ADFOCS Exercise Set #2

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Main Problems

1. In the **subgraph connectivity** problem, we are given a fixed graph G and a set S of nodes. The updates are nodes insertions and deletions to S. After every update, the algorithm should output whether the subgraph induced by S is connected or not. Prove that this problem admits no $O(m^{1-\epsilon})$ amortized update time assuming SETH.
2. Given a directed unweighted graph $\boldsymbol{G}$ with two special nodes **s** and **t**, the **maximum st-flow** is defined to be the maximum number of edge-disjoint paths from s to t in G. Prove that maintaining the value of maximum flow under edge insertions and deletions admits no $O(m^{1-\epsilon})$ amortized update time assuming SETH.
3. In the lecture we have seen a proof that maintaining whether the number of strongly connected components is at most 2 or not under edge insertions and deletions requires $m^{1-o(1)}$ update time. Prove that algorithms that can distinguish between the following two cases also requires $m^{1-o(1)}$ update time, for any constant k:
   - the number of strongly connected components is at most 2, and
   - the number of strongly connected components is at least k.

   Note that such algorithm may give an arbitrary answer when the input is not one of the above two cases.
4. In the **(S, T)-reachability** problem, we are given a directed graph and two sets of nodes S and T. Updates are edges insertions and deletions. After each update, we want to answer whether there is some $s \in S$ and $t \in T$ such that s cannot reach t (i.e. there is no directed path from s to t). Show that no algorithm admits $O(n^{2-\epsilon})$ update time for this problem, assuming SETH.
5. In the **Chan's Subset Union** problem, we are given n subsets $X_1, X_2, \ldots, X_n$ over a universe U and a set $S \subseteq \{1,2,\ldots,n\}$. Each update is to add or remove a number in $\{1,\ldots,n\}$ to and from S. After each update, the algorithm should output whether $\cup_{i\in S} X_i = U$. Let $m = \sum_i |X_i|$. Prove that no algorithm admits $O(m^{1-\epsilon})$ update time for this problem, assuming SETH.

Other Problems (to complete gaps from the lectures)

a) In the lecture, we sketched how the OMv conjecture implies the $OuMv$ conjecture. Show that the OMv conjecture implies the $\gamma - OuMv$ conjecture.
b) Prove that SETH implies the lower bounds of dynamic OV and 3OV stated in the lecture.