CrossMark

Applied Mathematics and Nonlinear Sciences 2(1) (2017) 195–200

Applied Mathematics and Nonlinear Sciences

http://journals.up4sciences.org

Odd mean labeling for two star graph

S. Sudhakar,[†] Silviya Francis, and V. Balaji.

Department of Mathematics, Sacred Heart College, Tirupattur, INDIA

Submission Info

Communicated by Wei Gao
Received 6th January 2017
Accepted 20th June 2017
Available online 20th June 2017

Abstract

In this paper further result on odd mean labeling is discussed. We prove that the two star $G = K_{1,m} \wedge K_{1,n}$ is an odd mean graph if and only if $|m-n| \leq 3$. The condition for a graph to be odd mean is that $p \leq q+1$, where p and q stands for the number of the vertices and edges in the graph respectively.

Keywords: Odd mean graph; path and star
AMS 2010 codes: 05C78

1 Introduction

All graphs in this paper are finite, simple and undirected. Terms not defined here are used in the sense of Harary [1]. In 1966, Rosa [3] introduced $\beta-$valuation of a graph. Golomb subsequently called such a labeling graceful. In 2015, Maheswari and Ramesh [2] proved the two star graph $G = K_{1,m} \wedge K_{1,n}$ is a mean graph if and only if $|m-n| \leq 4$. We prove the two star graph $G = K_{1,m} \wedge K_{1,n}$ is an odd mean graph if and only if $|m-n| \leq 3$.

2 Odd mean labeling

Definition 1. A graph $G = (V,E)$ with p vertices and q edges is said to be an odd mean graph if there exists a function f from the vertex set of G to $\{0,1,2,\cdots,2q-1\}$ such that the induced map f^* from the edge set of G to $\{1,3,5,\cdots,2q-1\}$ defined by

[†]Corresponding author.
Email address: pulibala70@gmail.com

UP4 Sciences ISSN 2444-8656 doi:10.21042/AMNS.2017.1.00016

$$f^*(e = uv) = \begin{cases} \dfrac{f(u)+f(v)}{2} & \text{if } f(u)+f(v) \text{ is even;} \\ \dfrac{f(u)+f(v)+1}{2} & \text{if } f(u)+f(v) \text{ is odd.} \end{cases}$$

then the resulting edges get distinct labels from the set $\{1,3,5,\cdots,2q-1\}$.

Definition 2 (c.f. [4])**.** A wedge is defined as an edge connecting two components of a graph, denoted as $\wedge$, $\omega(G\wedge) < \omega(G)$.

Remark 1. A single star graph $K_{1,n}$ is an odd mean graph only if $n \leq 2$.

Notation. $G = K_{1,m} \wedge K_{1,n}$ is the two star graph connected by a wedge.

Theorem 1. *The graph $G = K_{1,m} \wedge K_{1,n}$ is an odd mean graph if and only if $|m-n| \leq 3$.*

Proof.
Part A
Primarily, assume that $|m-n| \leq 3$.
To prove that $G = K_{1,m} \wedge K_{1,n}$ is an odd mean graph.
Without loss of generality, we assume that $m \leq n$. There are four cases viz. $n = m$, $n = m+1$, $n = m+2$, and $n = m+3$.
Case:1 $n = m$.
Consider the graph $G = K_{1,m} \wedge K_{1,n}$. Let $\{u\} \cup \{u_i : 1 \leq i \leq m\}$ and $\{v\} \cup \{v_j : 1 \leq j \leq m\}$ be the vertex set of first and second copies of $K_{1,m}$ respectively. Then $G = K_{1,m} \wedge K_{1,n}$ has $2m+1$ edges and $2m+2$ vertices.
The required vertex labeling $f : V(G) \to \{0,1,2,\cdots,2q-1\}$ is defined as follows,

$$\begin{array}{lll} f(u) = 0; & & f(v) = 2q-2; \\ f(u_i) = 4i-3 & \text{for} & 1 \leq i \leq m; \\ f(v_j) = 4j+2 & \text{for} & 1 \leq j \leq n-1; \\ f(v_n) = 2q-1. & & \end{array}$$

