Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
C. R. Acad. Sci. Paris, Ser. I 339 (2004) 293–298
COMPTES RENDUS MATHEMATIQUE

Numerical Analysis

Basis conversions among univariate polynomial representations ☆

Roberto Barrio [a], Juan Manuel Peña [b]

[a] *GME, Depto. Matemática Aplicada, Facultad de Ciencias, Universidad de Zaragoza, 50009 Zaragoza, Spain*
[b] *Depto. Matemática Aplicada, Facultad de Ciencias, Universidad de Zaragoza, 50009 Zaragoza, Spain*

Received 5 April 2004; accepted after revision 15 June 2004

Available online 28 July 2004

Presented by Philippe G. Ciarlet

Abstract

In this Note we provide a family of conversion algorithms relating Bernstein polynomials, monomials and the classical families of orthogonal polynomials, such as Jacobi, Gegenbauer, Legendre, Chebyshev, Laguerre and Hermite polynomials. ***To cite this article: R. Barrio, J.M. Peña, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Conversion de bases polynomiales univariées. Dans cette Note nous fournissons une famille d'algorithmes de conversion qui met en relation les polynômes de Bernstein, les monômes et les familles classiques de polynômes orthogonaux, tels que ceux de Jacobi, Gegenbauer, Legendre, Chebyshev, Laguerre ou Hermite. ***Pour citer cet article : R. Barrio, J.M. Peña, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Version française abrégée

Dans nombreuses applications théoriques et pratiques on utilise plusieurs représentations de polynômes et très fréquemment on transforme les unes en les autres [7,9,10]. Dans cet article on étudie le problème de la conversion et l'on donne de façon unifiée un ensemble de formules permettant de changer la plupart des bases univariées utilisées (Bernstein, les monômes et les familles classiques de polynômes orthogonaux : Jacobi, Gegenbauer, Legendre, Chebyshev, Laguerre et Hermite). Bien que dans la bibliographie existante on trouve plusieurs algorithmes de conversion [7,9,10], chacun d'eux s'obtient en utilisant des techniques différentes. Au contraire, notre approche

☆ The first author is supported by the Spanish Research Grant DGYCT BFM2003-02137 and the second author is supported by the Spanish Research Grant DGYCT BFM2003-03510.

E-mail addresses: rbarrio@unizar.es (R. Barrio), jmpena@unizar.es (J.M. Peña).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2004.06.017

permet d'obtenir toutes les formules de conversion entre des familles quelconques de suites polynomiales d'une façon très simple.

Pour déduire les algorithmes de conversion on comparera d'abord le comportement des représentations de Bernstein et celles de puissances sur le problème d'évaluation aux points extrêmes 0 et 1. Soit un polynôme $p_n(t)$ appartenant à l'espace des polynômes de degré inférieur ou égal à n, $t \in [0,1]$; alors, il peut être exprimé des deux façons, voir (1), où $b_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$ dénotent les polynômes de Bernstein de degré n. Rappelons que les polynômes de Bernstein sont très utiles pour le dessin géométrique assisté par ordinateur (CGAD) [4].

Quand on considère la représentation de puissances, les dérivées au point extrême 0 sont données par (2), tandis que dans le cas de la représentation de Bernstein, en utilisant la classique formule, (3), (cf., formule (4.19) de [4]) où Δ^r est l'opérateur itératif de différences progressives ($\Delta^r \gamma_j^{\mathrm{B}} = \Delta^{r-1}\gamma_{j+1}^{\mathrm{B}} - \Delta^{r-1}\gamma_j^{\mathrm{B}}$) et en utilisant le fait que $b_j^{n-r}(0) = 0$ pour tout $j > 0$, on obtient l'Éq. (4).

Mais, à l'extrêmité 1 la représentation de puissances ne parvient pas à offrir une formule simple pour les coefficients, ce qui diffère de la représentation de Bernstein : $\mathrm{d}^r p_n/\mathrm{d}t^r(1) = n!/(n-r)!\Delta^r \gamma_{n-r}^{\mathrm{B}}$ (il suffit d'utiliser (3) et le fait que $b_j^{n-r}(1) = 0$ pour tout $j < n-r$). D'un autre côté, $\gamma_0^{\mathrm{M}} = p(0) = \gamma_0^{\mathrm{B}}$ et d'après (2) et (4) on obtient la relation (5) entre les coefficients des deux représentations.

