Karl Bringmann **August 2018**

Exercises for ADFOCS 2018 - Sheet 2

Exercise 1 Metricity Problem: Given an $n \times n$ matrix A with entries in $\{0, \ldots, n^c\}$ (for some large constant $c > 0$), decide whether for all $i, j, k \in [n]$ we have $A_{ij} \leq A_{ik} + A_{kj}$.

Prove that **Metricity** is equivalent to **APSP** under subcubic reductions.

Exercise 2 $\mathbf{X + Y}$ problem: Given sets X and Y consisting of n integers, decide whether the set $X + Y = \{a + b \mid a \in X, b \in Y\}$ contains n^2 distinct integers or whether there are duplicates.

Show that if the $\mathbf{X + Y}$ problem can be solved in time $O(n^{2-\epsilon})$ for some $\epsilon > 0$, then **3SUM** can be solved in time $O(n^{2-\delta})$ for some $\delta > 0$.

Exercise 3 Hitting Set Problem: Given sets $S_1, \ldots, S_n, T_1, \ldots, T_n \subseteq \{1, \ldots, d\}$, determine whether there is a set S_i that intersects every set T_j (in this case S_i is called a "hitting set").

Clearly this problem can be solved in time $O(n^2 d)$. The **Hitting set Hypothesis (HSH)** states that this problem cannot be solved in time $O(n^{2-\varepsilon} \cdot \text{poly}(d))$.
Prove that **HSH** implies **OVH**.

Exercise 4 ZeroTriangle: Given a weighted directed graph $G = (V, E, w)$ with edge weights $w : E \to \{-n^c, \ldots, n^c\}$ (for some large constant $c > 0$), determine whether there are three vertices i, j, k such that $w(i, j) + w(j, k) + w(k, i) = 0$ holds.

Clearly this problem can be solved in time $O(n^3)$. Prove that if **ZeroTriangle** can be solved in time $O(n^{3-\varepsilon})$ (for some $\varepsilon > 0$) then:

a) **APSP** can be solved in time $O(n^{3-\delta})$ (for some $\delta > 0$), and

b) **3SUM** can be solved in time $O(n^{2-\delta})$ (for some $\delta > 0$).

Completion of Lecture:

Exercise 5 Prove that **MaxSubmatrix** is equivalent to **APSP** under subcubic reductions, i.e., complete the partial proof from the lecture.

Exercise 6 Construct a k-sum-free set $S \subseteq \{1, \ldots, U\}$ of size n over universe $U = n^{1+\varepsilon} k^{O(1/\varepsilon)}$, i.e., work out the details of the construction sketched in the lecture.