The corresponding edge labels are as follows: The edge label of uu_i is $2i-1$ for $1 \leq i \leq m$. The edge label of vv_j is $q+2j$ for $1 \leq j \leq n-1$. The edge label of vv_n is $2q-1$. Also, the wedge u_1v_n is $2m+1$. Therefore, the induced edge labels of $G = \{1,3,5,...,2m-1,2m+3,...,4m+1\}$ and has $2m+1$ distinct edges.
Hence the vertex labels and the induced edge labels of G are distinct.
Thus $G = K_{1,m} \wedge K_{1,n}$ is an odd mean graph when $n = m$.
Case:2 $n = m+1$.
Consider the graph $G = K_{1,m} \wedge K_{1,n}$. Let $\{u\} \cup \{u_i : 1 \leq i \leq m\}$ and $\{v\} \cup \{v_j : 1 \leq j \leq m+1\}$ be the vertex set of $K_{1,m}$ and $K_{1,n}$ respectively. Then $G = K_{1,m} \wedge K_{1,n}$ has $2m+3$ vertices and $2m+2$ edges.
The required vertex labeling $f : V(G) \to \{0,1,2,\cdots,2q-1\}$ is defined as follows,

$$\begin{array}{lll} f(u) = 0; & & f(v) = 2q-2; \\ f(u_i) = 4i-3 & \text{for} & 1 \leq i \leq m; \\ f(v_j) = 4j & \text{for} & 1 \leq j \leq n-1; \\ f(v_n) = 2q-1. & & \end{array}$$

The corresponding edge labels are as follows: The edge label of uu_i is $2i-1$ for $1 \leq i \leq m$. The edge label of vv_j is $q+2j-1$ for $1 \leq j \leq n-1$. The edge label of vv_n is $2q-1$. Also, the wedge u_1v_{n-1} is $2m+1$. Therefore, the induced edge labels of $G = \{1,3,5,...,2m-1,2m+3,...,4m+3\}$ and has $2m+2$ distinct edges.
Hence the vertex labels and the induced edge labels of G are distinct.

Thus $G = K_{1,m} \wedge K_{1,n}$ is an odd mean graph when $n = m+1$.
Case:3 $n = m+2$.
Consider the graph $G = K_{1,m} \wedge K_{1,n}$. Let $\{u\} \cup \{u_i : 1 \leq i \leq m\}$ and $\{v\} \cup \{v_j : 1 \leq j \leq m+2\}$ be the vertex set of $K_{1,m}$ and $K_{1,n}$ respectively. Then $G = K_{1,m} \wedge K_{1,n}$ has $2m+4$ vertices and $2m+3$ edges.
The required vertex labeling $f : V(G) \to \{0,1,2,\cdots,2q-1\}$ is defined as follows,

$$\begin{array}{llll} f(u) = 2; & & & f(v) = 2q-2; \\ f(u_1) = 0; & & & \\ f(u_i) = 4i-5 & \text{for} & 2 \leq i \leq m; & \\ f(v_j) = 4j-3 & \text{for} & 1 \leq j \leq n. & \end{array}$$

The corresponding edge labels are as follows: The edge label of uu_1 is1. uu_i is $2i-1$ for $2 \leq i \leq m$. The edge label of vv_j is $q+2j-2$ for $1 \leq j \leq n$. Also, the wedge u_1v_{n-1} is $2m+1$. Therefore, the induced edge labels of $G = \{1,3,5,...,2m+1,2m+3,...,4m+5\}$ and has $2m+3$ distinct edges.
Hence the vertex labels and the induced edge labels of G are distinct.
Thus $G = K_{1,m} \wedge K_{1,n}$ is an odd mean graph when $n = m+2$.
Case:4 $n = m+3$.
Consider the graph $G = K_{1,m} \wedge K_{1,n}$. Let $\{u\} \cup \{u_i : 1 \leq i \leq m\}$ and $\{v\} \cup \{v_j : 1 \leq j \leq m+3\}$ be the vertex set of $K_{1,m}$ and $K_{1,n}$ respectively. Then $G = K_{1,m} \wedge K_{1,n}$ has $2m+5$ vertices and $2m+4$ edges.
The required vertex labeling $f : V(G) \to \{0,1,2,\cdots,2q-1\}$ is defined as follows,

