer for 228 times. It is also noticeable that  $ACV$  outperforms  $CV$  with the number of accurate detection as high as 15, while  $CV$  outperforms  $ACV$  with no more than 6 accurate detection.

### 4.3. Quantile smoothing splines

In this section, we investigate applicability of the proposed method  $ACV$  in the problem of quantile smoothing splines. We consider the mean of the observations from a sinusoid curve with period 1, which is expressed as

$$y_i = \sin(2\pi x_i) + \epsilon_i, \quad i = 1, \dots, n,$$

where  $x_i$ s are *iid* from the standard uniform distribution with  $n = 200$ . We consider two distribution settings for the *iid*  $\epsilon_i$ 's: (i)  $N(0, 0.2^2)$ ; and (ii) shifted exponential distribution with the median 0 and the standard deviation 0.2. A fine grid search of the smoothing parameter is conducted to select the 'optimal' value by  $CV$  and  $ACV$  with  $L$  MC samples ( $L = 1000$ ) and  $K = 5$ . We adopt the empirical estimate of  $MSE$  by MC samples as,

$$MSE = \frac{1}{L} \sum_{l=1}^L \frac{1}{n} \sum_{i=1}^n (g(x_i) - \hat{g}^l(x_i))^2,$$

where  $g(x_i)$  is the true underlying function, and  $\hat{g}^l(x_i)$  indicates the fitted value at  $x_i$  from the  $l^{th}$  MC sample. The results at several quantiles are shown in Table 4, where  $reduction(\%)$  stands for the percentage of reduction in the average value of  $MSE$  achieved by  $ACV$  comparing to  $CV$ . Under the normal error distribution in case (i),

Table 4. Estimate of  $MSE$  and its standard error (in parentheses) based on 1000 MC samples for  $CV$  and  $ACV$ . All numbers are multiplied by  $10^4$ .

case (i)	$q=0.1$	$q=0.2$	$q=0.3$	$q=0.4$	$q=0.5$
$CV$	84.25(1.8)	63.39(1.2)	55.72(1.1)	51.63(1.0)	50.92(1.0)
$ACV$	83.13(1.8)	58.83(1.2)	50.55(1.0)	46.78(0.9)	45.10(0.8)
reduction(%)	1.32	7.21	9.28	9.4	11.42
case (ii)	$q=0.1$	$q=0.25$	$q=0.5$	$q=0.75$	$q=0.9$
$CV$	2.21(0.06)	5.02(0.12)	13.14(0.28)	35.84(0.08)	89.67(2.0)
$ACV$	1.81(0.04)	4.28(0.09)	11.81(0.23)	33.68(0.07)	90.41(2.3)
reduction(%)	18.28	14.75	10.08	6.03	-0.82

the result for  $q > 0.5$  is not reported because of the similarity to  $q < 0.5$ . The considerable reductions achieved by  $ACV$  indicate that its averaging strategy results in better selection of the penalization parameter. For both the cases, quantiles in the region of high conditional density of the errors lead to greater reduction (%) in  $MSE$ , comparing to the low density regions. The rationale is that fewer observations are available to estimate the targeted quantile in the low density regions, and thus the advantage of  $ACV$  is not as prominent. This can be seen at  $q = 0.9$  in case (ii) that the performances of these two methods are very similar with  $n = 200$ . To verify this finding, we increase the sample size to 400 and observe 5% reduction in  $MSE$  by using  $ACV$ . Thus, it is important to keep a sufficient number of observations in the hold-out fold to take advantage of  $ACV$ .

Another phenomenon often observed is data over-fitting by  $CV$ . In Figure 1, we demonstrate this point by two plots of the fitted curves that produce the maximum (left panel) and minimum (right panel) values of  $MSE_{ACV}/MSE_{CV}$  among 1000 MC samples, respectively. In the left panel,  $ACV$  in fact performs similarly to  $CV$ . In contrast, the right panel shows the obvious over-fitting by  $CV$ , manifested by its jagged curve.

## 5. A real example

Isaacs et al. (1983) established the reference percentiles of children for the serum concentrations of immunoglobulin-G ( $IgG$ ) in grams per liter, based on 298 children of ages 6 months to 6 years. They considered various polynomial regressions treating  $IgG$  as the response and  $age$  (in month) as an explanatory variable. The best-fit model provided by Isaacs et al. (1983) is  $IgG^{\frac{1}{2}} = 1.16 + 0.2715 * age^{\frac{1}{2}} - 0.01195 * age$  for the 50<sup>th</sup> percentile, which is drawn as a dashed line in Figure 2. Royston and Altman (1994) revisited this data and used additive polynomial regressions with smoother fitted lines to eliminate the jaggedness near the boundary ages of 6 months and 6 years. Herein, we target selecting an appropriate smoothing penalty parameter for quantile smoothing splines to obtain well-behaved reference percentiles.