D'après (5) on montre que $\gamma_1^{\mathrm{B}} = \gamma_1^{\mathrm{M}}/\binom{n}{1} + \gamma_0^{\mathrm{B}}$ et que $\gamma_2^{\mathrm{B}} = \gamma_2^{\mathrm{M}}/\binom{n}{2} + 2\gamma_1^{\mathrm{B}} - \gamma_0^{\mathrm{B}}$. Ensuite, on déduit sans peine l'Éq. (6), où l'on a posé $\tilde{\gamma}_k^{\mathrm{M},i} = \gamma_k^{\mathrm{B}}$ pour $k = 0, \ldots, i-1$, et $\tilde{\gamma}_i^{\mathrm{M},i} = -\gamma_i^{\mathrm{M}}/\binom{n}{i}$. Cela fournit une formule alternative pour changer la représentation en suite de puissances en celle de Bernstein.

Dans le Tableau 1 on détaille tous les algorithmes de conversion, tandis qu'au Tableau 2 on donne les coefficients $C_{m,k}, \pi_m$. Les algorithmes ont été obtenus selon le schéma détaillé dans les Éqs. (1), (2), (4), (5) et (6) et en appliquant les changements adéquats dans chaque cas (notons que pour une famille générale de polynômes orthogonaux les coefficients de la représentation polynomiale sont notés γ_i^{O}, tandis que dans le cas de deux familles de polynômes orthogonaux, ils sont notés par γ_i^{I} et γ_i^{II}).

1. Introduction

In many theoretical and practical applications several representations of polynomials are used and they are frequently converted into each other [7,9,10]. In this Note we deal with the conversion problem, providing in a unified way a complete set of formulas for changing from any of the most used univariate basis (Bernstein, monomials and the classical families of orthogonal polynomials: Jacobi, Gegenbauer, Legendre, Chebyshev, Laguerre and Hermite) into another one. Although in the literature there exist several conversion algorithms [7,9,10] any of them is obtained by using different techniques. On the contrary, the present approach permits to obtain all the conversions formulas among any family of polynomial series following a very simple idea.

2. Conversion algorithms

In order to derive the conversion algorithms, let us first compare the behaviour of Bernstein and power representations with respect to the problem of evaluating at the end points 0 and 1. Given a polynomial $p_n(t)$ in the space of polynomials of degree less than or equal to n on $t \in [0,1]$ it can be expressed in the following two ways:

$$p_n(t) = \sum_{i=0}^{n} \gamma_i^{\mathrm{M}} t^i, \qquad p_n(t) = \sum_{i=0}^{n} \gamma_i^{\mathrm{B}} b_i^n(t), \tag{1}$$

where $b_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$ denotes the Bernstein polynomials of degree n. Let us recall that Bernstein polynomials are very useful in computer aided geometric design [4].

Then, the derivatives at the end point 0 are given by

$$\frac{\mathrm{d}^r(p_n)}{\mathrm{d}t^r}(0) = r!\gamma_r^{\mathrm{M}}, \quad r = 0, \ldots, n, \tag{2}$$

in the case of the power representation. In the case of the Bernstein form, we have used the well-known formula (cf. formula (4.19) of [4])

$$\frac{\mathrm{d}^r(p_n(t))}{\mathrm{d}t^r} = \frac{n!}{(n-r)!}\sum_{j=0}^{n-r}\left(\Delta^r\gamma_j^{\mathrm{B}}\right)b_j^{n-r}(t), \quad r = 0, \ldots, n, \tag{3}$$

where Δ^r is the iterated forward difference operator ($\Delta^r\gamma_j^{\mathrm{B}} = \Delta^{r-1}\gamma_{j+1}^{\mathrm{B}} - \Delta^{r-1}\gamma_j^{\mathrm{B}}$) and the fact that $b_j^{n-r}(0) = 0$ for all $j > 0$, therefore we have obtained

$$\frac{\mathrm{d}^r(p_n)}{\mathrm{d}t^r}(0) = \frac{n!}{(n-r)!}\Delta^r\gamma_0^{\mathrm{B}}. \tag{4}$$