$$\begin{array}{llll} f(u) = 1; & & & f(v) = 2q-2; \\ f(v_1) = 0; & & & \\ f(u_i) = 4i+1 & \text{for} & 1 \leq i \leq m; & \\ f(v_j) = 4j-5 & \text{for} & 2 \leq j \leq n. & \end{array}$$

The corresponding edge labels are as follows: The edge label of uu_i is $2i+1$ for $1 \leq i \leq m$. The edge label of vv_1 is $q-1$ and vv_j is $q+2j-3$ for $2 \leq j \leq n$. Also, the wedge uv_1 is 1. Therefore, the induced edge labels of $G = \{1,3,5,...,2m-1,2m+3,...,4m+7\}$ and has $2m+4$ distinct edges.
Hence the vertex labels and the induced edge labels of G are distinct.
Thus $G = K_{1,m} \wedge K_{1,n}$ is an odd mean graph when $n = m+3$.
Therefore, $G = K_{1,m} \wedge K_{1,n}$ is an odd mean graph when $|m-n| \leq 3$.

Part B
Now we shall assume that $G = K_{1,m} \wedge K_{1,n}$ is an odd mean graph and prove that $|m-n| \leq 3$.
That is to prove the graph $G = K_{1,m} \wedge K_{1,n}$ is not an odd mean graph when $|m-n| > 3$.
Let us prove this by the method of induction. Consider the graph when $m = 1$ and $|m-n| = 4$ that is $n = 5$.
$G = K_{1,1} \wedge K_{1,5}$ then the vertex and edge set of G is given by,
$V(G) = \{u,v\} \cup \{u_1\} \cup \{v_j : 1 \leq j \leq 5\} \Rightarrow p = 8$
$E(G) = \{uu_1\} \cup \{vv_j : 1 \leq j \leq 5\} \cup \{w\} \Rightarrow q = 7$.
Now we shall prove that distinct vertex labels cannot be allotted to the vertices, on considering all the possible value for the vertex v (without loss of generality). The vertex labeling, $f : V(G) \to \{0,1,2,...,2q-1 = 13\}$.
Case:a
Let us first consider $f(v) = 13$.
The vertices should be labeled as such all the edges receives distinct odd label.
We should get 5 distinct odd labels for the edges of $K_{1,5}$.
The possibilities of the vertices are 0 or 1, 4 or 5, 8 or 9 and 12.
There are only four possibilities to label five vertices, which is not sufficient.
Therefore, G is not an odd mean graph when $f(v) = 13$.
Case:b
Let us first consider $f(v) = 12$.

The vertices should be labeled as such all the edges receives distinct odd label.
We should get 5 distinct odd labels for the edges of $K_{1,5}$.
The possibilities of the vertices are 1 or 2, 5 or 6, 9 or 10 and 13.
There are only four possibilities to label five vertices, which is not sufficient.
Therefore, G is not an odd mean graph when $f(v) = 12$.

Case:c

Let us first consider $f(v) = 11$.
The vertices should be labeled as such all the edges receives distinct odd label.
We should get 5 distinct odd labels for the edges of $K_{1,5}$.
The possibilities of the vertices are 2 or 3, 6 or 7 and 10.
There are only three possibilities to label five vertices, which is not sufficient.
Therefore, G is not an odd mean graph when $f(v) = 11$.