To find the ‘optimal’  $\hat{\lambda}$ , we examined 2000  $\log(\lambda)$  values within  $(-35, -0.2)$ . First, the data is randomly divided into 5 (nearly) equal parts, among which one contains 58 observations and each other part contains 60 observations. To account for the variability due to random split, we iterate the procedure 200 times for both  $ACV$  and  $CV$ . The penalization parameter values, denoted by  $\hat{\lambda}_{ACV}$  and  $\hat{\lambda}_{CV}$ , are obtained by averaging their respective 200 estimates of  $\lambda$ . The fitted conditional quantile lines of the  $IgG$  data

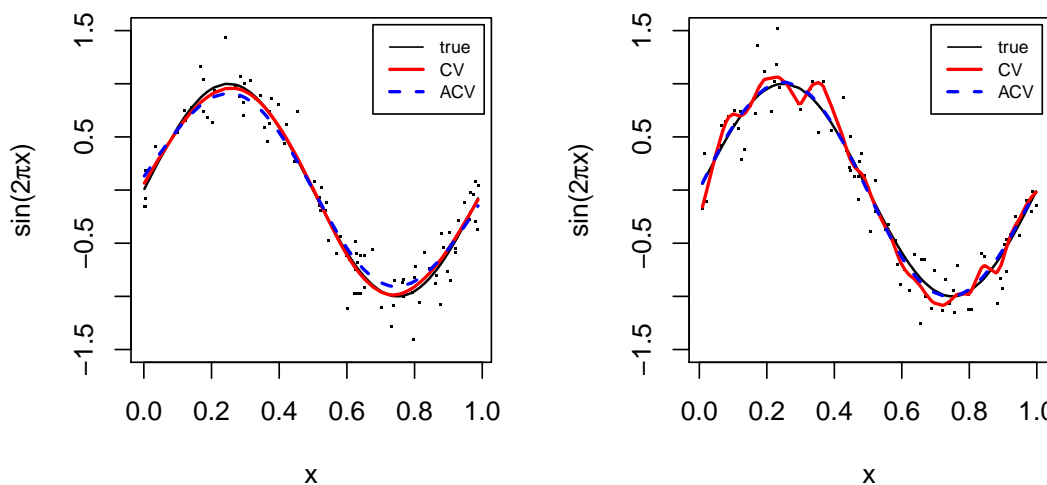


Figure 1. Under a normal error distribution, the fitted model selected by *ACV* performs relatively ‘worst’ compared to that selected by *CV* (left), and *ACV* performs relatively ‘best’ compared to *CV* (right) among 1000 simulated data sets.

Table 5. Average values of  $\log(\hat{\lambda}_{CV})$  and  $\log(\hat{\lambda}_{ACV})$  at various quantiles over 200 different random splits.

	$q=0.05$	$q=0.1$	$q=0.25$	$q=0.5$	$q=0.75$	$q=0.9$	$q=0.95$
<i>CV</i>	-15.49	-15.59	-16.73	-19.86	-4.02	-3.85	-3.84
<i>ACV</i>	-1.58	-1.62	-1.7	-1.64	-1.42	-1.5	-1.52

at  $q = 0.05, 0.1, 0.25, 0.5, 0.75, 0.9$ , and  $0.95$  are plotted in blue from the bottom to the top in Figure 2 .

In the data set, we find mild heteroscedasticity of increasing variance as *Age* increases. It is reflected by the fit from both *CV* and *ACV*. It is clear that the fitted line at  $q = 0.5$  provided by *CV* appears to be less smooth than *ACV*. We notice that *ACV* shares more similarity with the model given by Isaacs et al. (1983) which is the dashed line. At  $q = 0.25$ , the non-monotonic fitting line of *CV* implies that the *IgG* level of 3-year-old children is higher than that of 4-year-old children, which seems insensible from the perspective of biology. Overall, *ACV* provides smoother fit to data than *CV*. We also summarize the logarithm-transformed  $\hat{\lambda}_{CV}$  and  $\hat{\lambda}_{ACV}$  at various quantiles in Table 5. We can see that  $\hat{\lambda}_{CV}$  changes considerably across the quantiles, whereas  $\hat{\lambda}_{ACV}$  is stable (around -1.5) due to the averaging effect.