However, at the end point 1 the power representation does not provide an easy formula in terms of the coefficients, in contrast with the Bernstein representation: $\mathrm{d}^r p_n/\mathrm{d}t^r(1) = n!/(n-r)!\Delta^r\gamma_{n-r}^{\mathrm{B}}$ (use (3) and the fact that $b_j^{n-r}(1) = 0$ for all $j < n-r$). On the other hand, $\gamma_0^{\mathrm{M}} = p(0) = \gamma_0^{\mathrm{B}}$ and from (2) and (4) we can derive the following relationship between the coefficients of both representations:

$$\gamma_r^{\mathrm{M}} = \binom{n}{r}\Delta^r\gamma_0^{\mathrm{B}}, \quad r = 0, 1, \ldots, n. \tag{5}$$

From (5) we obtain $\gamma_1^{\mathrm{B}} = \gamma_1^{\mathrm{M}}/\binom{n}{1} + \gamma_0^{\mathrm{B}}$ and $\gamma_2^{\mathrm{B}} = \gamma_2^{\mathrm{M}}/\binom{n}{2} + 2\gamma_1^{\mathrm{B}} - \gamma_0^{\mathrm{B}}$, and so, it can be easily deduced that

$$\gamma_i^{\mathrm{B}} = -\Delta^i\tilde{\gamma}_0^{\mathrm{M},i} \tag{6}$$

if we define $\tilde{\gamma}_k^{\mathrm{M},i} = \gamma_k^{\mathrm{B}}$ for $k = 0, \ldots, i-1$, and $\tilde{\gamma}_i^{\mathrm{M},i} = -\gamma_i^{\mathrm{M}}/\binom{n}{i}$. This provides us an alternative formula to transform power series representation into Bernstein representation. We note that in the literature there are several conversion methods. For instance, let us mention that the transformation from the Bernstein basis to the monomials and the converse change can be done by the binomial formulas (10.5a) and (10.5b) of [6] and it can be also accomplished by blossoming. However, our approach provides a unified method of obtention of conversion formulas between many families of polynomials.

We can observe that the algorithm corresponding to (5) requires $\mathcal{O}(n)$ multiplications (n if we assume that the numbers $\binom{n}{i}$ are tabulated) and $n(n+1)/2$ subtractions. The algorithm (6) requires n divisions and n^2 subtractions. In order to calculate the number of subtractions, observe that, if we have calculated $\Delta^{i-1}\tilde{\gamma}_0^{\mathrm{M},i-1}$, then we can keep the information obtained with the differences of $\gamma_0^{\mathrm{B}}, \ldots, \gamma_{i-2}^{\mathrm{B}}$ and we obtain new differences with $\gamma_{i-1}^{\mathrm{B}}$ and $-\gamma_i^{\mathrm{M}}/\binom{n}{i}$. Therefore, the computational complexity of algorithms corresponding to (5) and (6) is lower than that obtained by the usual procedure [7] of determining the triangular matrix of change of basis between Bernstein polynomials and monomials and, afterwards, multiplying such matrix by a coefficients vector.

Let $p_n(t) = p_n((x+1)/2) = p_n^*(x)$ be the polynomial in Jacobi ($P_i^{(\alpha,\beta)}(x)$), Gegenbauer ($C_i^{\lambda}(x)$), Legendre ($P_i(x)$), Chebyshev ($T_i(x)$), Laguerre ($L_i(t)$) or Hermite ($H_i(x)$) basis:

$$\begin{aligned} p_n(t) &= \sum_{i=0}^{n}\gamma_i^{\mathrm{M}}t^i = \sum_{i=0}^{n}\gamma_i^{\mathrm{B}}b_i^n(t) = \sum_{i=0}^{n}\gamma_i^{\mathrm{L}}L_i(t) = \sum_{i=0}^{n}\gamma_i^{\mathrm{H}}H_i(t) \quad t \in [0,1] \\ &= \sum_{i=0}^{n}\gamma_i^{\mathrm{J}}P_i^{(\alpha,\beta)}(x) = \sum_{i=0}^{n}\gamma_i^{\mathrm{G}}C_i^{\lambda}(x) = \sum_{i=0}^{n}\gamma_i^{\mathrm{P}}P_i(x) = \sum_{i=0}^{n}\gamma_i^{\mathrm{T}}T_i(x) \quad x \in [-1,1]. \end{aligned} \tag{7}$$

Note that for the Bernstein and monomial basis we have taken the usual interval [0, 1], for Jacobi, Gegenbauer, Legendre and Chebyshev the canonical interval is [−1, 1], while for Laguerre and Hermite we have considered the interval $[0,1]$. The intervals of definition for Laguerre and Hermite are the intervals $[0,+\infty)$ and $(-\infty,+\infty)$, respectively, but in the conversion problem we consider a projection onto $[0,1]$. In the forthcoming, in order to simplify the notation, we denote by $p_n(x)$ the polynomial $p_n^*(x)$.

Following the above approach for the Bernstein and monomial conversion, in order to derive conversion algorithms for other families of polynomials, we need to obtain the values of the derivatives of the polynomial series at one end point. For example, given p_n the polynomial given by (7) and considering the case of Jacobi orthogonal polynomial series, we have the following properties (see [8]):

$$\frac{\mathrm{d}^m P_i^{(\alpha,\beta)}(x)}{\mathrm{d}x^m} = 1/2^m (i+\alpha+\beta+1)_m P_{i-m}^{(\alpha+m,\beta+m)}(x),$$
$$P_i^{(\alpha,\beta)}(-1) = (-1)^i \binom{i+\beta}{i}, \qquad P_i^{(\alpha,\beta)}(1) = \binom{i+\alpha}{i}.$$