Case:d

Let us first consider $f(v) = 10$.
The vertices should be labeled as such all the edges receives distinct odd label.
We should get 5 distinct odd labels for the edges of $K_{1,5}$.
The possibilities of the vertices are 0, 3 or 4, 7 or 8 and 11 or 12.
There are only four possibilities to label five vertices, which is not sufficient.
Therefore, G is not an odd mean graph when $f(v) = 12$.

Case:e

Let us first consider $f(v) = 9$.
The vertices should be labeled as such all the edges receives distinct odd label.
We should get 5 distinct odd labels for the edges of $K_{1,5}$.
The possibilities of the vertices are 0 or 1, 4 or 5, 8 and 12 or 13.
There are only four possibilities to label five vertices, which is not sufficient.
Therefore, G is not an odd mean graph when $f(v) = 9$.

Case:f

Let us first consider $f(v) = 8$.
The vertices should be labeled as such all the edges receives distinct odd label.
We should get 5 distinct odd labels for the edges of $K_{1,5}$.
The possibilities of the vertices are 1 or 2, 5 or 6, 9 or 10 and 13.
There are only four possibilities to label five vertices, which is not sufficient.
Therefore, G is not an odd mean graph when $f(v) = 8$.

Case:g

Let us first consider $f(v) = 7$.
The vertices should be labeled as such all the edges receives distinct odd label.
We should get 5 distinct odd labels for the edges of $K_{1,5}$.
The possibilities of the vertices are 2 or 3, 6 or 7 and 10 or 11.
There are only three possibilities to label five vertices, which is not sufficient.
Therefore, G is not an odd mean graph when $f(v) = 7$.

Case:h

Let us first consider $f(v) = 6$.
The vertices should be labeled as such all the edges receives distinct odd label.
We should get 5 distinct odd labels for the edges of $K_{1,5}$.
The possibilities of the vertices are 0, 3 or 4, 7 or 8 and 11 or 12.
There are only four possibilities to label five vertices, which is not sufficient.
Therefore, G is not an odd mean graph when $f(v) = 6$.

Case:i

Let us first consider $f(v) = 5$.
The vertices should be labeled as such all the edges receives distinct odd label.
We should get 5 distinct odd labels for the edges of $K_{1,5}$.
The possibilities of the vertices are 0 or 1, 4 or 5, 8 or 9 and 12 or 13.
There are only four possibilities to label five vertices, which is not sufficient.
Therefore, G is not an odd mean graph when $f(v) = 5$.

Case:j

Let us first consider $f(v) = 4$.
The vertices should be labeled as such all the edges receives distinct odd label.
We should get 5 distinct odd labels for the edges of $K_{1,5}$.
The possibilities of the vertices are 1 or 2, 5 or 6, 9 or 10 and 13.
There are only four possibilities to label five vertices, which is not sufficient.
Therefore, G is not an odd mean graph when $f(v) = 4$.

Case:k

Let us first consider $f(v) = 3$.
The vertices should be labeled as such all the edges receives distinct odd label.
We should get 5 distinct odd labels for the edges of $K_{1,5}$.
The possibilities of the vertices are 2, 6 or 7 and 10 or 11.
There are only three possibilities to label five vertices, which is not sufficient.
Therefore, G is not an odd mean graph when $f(v) = 3$.

Case:l

Let us first consider $f(v) = 2$.
The vertices should be labeled as such all the edges receives distinct odd label.
We should get 5 distinct odd labels for the edges of $K_{1,5}$.
The possibilities of the vertices are 0, 3 or 4, 7 or 8 and 11 or 12.
There are only four possibilities to label five vertices, which is not sufficient.
Therefore, G is not an odd mean graph when $f(v) = 2$.

Case:m

Let us first consider $f(v) = 1$.
The vertices should be labeled as such all the edges receives distinct odd label.
We should get 5 distinct odd labels for the edges of $K_{1,5}$.
The possibilities of the vertices are 0, 4 or 5, 8 or 9 and 12 or 13.
There are only four possibilities to label five vertices, which is not sufficient.
Therefore, G is not an odd mean graph when $f(v) = 1$.