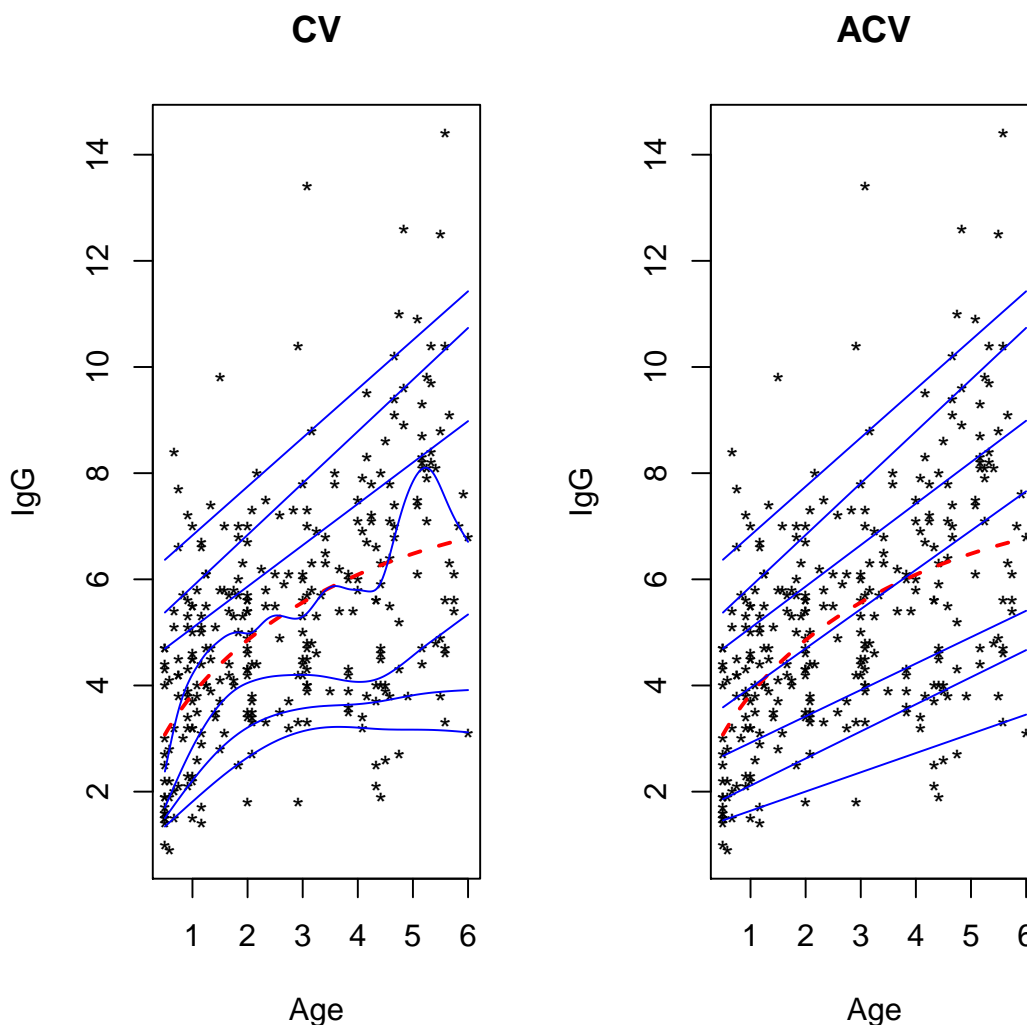


Figure 2. Fitted quantile smoothing splines at  $q=0.05, 0.1, 0.25, 0.5, 0.75, 0.9,$  and  $0.95$ , highlighted in blue from the bottom to the top, in the immunoglobulin-G data set. The dashed fitting line is obtained from the model given by Isaacs et al. (1983). The results of *CV* and *ACV* are shown in the left and right panels, respectively.

## 6. Conclusion

In this work, we propose a  $K$ -fold averaging cross-validation procedure for model selection and parameter estimation. We establish its theoretical property and show its promise via empirical investigation. Since cross-validation is actively employed in many areas of statistics, *ACV* can also be applied to a broad range of modeling procedures. For example, *ACV* can be easily used with penalized model selection method, for which selection of penalty parameters is of interest. Demonstrated through simulations with usage of LASSO, the *ACV* method outperforms the *CV* method in terms of mean squared error

and selection accuracy. We also investigate its applicability in quantile smoothing splines, and demonstrate its capability of providing more smooth data fit than the traditional CV method.

One shall pay attention to the size of the test set and selection of  $K$  when implementing  $ACV$ . A large value of  $K$  combined with a small sample size may cause insufficient fit of the data since  $K$ -fold  $ACV$  performs model selection based on  $1/K$  of the data. We recommend including a sufficient number of observations in a test set. Note that the desirable size of the test set also depends on complexity of the underlying true model.