Table 1
Conversion algorithms

Tableau 1
Algorithmes de conversion

MONOMIAL ⟶ BERNSTEIN	BERNSTEIN ⟶ MONOMIAL
$\gamma_0^{\mathrm{B}} = \gamma_0^{\mathrm{M}}$ **for** $i = 1$ **to** n **for** $k = 0$ **to** $i-1$ $\tilde\gamma_k^{\mathrm{M},i} = \gamma_k^{\mathrm{B}}$ **end** $\tilde\gamma_i^{\mathrm{M},i} = -\gamma_i^{\mathrm{M}}/\binom{n}{i}$ $\gamma_i^{\mathrm{B}} = -\Delta^i \tilde\gamma_0^{\mathrm{M},i}$ **end**	**for** $i = 0$ **to** n $\gamma_i^{\mathrm{M}} = \binom{n}{i}\Delta^i \gamma_0^{\mathrm{B}}$ **end**
ORTHOGONAL ⟶ BERNSTEIN	BERNSTEIN ⟶ ORTHOGONAL
$\gamma_0^{\mathrm{B}} = \sum_{k=0}^n C_{0,k}\gamma_k^{\mathrm{O}}$ **for** $i = 1$ **to** n **for** $k = 0$ **to** $i-1$ $\tilde\gamma_k^i = \gamma_k^{\mathrm{B}}$ **end** $\tilde\gamma_i^i = -\pi_i \frac{(n-i)!}{n!}\sum_{r=i}^n C_{i,r}\gamma_r^{\mathrm{O}}$ $\gamma_i^{\mathrm{B}} = -\Delta^i \tilde\gamma_0^i$ **end**	$\gamma_n^{\mathrm{O}} = \Delta^n \gamma_0^{\mathrm{B}} \frac{n!}{\pi_n C_{n,n}}$ **for** $i = n-1$ **to** 0 **by** -1 $\gamma_i^{\mathrm{O}} = \left(\frac{\Delta^i\gamma_0^{\mathrm{B}}}{\pi_i}\frac{n!}{(n-i)!} - \sum_{k=i+1}^n C_{i,k}\gamma_k^{\mathrm{O}}\right)/C_{i,i}$ **end**
ORTHOGONAL ⟶ MONOMIAL	MONOMIAL ⟶ ORTHOGONAL
for $i = 0$ **to** n $\gamma_i^{\mathrm{M}} = \frac{\pi_i}{i!}\sum_{r=i}^n C_{i,r}\gamma_r^{\mathrm{O}}$ **end**	$\gamma_n^{\mathrm{O}} = \frac{n!}{\pi_n}\frac{\gamma_n^{\mathrm{M}}}{C_{n,n}}$ **for** $i = n-1$ **to** 0 **by** -1 $\gamma_i^{\mathrm{O}} = \left(\frac{i!\gamma_i^{\mathrm{M}}}{\pi_i} - \sum_{k=i+1}^n C_{i,k}\gamma_k^{\mathrm{O}}\right)/C_{i,i}$ **end**
ORTHOGONAL II ⟶ ORTHOGONAL I	
$\gamma_n^{\mathrm{I}} = \frac{\pi_n^{\mathrm{II}}}{\pi_n^{\mathrm{I}}}\frac{C_{n,n}^{\mathrm{II}}}{C_{n,n}^{\mathrm{I}}}\gamma_n^{\mathrm{II}}$ **for** $i = n-1$ **to** 0 **by** -1 $\gamma_i^{\mathrm{I}} = \left(\frac{\pi_i^{\mathrm{II}}}{\pi_i^{\mathrm{I}}}\sum_{k=i}^n C_{i,k}^{\mathrm{II}}\gamma_k^{\mathrm{II}} - \sum_{k=i+1}^n C_{i,k}^{\mathrm{I}}\gamma_k^{\mathrm{I}}\right)/C_{i,i}^{\mathrm{I}}$ **end**	

Now, we can easily obtain

$$\left.\frac{\mathrm{d}^m p_n(x)}{\mathrm{d}t^m}\right|_{x=-1} = \sum_{k=m}^{n} \left\{(-1)^{k-m}(k+\alpha+\beta+1)_m \binom{k+\beta}{k-m}\right\} \gamma_k^{\mathrm{J}},$$

$$\left.\frac{\mathrm{d}^m p_n(x)}{\mathrm{d}t^m}\right|_{x=1} = \sum_{k=m}^{n} \left\{(k+\alpha+\beta+1)_m \binom{k+\alpha}{k-m}\right\} \gamma_k^{\mathrm{J}},$$

where $(n)_i = n\cdot(n+1)\cdots(n+i-1)$ stands for the Pochhammer symbol.

The rest of the cases may be obtained just by following the same arguments and taking into account the formulas for each particular family of orthogonal polynomials [1,8].

Taking into account the previous result, we can unify the notation of the derivatives at the left end of the corresponding interval $[\ell, 1]$ ($\ell = -1$ or $\ell = 0$) of the different families of orthogonal polynomials. Let $p_n(u) = \sum_{i=0}^{n} \gamma_i \mathrm{P}_i(u)$ (with $u = x$ or $u = t$) be a polynomial expressed in terms of any of such families and so

$$\left.\frac{\mathrm{d}^m p_n(u)}{\mathrm{d}t^m}\right|_{u=\ell} = \pi_m \sum_{k=m}^{n} C_{m,k}\gamma_k.$$

In Table 1, we detail all the conversion algorithms and the coefficients $C_{m,k}, \pi_m$ are provided in Table 2. The algorithms have been obtained following the scheme detailed in Eqs. (1), (2), (4), (5) and (6) and performing the necessary changes on each case (for a general family of orthogonal polynomials the coefficients of the polynomial representation are denoted by γ_i^{O}, and in the case of two families of orthogonal polynomials γ_i^{I} and γ_i^{II}).