Case:n

Let us first consider $f(v) = 0$.
The vertices should be labeled as such all the edges receives distinct odd label.
We should get 5 distinct odd labels for the edges of $K_{1,5}$.
The possibilities of the vertices are 1 or 2, 5 or 6, 9 or 10 and 13.
There are only four possibilities to label five vertices, which is not sufficient.
Therefore, G is not an odd mean graph when $f(v) = 0$.

Therefore, $G = K_{1,1} \wedge K_{1,5}$ is not an odd mean graph.
By the method of induction assume that the result is true for all $m \leq k-1$. Now consider the graph when $m = k$.
$G = K_{1,k} \wedge K_{1,k+4}$, then the vertex and edge set of G is given by,
$V(G) = \{u, v\} \cup \{u_i : 1 \leq i \leq k\} \cup \left\{v_j : 1 \leq j \leq k+4\right\} \Rightarrow p = 2k+6$
$E(G) = \{uu_i : 1 \leq i \leq k\} \cup \left\{vv_j : 1 \leq j \leq k+4\right\} \cup \{w\} \Rightarrow q = 2k+5$.
The vertex labeling, $f : V(G) \to \{0, 1, 2, ..., 2q-1 = 4k+9\}$.

By the induction method we know that when $m = k-1$ the result is true, that is the graph $G = K_{1,k-1} \wedge K_{1,k+3}$ with $2k+4$ vertices and $2k+3$ edges and vertex label $f : V(G) \rightarrow \{0,1,2,...,2q-1 = 4k+5\}$has only $2k+3$ choices of labels to label $2k+4$ vertices. Then the graph $G = K_{1,k} \wedge K_{1,k+4}$ with $2k+6$ vertices and $2k+5$ edges and vertex label $f : V(G) \rightarrow \{0,1,2,...,2q-1 = 4k+9\}$ will have $2k+3+2 = 2k+5$ choices of label to label $2k+6$ vertices. Hence $G = K_{1,m} \wedge K_{1,n}$ is not an odd mean graph when $|m-n| = 4$.
We have proved that $G = K_{1,m} \wedge K_{1,n}$ is not an odd mean graph when $|m-n| = 4$, it may also be noted that when the difference between m and n increases the choices will be still less and G will not be an odd mean graph for greater differences.
Therefore $G = K_{1,m} \wedge K_{1,n}$ is not an odd mean graph when $|m-n| \geq 3$.
Hence the theorem.

3 Applications

The odd mean labeling is applied on a graph (network) in order to enhance fastness, efficient communication and various issues,

1. A protocol, with secured communication can be achieved, provided the graph (network) is sufficiently connected.
2. To find an efficient way for safer transmissions in areas such as Cellular telephony, Wi-Fi, Security systems and many more.
3. Channel labeling can be used to determine the time at which sensor communicate.

Researchers may get the use of odd mean labeling in their research concerned with the above discussed issues.

Acknowledgements

One of the authors (Dr. V. Balaji) acknowledges University Grants Commission, SERO, Hyderabad and India for financial assistance (No. F MRP 5766 / 15 (SERO / UGC)).

References

[1] F. Harary, (1969), *Graph Theory*, Addison-Wesley, Reading.
[2] V. Maheswari, D. S. T. Ramesh, S. Francis and V. Balaji, (2015) *On Mean Labeling*, Bulletin of Kerala Mathematics Association, 12, No 1, 54-64.
[3] A. Rosa, (1967) *On certain valuations of the vertices of a graph*, Theory of Graphs (Internat. Symposium, Rome, July 1966), Gordon and Breach, N. Y. and Dunod Paris, 349-355.
[4] Silviya Francis and V. Balaji, (2016), *On limits of mean labeling for three star graphs*, Advances in Theoretical and Applied Mathematics, 11, No 4, 471-484.