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## Appendix

### *Proof of Theorem 2.1*

**Proof:**

$$tr(\mathbf{H}'_{ACV}\mathbf{H}_{ACV}) = \frac{1}{K^2}tr\left(\left(\sum_{k=1}^K \mathbf{P}_k\right)' \left(\sum_{k=1}^K \mathbf{P}_k\right)\right) \leq \frac{1}{K}tr\left(\sum_{k=1}^K \mathbf{P}'_k \mathbf{P}_k\right) \quad (1)$$

Since  $\mathbf{P}'_k \mathbf{P}_k = \mathbf{P}_k$ , and  $tr(\mathbf{P}_k) = |M_k|$ , we have,

$$\frac{\sigma^2}{n}tr(\mathbf{H}'_{ACV}\mathbf{H}_{ACV}) \leq \frac{\sigma^2}{n} \frac{1}{K} \sum_{k=1}^K |M_k|, \quad (2)$$

where  $\frac{1}{K} \sum_{k=1}^K |M_k| \equiv d^*$  is the mean number of selected variables from ACV. Thus,  $|d - d^*| \leq p$ , which leads to  $|d - d^*| \frac{\sigma^2}{n} = O(n^{-1})$ . The second term on the RHS of (5) will disappear when  $M_f$  is in Category I, which completes the proof of (a).

When  $M_f$  is in Category II, the second term on the RHS of (5) will be

$$\frac{1}{n} \beta' \mathbf{X}' (\mathbf{I}_n - \mathbf{H}_{ACV})' (\mathbf{I}_n - \mathbf{H}_{ACV}) \mathbf{X} \beta \leq \max_{1 \leq k \leq K} \frac{1}{n} \beta' \mathbf{X}' (\mathbf{I}_n - \mathbf{P}_k) \mathbf{X} \beta. \quad (3)$$

When it is subtracted by the second term on the RHS of equation (6), we have

$$\frac{1}{n} \beta' \mathbf{X}' (\mathbf{I}_n - \mathbf{H}_{ACV})' (\mathbf{I}_n - \mathbf{H}_{ACV}) \mathbf{X} \beta - \frac{1}{n} \beta' \mathbf{X}' (\mathbf{I}_n - \mathbf{P}_{CV}) \mathbf{X} \beta \leq \max_{1 \leq k \leq K} \frac{1}{n} \beta' \mathbf{X}' (\mathbf{P}_{CV} - \mathbf{P}_k) \mathbf{X} \beta. \quad (4)$$

By Assumption B, the RHS of equation (4) is  $o(1)$ , which completes the proof of (b).  $\square$

### *Proof of Lemma 2.2*

**Proof:** First, define  $\max_{1 \leq k \leq K} A_k$  to be  $A_k$  with maximum determinant for  $k = 1, \dots, K$ , where  $A_k$  is a positive definite matrix. Note that  $\det(A_k) > 0$  as a property of positive

definite matrix, and this is the case here since we assume  $\mathbf{X}_k$  is full column rank. From the facts that the value of correlation is always between -1 and 1, and  $\det(A_k A_{k'}) = \det(A_k) \det(A_{k'})$ , we have

$$\begin{aligned} \det(\text{cov}(\hat{\beta}_k, \hat{\beta}_{k'})) &\leq \det\left(\sqrt{\text{var}(\hat{\beta}_k)}\sqrt{\text{var}(\hat{\beta}_{k'})}\right) = \det\left(\sqrt{\text{var}(\hat{\beta}_k)}\right) \det\left(\sqrt{\text{var}(\hat{\beta}_{k'})}\right) \\ &\leq \det\left(\max_{1 \leq k \leq K} \text{var}(\hat{\beta}_k)\right), \end{aligned}$$

where the last inequality directly comes from the definition. Therefore, we have,

$$\begin{aligned} \det(\text{var}(\frac{1}{K} \sum_{k=1}^K \hat{\beta}_k)) &= \det\left(\frac{1}{K^2} \sum_{k=1}^K \text{var}(\hat{\beta}_k) + \frac{2}{K^2} \sum_{1 \leq k < k' \leq K} \text{cov}(\hat{\beta}_k, \hat{\beta}_{k'})\right) \\ &\leq \det\left(\frac{1}{K} \max_{1 \leq k \leq K} \text{var}(\hat{\beta}_k) + \frac{K(K-1)}{K^2} \max_{1 \leq k \leq K} \text{var}(\hat{\beta}_k)\right) \\ &= \det\left(\max_{1 \leq k \leq K} \text{var}(\hat{\beta}_k)\right) = \det(\text{var}(\hat{\beta}_{CV})) \end{aligned}$$

When  $\hat{\beta}_k$ s are identical for  $k = 1, \dots, K$ , the above inequality becomes equality. Further, when the selected models from all the folds  $\{\hat{\beta}_k\}$  are identical, it is in fact the same as a selected model by the traditional  $K$ -fold  $CV$ . Thus, we have  $\hat{\beta}_{ACV} = \hat{\beta}_{CV}$ , which confirms the last equality. □