Let us finish with some comments on the numerical stability of the algorithms. On one hand, it is known that the alternative method of calculating the corresponding matrix of change of basis has the handicap of the exponential growth of the condition number of the matrix [5]. On the other hand, our algorithms use linear recurrences and forward differences, which present a reasonable stability behaviour [3,2]. Finally, just as a brief example, we have performed using MATLAB a numerical test on the conversions between Bernstein and power series. As a test example we have considered the polynomials with coefficients $\gamma_i^{\mathrm{M}} = 1/(i+1)$ or $\gamma_i^{\mathrm{B}} = 1/(i+1)$ and with degrees $n = 5, 10, 15, 20$. In the conversion from monomials to the Bernstein basis, the maximum relative error in the coefficients is 3.82×10^{-16} $(n=5)$, 1.63×10^{-15} $(n=10)$, 2.56×10^{-13} $(n=15)$ and 2.26×10^{-12} $(n=20)$ and from the Bernstein basis to monomials 4.44×10^{-16} $(n=5)$, 1.17×10^{-14} $(n=10)$, 3.12×10^{-13} $(n=15)$ and 3.96×10^{-12} $(n=20)$.

Table 2
Coefficients of the conversion algorithms

Tableau 2
Coefficients des algorithmes de conversion

$\mathrm{P}_i(x)$	$C_{m,k}$	π_m
Jacobi ($P_i^{(\alpha,\beta)}(x)$)	$(-1)^{k-m}(k+\alpha+\beta+1)_m\binom{k+\beta}{k-m}$	1
Gegenbauer ($C_i^{\lambda}(x)$)	$(-1)^{k-m}\binom{2\lambda+m+k-1}{k-m}$	$2^{2m}(\lambda)_m$
Legendre ($P_i(x)$)	$(-1)^{k-m}\binom{m+k}{k-m}$	$2^{2m}(1/2)_m$
Chebyshev ($T_i(x)$)	$\begin{cases} C_{0,k} = (-1)^k \\ C_{m,k} = k(-1)^{k-m}\binom{m+k-1}{k-m}, & m>0 \end{cases}$	$\begin{cases} \pi_0 = 1 \\ \pi_m = 2^{2m-1}(m-1)! \end{cases}$
Laguerre ($L_i(t)$)	$\binom{k}{k-m}$	$(-1)^m$
Hermite ($H_i(t)$)	$\begin{cases} 0, & \text{for } k-m=2j+1 \\ (-1)^j\ (j+1)_{j+m}, & \text{for } k-m=2j \end{cases}$	2^m

References

[1] R. Barrio, J.M. Peña, Numerical evaluation of the p-th derivative of the Jacobi series, Appl. Numer. Math. 43 (2002) 335–357.
[2] R. Barrio, J.M. Peña, Evaluation of the derivative of a polynomial in Bernstein form, Appl. Math. Comput., in press.
[3] R. Barrio, B. Melendo, S. Serrano, On the numerical evaluation of linear recurrences, J. Comput. Appl. Math. 150 (2003) 71–86.
[4] G. Farin, Curves and Surfaces for Computer Aided Geometric Design, fourth ed., Academic Press, San Diego, 1996.
[5] R.T. Farouki, Legendre–Bernstein basis transformations, J. Comput. Appl. Math. 119 (2000) 145–160.
[6] J. Hoschek, D. Lasser, in: A.K. Peters (Ed.), Fundamentals of Computer Aided Geometric Design, Wellesley, 1993.
[7] Y.M. Li, X.Y. Zhang, Basis conversion among Bezier, Tchebyshev and Legendre, Comput. Aided Geom. Design 15 (1998) 637–642.
[8] W. Magnus, F. Oberhettinger, R.P. Soni, Formulas and Theorems for the Special Functions of Mathematical Physics, Springer-Verlag, 1966.
[9] A. Ronveaux, A. Zarzo, E. Godoy, Recurrence relations for connection coefficients between two families of orthogonal polynomials, J. Comput. Appl. Math. 62 (1995) 67–73.
[10] B.Y. Ting, Y.L. Luke, Conversion of polynomials between different polynomial bases, IMA J. Numer. Anal. 1 (1981) 229–234.